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DISTRIBUTIONAL DERIVATIVES ON A REGULAR OPEN SURFACE WITH PHYSICAL APPLICATIONS

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Abstract. The spatial derivatives of Schwartz-Sobolev distributions which display singularities of arbitrary order on an arbitrary regular *open* surface are investigated. The contributions of the present investigation to literature are i) an approach alternative to the derivation of the distributional derivatives of multilayers by Estrada and Kanwal; ii) an extension of the available results for *closed* surfaces to *open* surfaces featuring boundary distributions of *arbitrary* order. The end results are applied in the distributional investigation of Maxwell equations in presence of single and double layer sources located on a regular open surface.

Keywords: Schwartz-Sobolev distributions, vector analysis, electromagnetic theory, boundary relations, single and double layer structures.

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1. Introduction

In this work we provide a systematic investigation of spatial derivatives of field quantities which display arbitrary singular behavior on a regular open surface in a Schwartz-Sobolev space setting. Our approach and notation conform to the teachings of Ricardo Estrada and Ram Kanwal in the area regarding the study of propagation of wavefronts and multilayers, which go back to year 1980 [1] and are summarized in the textbook [2]. The contributions of the present investigation to literature are

(1) an approach alternative to the derivation of the distributional derivatives of multilayers by Estrada and Kanwal [3,Proposition 1];
(2) an extension of the available results for *closed* surfaces to *open* surfaces featuring boundary distributions of *arbitrary* order.

Following the differentiation of the singular and regular components of arbitrary distributions, the end results are applied in the distributional investigation of Maxwell equations in presence of single and double layer sources located on a regular open surface. Throughout the text R_n stands for n dimensional Euclidean space.

In a Schwartz-Sobolev space setting we may assume arbitrary scalar and vector field quantities to be expressed in the conventional form

$$V(\vec{r};t) = \{V(\vec{r};t)\} + [V(\vec{r};t)]_S \, , \vec{A}(\vec{r};t) = \left\{\vec{A}(\vec{r};t)\right\} + \left[\vec{A}(\vec{r};t)\right]_S . \tag{1.1}$$

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The "regular component" of a distribution given in curly brackets is defined outside the singularity domain and assumed to be of L^1_{loc}, the class of locally integrable functions in the Lebesgue sense, in any compact subspace of R_3. The "singular component" of a distribution with representation $[]_S$ is assumed to be of D', the class of Schwartz-Sobolev distributions, in the present case concentrated on a regular open surface S. The singular components are constructed through the Dirac delta distributions and their spatial and temporal derivatives of every order which are postulated to represent the source quantities successfully for all types of polarization (molecular displacement) mechanisms in classical electromagnetic theory. The spatial derivatives of locally integrable functions in L^1_{loc} may generate distributions in D' as addressed in Theorem 4.1.

2. Schwartz-Sobolev Description of Surface Distributions

Let us consider an arbitrary regular open surface S in R_3 as depicted in Fig.1., where $Ox_1x_2x_3$ is the Cartesian reference frame and (n, v, λ) signifies the local curvilinear coordinate system for S such that the surface is spanned by (v, λ), while n denotes the normal curve.

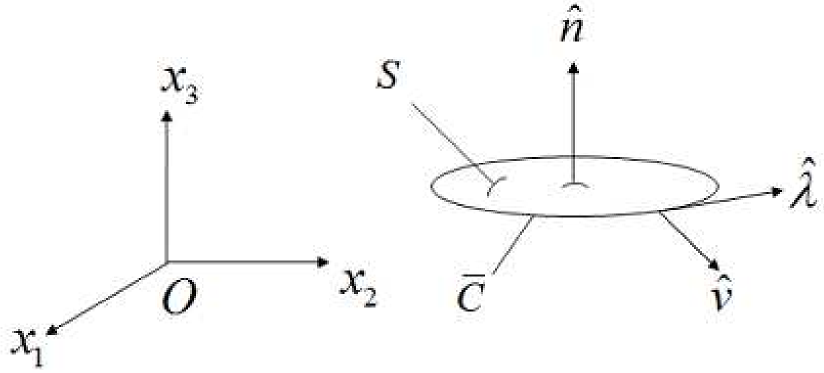

Figure 1. Local and Cartesian reference frames of an arbitrary regular open surface

We assume $(\hat{n}, \hat{v}, \hat{\lambda})$ constitute a right-handed triple of orthogonal unit vectors such that $\hat{\lambda}$ is the unit tangent to the boundary curve $\bar{C} = \partial S$, and $\hat{v}$ is located in the plane tangent to S but is normal to $\bar{C}$.

To ensure the smooth behavior of the regular components in (1.1) in presence of a surface of singularity S, one requires them to be *regular singular* functions ([2], Sec. 5.5) in a subspace $E \subset L^1_{loc}$ defined with the properties

(1) $\{V(\vec{r};t)\}$ and $\left\{\vec{A}(\vec{r};t)\right\}$ have spatial derivatives of all orders outside S, and

(2) $\{V(\vec{r};t)\}$, $\left\{\vec{A}(\vec{r};t)\right\}$ and all their spatial derivatives have bounded boundary values as one approaches from both sides of S.

The surface distribution of order $k+1$, $d^k_n\delta(S)$, is described by the inner product

$$\left\langle d^k_n\delta(S), \phi(\vec{r};t)\right\rangle = (-1)^k\left\langle \delta(S), \frac{d^k}{dn^k}\phi(\vec{r};t)\right\rangle = (-1)^k\int_{-\infty}^{\infty}\int_S \frac{d^k}{dn^k}\phi(\vec{r};t)dS\,dt, \forall k. \quad (2.1)$$

where $\phi(\vec{r};t) \in D$ is a test function infinitely differentiable (in C^∞) with a compact support and dS is the surface measure on S. They have the MKSA unit $[m^{-k-1}]$. In virtue of the functional spaces mentioned so far we can make the topological remark $C^\infty_0 = D \subset C^\infty \subset C^l \subset C \subset E \subset L^1_{loc} \subset D'$, where C^l represents the space of continuous functions which have continuous $l > 1$ derivatives, and D' is the dual space of D. For algebraic purposes we prefer to express an open surface S as a finite section of a closed surface $\bar{S}$, which reads

$$d_n^k \delta(S) = H[S]\, d_n^k \delta(\bar{S}), \tag{2.2}$$

where

$$H[S] = \begin{cases} 1, \vec{r} \in S \\ 0, \vec{r} \notin S \end{cases} \tag{2.3}$$

represents the characteristic function of S. Accordingly, the singular component of a surface distribution of arbitrary order can be constructed through "multilayers" in the most general form

$$[V(\vec{r};t)]_S = H[S] \sum_{k=0}^{\infty} V_k(\vec{r}_S;t) d_n^k \delta(\bar{S}),\ \left[\vec{A}(\vec{r};t)\right]_S = H[S] \sum_{k=0}^{\infty} \vec{A}_k(\vec{r}_S;t) d_n^k \delta(\bar{S}) \tag{2.4}$$

with the definition

$$\begin{aligned} \left\langle \vec{A}_k(\vec{r}_S;t) d_n^k \delta(\bar{S}), \phi(\vec{r};t) \right\rangle &= (-1)^k \left\langle \delta(\bar{S}), \frac{d^k}{dn^k} \left(\tilde{\vec{A}}_k(\vec{r};t) \phi(\vec{r};t) \right) \right\rangle \\ &= (-1)^k \left\langle \delta(\bar{S}), \tilde{\vec{A}}_k(\vec{r};t) \frac{d^k}{dn^k} \phi(\vec{r};t) \right\rangle \\ &= (-1)^k \int_{-\infty}^{\infty} \int_{\bar{S}} \vec{A}_k(\vec{r}_S;t) \frac{d^k}{dn^k} \phi(\vec{r};t) dS\, dt \end{aligned}$$

Here $\vec{r}_S$ is the position vector for any point on S; $V_k(\vec{r}_S;t)$, $\vec{A}_k(\vec{r}_S;t)$ are smooth, locally integrable "density functions" with supports defined on S; and $\tilde{\vec{A}}_k(\vec{r};t)$ is any extension of $\vec{A}_k(\vec{r}_S;t)$ to a neighborhood of S in R_3 with the properties

$$\frac{d^k}{dn^k} \tilde{\vec{A}}_k(\vec{r};t) = \vec{0}, \tilde{\vec{A}}_k(\vec{r}_S;t) d_n^k \delta(\bar{S}) = \vec{A}_k(\vec{r}_S;t) d_n^k \delta(\bar{S}), \forall k.$$

Theorem 2.1. *Every distribution that has compact support is of finite order.*

This general regularity theorem is quite well known and many alternative approaches to its proof are available in literature (cf.[4, Ch.3], [5, Ch.3 Sec.6]).

The reflection of this theorem for a surface type distribution whose support lies on a regular surface S is that the field quantities in (2.4) have the unique representation

$$\begin{aligned} V(\vec{r};t) &= \{V(\vec{r};t)\} + H[S] \sum_{k=0}^{N} V_k(\vec{r}_S;t) d_n^k \delta(\bar{S}), \\ \vec{A}(\vec{r};t) &= \left\{\vec{A}(\vec{r};t)\right\} + H[S] \sum_{k=0}^{N} \vec{A}_k(\vec{r}_S;t) d_n^k \delta(\bar{S}) \end{aligned} \tag{2.5}$$

where N is a *finite* number.

Let (u_1, u_2) be real valued orthogonal Gaussian curves of S with unit tangent vectors $\hat{u}_1$,$\hat{u}_2$ along the curves $u_1 =$ const. and $u_2 =$ const. and $\hat{n}$ is the unit normal of S such that $(\hat{u}_1, \hat{u}_2, \hat{n})$constitute a right handed system.

Definition 2.1. *Let* $\psi(u_1, u_2;t)$ *and* $\vec{A}(u_1, u_2;t) = A_1(u_1, u_2;t)\hat{u}_1 + A_2(u_1, u_2;t)\hat{u}_2 + A_n(u_1, u_2;t)\hat{n}$ *be regular scalar/vector density functions defined only on the surface* S.

Then $\frac{D\psi}{Dx_i} = \hat{x}_i \cdot \left(\frac{\hat{u}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial\psi}{\partial u_2} \right)$, $i = 1, 2, 3$ *denote the spatial derivatives as computed by an observer on the surface.*

The present notation $\frac{D}{Dx_i}$ is given as $\frac{\delta}{\delta x_i}$ in [2] and the references therein. We have introduced the new notation intentionally so that it does not mix with the Dirac delta distribution symbol.

Theorem 2.2. *Directional (normal) derivatives of a first order surface distribution* $\delta(S)$, *namely*

$$\delta^{(k+1)}(\bar{S}) \equiv \hat{n} \cdot grad\delta^{(k)}(\bar{S}) = \frac{d^{k+1}}{dn^{k+1}}\delta(\bar{S}), \forall k, \tag{2.6}$$

are described by

$$\begin{aligned} \left\langle \delta^{(k)}(\bar{S}), \phi(\vec{r};t) \right\rangle &= (-1)^k \left\langle \delta(\bar{S}), \left(\frac{d}{dn} - 2\Omega\right)^k \phi(\vec{r};t) \right\rangle \\ &= (-1)^k \int_{-\infty}^{\infty} \int_{\bar{S}} \left(\frac{d}{dn} - 2\Omega\right)^k \phi(\vec{r};t) d\bar{S}\, dt, \forall k \end{aligned} \tag{2.7}$$

where 2Ω *is the first curvature (and* Ω*, the mean curvature) of* S.

Proof. The first order derivative is calculated by using Einstein summation notation as

$$\begin{aligned} \left\langle \delta^{(1)}(\bar{S}), \phi \right\rangle &= \left\langle n_i \frac{\partial}{\partial x_i}\delta(\bar{S}), \phi \right\rangle = \left\langle \frac{\partial}{\partial x_i}\delta(\bar{S}), n_i\phi \right\rangle = -\left\langle \delta(\bar{S}), \frac{\partial}{\partial x_i}(n_i\phi) \right\rangle \\ &= -\left\langle \delta(\bar{S}), n_i\frac{\partial\phi}{\partial x_i} + \phi\frac{Dn_i}{Dx_i} \right\rangle = -\left\langle \delta(\bar{S}), \frac{d\phi}{dn} - 2\Omega\phi \right\rangle = \left\langle 2\Omega\delta(\bar{S}) + d_n\delta(\bar{S}), \phi \right\rangle \end{aligned}$$

i.e.,

$$\delta^{(1)}(\bar{S}) = 2\Omega\delta(\bar{S}) + d_n\delta(\bar{S}). \tag{2.8}$$

Successive applications of the property (2.8) yields the desired result (2.7) since $\frac{d\Omega}{dn} = 0$. □

Theorem 2.3. *The linear relation between the two sets of distributions* $\delta^{(k)}(\bar{S})$ *and* $d_n^k\delta(\bar{S})$ *are given by*

$$\begin{aligned} \delta^{(k)}(\bar{S}) &= \sum_{m=0}^{k} \binom{k}{m} (2\Omega)^{k-m}\, d_n^m\delta(\bar{S}), \\ d_n^k\delta(\bar{S}) &= \sum_{m=0}^{k} \binom{k}{m} (-2\Omega)^{k-m}\, \delta^{(m)}(\bar{S}), \forall k \end{aligned} \tag{2.9}$$

Proof. From (2.7) one can write

$$\begin{aligned}
\left\langle \delta^{(k)}(\bar{S}),\phi \right\rangle &= (-1)^k \left\langle \delta(\bar{S}), \left(\frac{d}{dn} - 2\Omega\right)^k \phi \right\rangle \\
&= (-1)^k \left\langle \delta(\bar{S}), \sum_{m=0}^{k} \binom{k}{m} (-2\Omega)^{k-m} \frac{d^m\phi}{dn^m} \right\rangle \\
&= (-1)^k \left\langle \sum_{m=0}^{k} \binom{k}{m} (-2\Omega)^{k-m} (-1)^m d_m^n \delta(\bar{S}),\phi \right\rangle \\
&= \left\langle \sum_{m=0}^{k} \binom{k}{m} (2\Omega)^{k-m} d_m^n \delta(\bar{S}),\phi \right\rangle
\end{aligned}$$

A similar binomial expansion yields

$$\left\langle d_n^k \delta(\bar{S}),\phi \right\rangle = \left\langle \left(\frac{d}{dn} - 2\Omega\right)^k \delta(\bar{S}),\phi \right\rangle = \left\langle \sum_{m=0}^{k} \binom{k}{m} (-2\Omega)^{k-m} \delta^{(m)}(\bar{S}),\phi \right\rangle.$$

Regarding the unique solutions of the distributional forms of Maxwell equations in Sec.6, it requires to provide the following theorem. □

Theorem 2.4. *The unique solution of the relation*

$$\{V(\vec{r};t)\} + H[S] \sum_{k=0}^{N} V_k(\vec{r}_S;t) d_n^k \delta(\bar{S}) = 0 \tag{2.10}$$

with $V_k(\vec{r}_S;t)$, $\forall k$ *being smooth density functions is*

$$\{V(\vec{r};t)\} = 0 \quad \textit{and} \quad V_k(\vec{r}_S;t) = 0, \forall k \tag{2.11}$$

Proof. Since the regular and singular components of a distribution, by definition, have nonintersecting supports, (2.10) is automatically decomposed as $\{V(\vec{r};t)\} = 0$ and

$$H[S] \sum_{k=0}^{N} V_k(\vec{r}_S;t) d_n^k \delta(\bar{S}) = 0. \tag{2.12}$$

Without losing generality, we may consider $\bar{S}$ as a regular surface with orthogonal Gaussian curves (u_1,u_2,n) with metric coefficients (h_1,h_2,h_n) such that $\bar{S} = \{(u_1,u_2,n) | \, n = \alpha = const\}$. Then one has $\delta(\bar{S}) = \frac{1}{h_n}\delta(n-\alpha)$. For an arbitrary nonzero constant a let us invoke a scaling $n' = an$, $\alpha' = a\alpha$ such that $\bar{S}$ coincides with $\bar{S}' = \{(u_1,u_2,n') | \, n' = \alpha' = const\}$, which requires

$$H[S'] \sum_{k=0}^{N} V_k(\vec{r}_{S'};t) d_{n'}^k \delta(\bar{S}') = 0. \tag{2.13}$$

On the other hand we have the obvious relations

$$\vec{r}_S = \vec{r}_{S'}, H[S] = H[S'] \tag{2.14}$$

and the scaling property

$$d_{n'}^k \delta(\bar{S}') = \frac{1}{a^{k+1}} d_n^k \delta(\bar{S}), \forall k \tag{2.15}$$

to be proved immediately. Inserting (2.14), (2.15) into (2.13) we get

$$H[S]\sum_{k=0}^{N}\frac{1}{a^{k+1}}V_k(\vec{r}_S;t)d_n^k\delta(\bar{S})=0. \tag{2.16}$$

A comparison of (2.15) and (2.16) reveals that these two equations can be satisfied simultaneously for any nonzero constant a iff $V_k(\vec{r}_S;t)=0,\forall k$. This can be seen easily when we apply a different inner product

$$\begin{aligned}\left\langle V_k(\vec{r}_S;t)H[S]d_n^k\delta(\bar{S}),\phi(\vec{r};t)\right\rangle &= \oint V_k(\vec{r}_S;t)H[S]d_n^k\delta(\bar{S})\phi(\vec{r};t)dn\\ &=(-1)^kV_k(\vec{r}_S;t)\frac{d^k\phi}{dn^k}(\vec{r}_S;t),\forall k\end{aligned}$$

through which (2.12) and (2.16) can be rewritten as

$$\sum_{k=0}^{N}(-1)^kV_k(\vec{r}_S;t)\frac{d^k\phi}{dn^k}(\vec{r}_S;t)=0,\sum_{k=0}^{N}\frac{(-1)^k}{a^{k+1}}V_k(\vec{r}_S;t)\frac{d^k\phi}{dn^k}(\vec{r}_S;t)=0, \tag{2.17}$$

respectively. Since the test function $\phi(\vec{r};t)$ and constant a are arbitrary, the linear system of equations (2.17) can hold iff $V_k(\vec{r}_S;t)=0,\forall k$. To conclude let us also prove the scaling property (2.15) as follows:

$$\begin{aligned}\left\langle\delta(\bar{S}'),\phi(\vec{r};t)\right\rangle &= \left\langle\frac{1}{h_n}\delta(n'-\alpha'),\phi\left(u_1,u_2,\frac{n'}{a};t\right)\right\rangle\\ &=\int_{-\infty}^{\infty}\oint\delta(n'-\alpha')\phi\left(u_1,u_2,\frac{n'}{a};t\right)h_1h_2\frac{dn'}{a}du_1\,du_2d\\ &=\frac{1}{a}\int_{-\infty}^{\infty}\oint\phi(u_1,u_2,\alpha;t)h_1h_2du_1\,du_2dt=\left\langle\frac{1}{a}\delta(\bar{S}),\phi(\vec{r};t)\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle d_{n'}^k\delta(\bar{S}'),\phi(\vec{r};t)\right\rangle &= (-1)^k\left\langle\delta(\bar{S}'),\frac{d^k}{dn'^k}\phi(\vec{r};t)\right\rangle=(-1)^k\left\langle\frac{1}{a}\delta(\bar{S}),\frac{1}{a^k}\frac{d^k}{dn^k}\phi(\vec{r};t)\right\rangle\\ &=\left\langle\frac{(-1)^k}{a^{k+1}}d_n\delta(\bar{S}),\phi(\vec{r};t)\right\rangle.\end{aligned}$$

□

Corollary 2.1. *The unique solution of the relation*

$$\{V(\vec{r};t)\}+H[S]\sum_{k=0}^{N}V_k(\vec{r}_S;t)\delta^{(k)}(\bar{S})=0 \tag{2.18}$$

with $V_k(\vec{r}_S;t)$ being smooth density functions is

$$\{V(\vec{r};t)\}=0 \quad \text{and} \quad V_k(\vec{r}_S;t)=0,\forall k. \tag{2.19}$$

Corollary 2.1 can be verified in virtue of the linear relation between the two sets of distributions in Theorem 2.3.

3. Distributional Derivatives of the Singular Components

3.1. **Vector Operators Acting on Density Functions.** The surficial gradient, divergence and curl operators acting on arbitray scalar/vector density functions $\psi(u_1, u_2; t)$ and $\vec{A}(u_1, u_2; t)$ in Definition 2.1 are given by the following theorem, a proof of which can be found in many respected textbooks (cf. [6],[7, Ch.12] as earliest accounts, where the density functions are termed as "point functions").

Theorem 3.1.

$$grad_S\psi = \frac{\hat{u}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial\psi}{\partial u_2} \tag{3.1}$$

$$div_S\vec{A} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1) + \frac{\partial}{\partial u_2}(h_1A_2)\right] - 2\Omega A_n \tag{3.2}$$

$$curl_S\vec{A} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\hat{n} + \frac{A_2}{\alpha_2}\hat{u}_1 - \frac{A_1}{\alpha_1}\hat{u}_2 + grad_SA_n \times \hat{n}. \tag{3.3}$$

The metric coefficients h_1,h_2 of the curves are related to the principle radii of curvature $\alpha_{1,2}$ through

$$\frac{1}{\alpha_1} = -\frac{1}{h_1}\frac{dh_1}{dn}, \frac{1}{\alpha_2} = -\frac{1}{h_2}\frac{dh_2}{dn} \tag{3.4}$$

and the first curvature of S satisfies

$$2\Omega = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} = -\mathrm{div}_S(\hat{n}). \tag{3.5}$$

Let us write the tangential component of $\vec{A}$ as $\vec{A}_t$. Then for this component one also has

$$div_S\vec{A}_t = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1) + \frac{\partial}{\partial u_2}(h_1A_2)\right] \tag{3.6}$$

$$\begin{aligned} curl_S\vec{A}_t &= \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\hat{n} + \frac{A_2}{R_2}\hat{u}_1 - \frac{A_1}{R_1}\hat{u}_2 \\ &= \hat{n}\, div_S(\vec{A}_t \times \hat{n}) + \frac{A_2}{\alpha_2}\hat{u}_1 - \frac{A_1}{\alpha_1}\hat{u}_2 \end{aligned} \tag{3.7}$$

$$div_S(\vec{A}_t \times \hat{n}) = \hat{n} \cdot curl_S\vec{A}_t \tag{3.8}$$

$$curl_S\left(\hat{n} \times \vec{A}_t\right) = \hat{n}\, div_S\vec{A}_t + \frac{A_1}{\alpha_2}\hat{u}_1 + \frac{A_2}{\alpha_1}\hat{u}_2 \tag{3.9}$$

$$div_S\vec{A}_t = \hat{n} \cdot curl_S\left(\hat{n} \times \vec{A}_t\right) \tag{3.10}$$

3.2. **Distributional Derivatives of the Characteristic Function.**

Lemma 3.1. *On a regular surface S with boundary $\bar{C} = \partial S$ the characteristic function $H[S]$ has the property*

$$\overline{grad}_S H[S] = -\hat{v}\delta(\bar{C}) \tag{3.11}$$

where $\delta(\bar{C})$ represents the distribution in R_2 with support on the boundary curve $\bar{C}$ and is defined by

$$\left\langle\delta(\bar{C}), \phi\right\rangle = \oint_{\bar{C}} \phi(\vec{r}_{\bar{C}})d\bar{C}. \tag{3.12}$$

In (3.12) and the rest of the analysis the bar sign signifies the distributional derivatives of density functions on a surface, a notation which was first introduced and used in [8].

Proof. The surficial gradient in (3.1) is related to an adjoint surficial gradient

$$\nabla_S^*\psi = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(\hat{u}_1h_2\psi) + \frac{\partial}{\partial u_2}(\hat{u}_2h_1\psi)\right], \psi \in D' \tag{3.13}$$

through

$$grad_S\psi = \nabla_S^*\psi - \hat{n}2\Omega\psi$$

or equivalently

$$\langle grad_S\psi, \phi\rangle = \langle \nabla_S^*\psi, \phi\rangle - \langle \hat{n}2\Omega\psi, \phi\rangle \tag{3.14}$$

and the distributional adjoint relation

$$\langle \nabla_S^*\psi, \phi\rangle = -\langle \psi, grad_S\phi\rangle. \tag{3.15}$$

Combining (3.14) and (3.15) one gets

$$\langle grad_S\psi, \phi\rangle = -\langle \psi, grad_S\phi\rangle - \langle \hat{n}2\Omega\psi, \phi\rangle. \tag{3.16}$$

Accordingly, for $\psi = H[S]$ we have

$$\left\langle \overline{grad}_S H[S], \phi\right\rangle = -\langle H[S], grad_S\phi\rangle - \langle \hat{n}2\Omega H[S], \phi\rangle. \tag{3.17}$$

A direct substitution of the standard surface gradient theorem (cf. [9, p.278, eq.(402)])

$$\int_S grad_S\phi dS + \int_S \hat{n}2\Omega\phi dS = \oint_{\bar{C}=\partial S} \phi\hat{v}dC, \tag{3.18}$$

which can be written in distributional form in R_2 as

$$\langle H[S], grad_S\phi\rangle + \langle \hat{n}2\Omega H[S], \phi\rangle = \left\langle \hat{v}\delta(\bar{C}), \phi\right\rangle, \tag{3.19}$$

into (3.17) yields the desired result (3.11). □

Lemma 3.2. *On an arbitrary regular surface one has the property*

$$\left(\overline{grad}_S H[S]\right)\delta(\bar{S}) = -\hat{v}\delta(\bar{C}), \tag{3.20}$$

where $\delta(\bar{C})$ represents the distribution in R_3 with support on the boundary curve $\bar{C}$.

Proof. $\bar{C}$ in R_2 corresponds to a cylinder $S_\perp$ with cross section $\bar{C}$ and unit normal $\hat{v}$ in R_3. If we denote the distribution for this cylinder by $\delta(S_\perp)$, $\bar{C}$ in R_3 can be considered as the intersection of the surfaces S and $S_\perp$; namely, $\delta(\bar{C}) = \delta(S)\delta(S_\perp)$, which also verifies (3.20). □

3.3. **Distributional Derivatives of Dirac Delta Singularities of Arbitrary Order.**

Theorem 3.2. *The surface distribution $\delta^{(k)}(\bar{S})$, $\forall k$ has zero local spatial partial derivatives:*

$$\frac{D}{Dx_i}\delta^{(k)}(\bar{S}) = 0, i = 1,2,3 \quad or \quad grad_S\delta^{(k)}(\bar{S}) = \vec{0}. \tag{3.21}$$

Proof. One can directly employ (3.16) as

$$\left\langle grad_S\delta^{(k)}(\bar{S}), \phi\right\rangle = -\left\langle \delta^{(k)}(\bar{S}), grad_S\phi\right\rangle - \left\langle \hat{n}2\Omega\delta^{(k)}(\bar{S}), \phi\right\rangle$$

$$= -(-1)^k\left\langle \delta(\bar{S}), grad_S\left(\frac{d}{dn} - 2\Omega\right)^k\phi\right\rangle - (-1)^k\left\langle \delta(\bar{S}), \hat{n}2\Omega\left(\frac{d}{dn} - 2\Omega\right)^k\phi\right\rangle,$$

which, upon substituting the surface gradient theorem for $\left(\frac{d}{dn}-2\Omega\right)^k\phi$ on a closed surface; namely,

$$\left\langle \delta(\bar{S}), grad_S\left(\frac{d}{dn}-2\Omega\right)^k\phi\right\rangle = -\left\langle \hat{n}2\Omega\delta(\bar{S}), \left(\frac{d}{dn}-2\Omega\right)^k\phi\right\rangle,$$

provides the desired result (3.21). For the particular case $k=0$, an alternative proof for the surface gradient theorem

$$\left\langle \delta(\bar{S}), grad_S\phi\right\rangle = -\left\langle \hat{n}2\Omega\delta(\bar{S}), \phi\right\rangle$$

is available in ([2, Sec. 5.5, Theorem 2]). □

Theorem 3.3.

$$\frac{\partial}{\partial x_i}\delta^{(k)}(\bar{S}) = n_i\delta^{(k+1)}(\bar{S}), \forall k \tag{3.22}$$

Proof.

$$\begin{aligned} 0 &= \left\langle \tfrac{D}{Dx_i}\delta^{(k)}(\bar{S}), \phi\right\rangle = -\left\langle \delta^{(k)}(\bar{S}), \tfrac{D\phi}{Dx_i}\right\rangle = -\left\langle \delta^{(k)}(\bar{S}), \left(\tfrac{\partial}{\partial x_i} - n_i\tfrac{d}{dn}\right)\phi\right\rangle \\ &= \left\langle \left(\tfrac{\partial}{\partial x_i} - n_i\tfrac{d}{dn}\right)\delta^{(k)}(\bar{S}), \phi\right\rangle = \left\langle \tfrac{\partial}{\partial x_i}\delta^{(k)}(\bar{S}) - n_i\delta^{(k+1)}(\bar{S}), \phi\right\rangle \end{aligned}$$

□

Corollary 3.1.

$$grad\delta^{(k)}(\bar{S}) = \hat{n}\delta^{(k+1)}(\bar{S}), \forall k. \tag{3.23}$$

Lemma 3.3. *On an arbitrary regular surface with boundary $\bar{C} = \partial S$ one has the property*

$$\left(\overline{grad}_S H[S]\right)\delta^{(k)}(\bar{S}) = -\hat{v}\delta_n^{(k)}(\bar{C}), \forall k \tag{3.24}$$

with boundary distributions of arbitrary order $\delta_n^{(k)}(\bar{C}) = \frac{d^k}{dn^k}\delta(\bar{C})$.

Corollary 3.2.

$$\begin{aligned} grad(V_S(\vec{r}_S;t)H[S]\delta(\bar{S})) &= (grad_S V_S)\,H[S]\delta(\bar{S}) + \hat{n}V_S H[S]\delta^{(1)}(\bar{S}) - \hat{v}V_S\delta(\bar{C}) \\ &= (grad_S V_S + 2\Omega\hat{n}V_S)\,H[S]\delta(\bar{S}) + \hat{n}V_S H[S]d_n\delta(\bar{S}) - \hat{v}V_S\delta(\bar{C}) \end{aligned} \tag{3.25}$$

$$\begin{aligned} div(\vec{A}_S(\vec{r}_S;t)H[S]\delta(\bar{S})) &= \left(div_S\vec{A}_S\right)H[S]\delta(\bar{S}) + \hat{n}\cdot\vec{A}_S H[S]\delta^{(1)}(\bar{S}) - \hat{v}\cdot\vec{A}_S\delta(\bar{C}) \\ &= \left(div_S\vec{A}_S + 2\Omega\hat{n}\cdot\vec{A}_S\right)H[S]\delta(\bar{S}) + \hat{n}\cdot\vec{A}_S H[S]d_n\delta(\bar{S}) - \hat{v}\cdot\vec{A}_S\delta(\bar{C}) \\ &= \hat{n}\cdot curl_S(\hat{n}\times\vec{A}_S)H[S]\delta(\bar{S}) + \hat{n}\cdot\vec{A}_S H[S]d_n\delta(\bar{S}) + (\vec{A}_S\times\hat{n})\cdot\hat{\lambda}\delta(\bar{C}) \end{aligned} \tag{3.26}$$

$$\begin{aligned} &curl(\vec{A}_S(\vec{r}_S;t)H[S]\delta(\bar{S})) \\ &= \left(curl_S\vec{A}_S\right)H[S]\delta(\bar{S}) + \hat{n}\times\vec{A}_S H[S]\delta^{(1)}(\bar{S}) - \hat{v}\times\vec{A}_S\delta(\bar{C}) \\ &= \left(curl_S\vec{A}_S + 2\Omega\hat{n}\times\vec{A}_S\right)H[S]\delta(\bar{S}) + \hat{n}\times\vec{A}_S H[S]d_n\delta(\bar{S}) - \hat{v}\times\vec{A}_S\delta(\bar{C} \\ &= \left(grad_S(\vec{A}_S\cdot\hat{n})\times\hat{n} + \hat{n}(\hat{n}\cdot curl_S\vec{A}_S) + grad_S(\hat{n})\cdot(\vec{A}_S\times\hat{n})\right)H[S]\delta(\bar{S}) \\ &\quad + \hat{n}\times\vec{A}_S H[S]d_n\delta(\bar{S}) + \left((\hat{n}\cdot\vec{A}_S)\hat{\lambda} - (\hat{\lambda}\cdot\vec{A}_S)\hat{n}\right)\delta(\bar{C}) \end{aligned} \tag{3.27}$$

where

$$grad_S(\hat{n}) = \frac{\hat{u}_1}{h_1}\frac{\partial\hat{n}}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial\hat{n}}{\partial u_2} = -\frac{1}{\alpha_1}\hat{u}_1\hat{u}_1 - \frac{1}{\alpha_2}\hat{u}_2\hat{u}_2.$$

The validity of the results (3.25)-(3.27) can also be checked from the end results in [10, eq.(8-10)] derived based on differential geometrical methods. The notational correspondence between the two papers is as $\delta(\bar{S}) = \delta_{\bar{S}}$, $\delta(\bar{C}) = \delta_{\partial\bar{S}}$, $\hat{\lambda} = \hat{l}$; and since a vector test function $\vec{w}$ is employed in [10] as opposed to our scalar ϕ, we also have the equivalence $\nabla_{\hat{n}}(\cdot)\delta_{\bar{S}} = -d_n\delta(\bar{S})$ and $\nabla_{\hat{n}}(\hat{n}\cdot)\delta_{\bar{S}} = -d_n\delta(\bar{S})$.

Corollary 3.3. *A combination of* (2.8) *and* (3.22) *reads*

$$\frac{\partial}{\partial x_i} d_n\delta(\bar{S}) = n_i 2\Omega\delta(\bar{S}) + n_i d_n\delta(\bar{S}). \tag{3.28}$$

The spatial derivatives of $d_n^k\delta(\bar{S})$ were introduced for the first time in [3] in a closed form utilizing mathematical induction method and "the fundamental magnitudes of arbitrary order" of a surface, a concept which was introduced in the same paper. For an explicit representation of these derivatives we provide the following theorem.

Theorem 3.4.

$$\frac{\partial}{\partial x_i} d_n^k\delta(\bar{S}) = k\Lambda_i d_n^{k-1}\delta(\bar{S}) + 2\Omega n_i d_n^k\delta(\bar{S}) + n_i d_n^{k+1}\delta(\bar{S}), k \geq 1 \tag{3.29}$$

where we define

$$\Lambda_i = \frac{D}{Dx_i}\left(-2\Omega\right). \tag{3.30}$$

Proof. We insert (3.22) in (2.9) as

$$\begin{aligned}
\frac{\partial}{\partial x_i} d_n^k\delta(\bar{S}) &= \sum_{m=0}^{k} \binom{k}{m} \frac{\partial}{\partial x_i}\left[\left(-2\Omega\right)^{k-m}\delta^{(m)}(\bar{S})\right] \\
&= \sum_{m=0}^{k} \binom{k}{m} \left[\left(\frac{D}{Dx_i}\left(-2\Omega\right)^{k-m}\right)\delta^{(m)}(\bar{S}) + \left(-2\Omega\right)^{k-m} n_i\delta^{(m+1)}(\bar{S})\right] \\
&= \sum_{m=0}^{k} \binom{k}{m} \left[\left(\frac{D}{Dx_i}\left(-2\Omega\right)^{k-m}\right)\sum_{r=0}^{m}\binom{m}{r}\left(2\Omega\right)^{m-r} d_n^r\delta(\bar{S})\right. \\
&\quad \left. + \left(-2\Omega\right)^{k-m} n_i \sum_{r=0}^{m+1}\binom{m+1}{r}\left(2\Omega\right)^{m-r+1} d_n^r\delta(\bar{S})\right] \\
&= \Lambda \sum_{m=0}^{k-1}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\left(k-m\right)\left(-2\Omega\right)^{k-m-1}\left(2\Omega\right)^{m-r} d_n^r\delta(\bar{S}) \\
&\quad + n_i \sum_{m=0}^{k}\sum_{r=0}^{m+1}\binom{k}{m}\binom{m+1}{r}\left(-2\Omega\right)^{k-m}\left(2\Omega\right)^{m-r+1} d_n^r\delta(\bar{S})
\end{aligned}$$

and change the order of summation to get

$$\frac{\partial}{\partial x_i} d_n^k \delta(\bar{S}) = \Lambda_i \left(-2\Omega\right)^{k-1} \sum_{r=0}^{k-1} d_n^r \delta(\bar{S}) \left(2\Omega\right)^{-r} \sum_{m=r}^{k-1} (-1)^m \binom{k}{m} \binom{m}{r} (k-m)$$
$$- n_i \left(-2\Omega\right)^{k+1} \delta(\bar{S}) \sum_{m=0}^{k} (-1)^m \binom{k}{m}$$
$$- n_i \left(-2\Omega\right)^{k+1} \sum_{r=1}^{k+1} d_n^r \delta(\bar{S}) \left(2\Omega\right)^{-r} \sum_{m=r-1}^{k} (-1)^m \binom{k}{m} \binom{m+1}{r}$$

Incorporating the property $\sum\limits_{m=0}^{k} (-1)^m \binom{k}{m} = 0$ trivializes the second term at r.h.s. and one gets

$$\frac{\partial}{\partial x_i} d_n^k \delta(\bar{S}) = n_i d_n^{k+1} \delta(\bar{S})$$
$$+ \Lambda_i \left(-2\Omega\right)^{k-1} \sum_{r=0}^{k-1} d_n^r \delta(\bar{S}) \left(2\Omega\right)^{-r} \sum_{m=r}^{k-1} (-1)^m \binom{k}{m} \binom{m}{r} (k-m) \tag{3.31}$$
$$- n_i \left(-2\Omega\right)^{k+1} \sum_{r=1}^{k} d_n^r \delta(\bar{S}) \left(2\Omega\right)^{-r} \sum_{m=r-1}^{k} (-1)^m \binom{k}{m} \binom{m+1}{r}$$

On the other hand we have the easily verifiable binomial relations

$$\sum_{m=r}^{k-1} (-1)^m \binom{k}{m} \binom{m}{r} (k-m) = \begin{cases} 0, r \le k-2 \\ (-1)^{k-1} k, r = k-1 \end{cases},$$
$$\sum_{m=r-1}^{k} (-1)^m \binom{k}{m} \binom{m+1}{r} = \begin{cases} 0, r \le k-1 \\ (-1)^k, r = k \end{cases}$$

which, upon substitution into (3.31), yield the end result (3.29). □

Corollary 3.4.

$$grad\left(d_n^k \delta(\bar{S})\right) = k\vec{\Lambda} d_n^{k-1} \delta(\bar{S}) + 2\Omega \hat{n} d_n^k \delta(\bar{S}) + \hat{n} d_n^{k+1} \delta(\bar{S}), k \ge 1 \tag{3.32}$$

where we define

$$\vec{\Lambda} = grad_S\left(-2\Omega\right). \tag{3.33}$$

In Table 1 we present the mentioned fundamental magnitudes of certain canonical surfaces.

Table 1. An illustration of the fundamental magnitudes of certain canonical surfaces.

(u_1, u_2, n)	Geometry	$\delta(\bar{S})$	$\hat{n}$	2Ω	$\vec{\Lambda}$
(φ, z, ρ)	cylinder $\rho = \rho_0$	$\delta\left(\rho - \rho_0\right)$	$\hat{\rho}$	$-\frac{1}{\rho_0}$	$\vec{0}$
(θ, φ, r)	sphere $r = r_0$	$\delta\left(r - r_0\right)$	$\hat{r}$	$-\frac{2}{r_0}$	$\vec{0}$
(φ, r, θ)	cone $\theta = \theta_0$	$\frac{1}{r}\delta\left(\theta - \theta_0\right)$	$\hat{\theta}$	$-\frac{\cos\theta_0}{r\sin\theta_0}$	$-\frac{\cos\theta_0}{r^2\sin\theta_0}\hat{r}$

Lemma 3.4. *On an arbitrary regular surface with boundary $\bar{C} = \partial S$ one has the property*

$$\left(\overline{grad}_S H[S]\right) d_n^k \delta(\bar{S}) = -\hat{v} d_n^k \delta(\bar{C}), \forall k \tag{3.34}$$

with boundary distributions of arbitrary order defined by

$$\left\langle d_n^k \delta(\bar{C}), \phi \right\rangle = (-1)^k \left\langle \delta(\bar{C}), \frac{d^k \phi}{dn^k} \right\rangle = (-1)^k \int_{-\infty}^{\infty} \int_{\bar{C}} \frac{d^k}{dn^k} \phi(\vec{r}; t) d\bar{C}\, dt, \forall k \tag{3.35}$$

Corollary 3.5.

$$\begin{aligned} grad\left(H[S] d_n^k \delta(\bar{S})\right) = &- \hat{v} d_n^k \delta(\bar{C}) \\ &+ H[S]\left(k\vec{\Lambda} d_n^{k-1}\delta(\bar{S}) + 2\Omega \hat{n} d_n^k \delta(\bar{S}) + \hat{n} d_n^{k+1}\delta(\bar{S})\right), k \geq 1. \end{aligned} \tag{3.36}$$

The generalized derivatives of multilayers of arbitrary order were obtained in closed form in [11]. In what follows we provide an explicit expression of the second order partial derivatives of $d_n^k \delta(\bar{S})$. Higher order partial derivatives can also be obtained in a straightforward manner by successive applications of the recursive relation (3.29).

Theorem 3.5.

$$\begin{aligned} \frac{\partial^2}{\partial x_j \partial x_i} d_n \delta(\bar{S}) = &\left[\Lambda_{ij} + 2\Omega\left(\Lambda_i n_j + \Lambda_j n_i\right)\right] \delta(\bar{S}) \\ &+ \left[\left(\Lambda_i n_j + \Lambda_j n_i\right) + 2\Omega\left(\mu_{ij} + 2\Omega n_i n_j\right)\right] d_n \delta(\bar{S}) \\ &+ \left(\mu_{ij} + 4\Omega n_i n_j\right) d_n^2 \delta(\bar{S}) + n_i n_j d_n^3 \delta(\bar{S}) \end{aligned} \tag{3.37}$$

$$\begin{aligned} \tfrac{\partial^2}{\partial x_j \partial x_i} d_n^k \delta(\bar{S}) = &\, k(k-1)\Lambda_i \Lambda_j d_n^{k-2}\delta(\bar{S}) + k\left[\Lambda_{ij} + 2\Omega\left(\Lambda_i n_j + \Lambda_j n_i\right)\right] d_n^{k-1}\delta(\bar{S}) \\ &+ \left[k\left(\Lambda_i n_j + \Lambda_j n_i\right) + 2\Omega\left(\mu_{ij} + 2\Omega n_i n_j\right)\right] d_n^k \delta(\bar{S}) \qquad , k \geq 2 \\ &+ \left(\mu_{ij} + 4\Omega n_i n_j\right) d_n^{k+1}\delta(\bar{S}) + n_i n_j d_n^{k+2}\delta(\bar{S}) \end{aligned} \tag{3.38}$$

where

$$\Lambda_{ij} = \frac{D\Lambda_i}{Dx_j} = \frac{D^2}{Dx_j Dx_i}\left(-2\Omega\right), \mu_{ij} = \frac{Dn_i}{Dx_j} \tag{3.39}$$

The surface quantities μ_{ij}, called the second fundamental forms of a surface in Cartesian notation, were first introduced in [3].

Corollary 3.6. *For $i = j$ one has*

$$\begin{aligned} \tfrac{\partial^2}{\partial x_i^2} d_n \delta(\bar{S}) = &\left(\Lambda_{ii} + 4\Omega\Lambda_i n_i\right)\delta(\bar{S}) + \left[2\Lambda_i n_i + 2\Omega\left(\mu_{ii} + 2\Omega n_i^2\right)\right] d_n \delta(\bar{S}) \\ &+ \left(\mu_{ii} + 4\Omega n_i^2\right) d_n^2 \delta(\bar{S}) + n_i^2 d_n^3 \delta(\bar{S}) \end{aligned} \tag{3.40}$$

$$lap\left(d_n \delta(\bar{S})\right) = \left(div_S \vec{\Lambda}\right)\, \delta(\bar{S}) + 2\Omega d_n^2 \delta(\bar{S}) + d_n^3 \delta(\bar{S}) \tag{3.41}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} d_n^k \delta(\bar{S}) = &\, k(k-1)\Lambda_i^2 d_n^{k-2}\delta(\bar{S}) + k\left(\Lambda_{ii} + 4\Omega\Lambda_i n_i\right) d_n^{k-1}\delta(\bar{S}) \\ &+ \left[2k\Lambda_i n_i + 2\Omega\left(\mu_{ii} + 2\Omega n_i^2\right)\right] d_n^k \delta(\bar{S}) + \left(\mu_{ii} + 4\Omega n_i^2\right) d_n^{k+1}\delta(\bar{S}) \\ &+ n_i^2 d_n^{k+2}\delta(\bar{S}), k \geq 2 \end{aligned} \tag{3.42}$$

$$\begin{aligned} lap\left(d_n^k \delta(\bar{S})\right) = &\, k(k-1)\left|\vec{\Lambda}\right|^2 d_n^{k-2}\delta(\bar{S}) + k\,\left(div_S \vec{\Lambda}\right)\, d_n^{k-1}\delta(\bar{S}) \\ &+ 2\Omega d_n^{k+1}\delta(\bar{S}) + d_n^{k+2}\delta(\bar{S}), k \geq 2 \end{aligned} \tag{3.43}$$

where we have incorporated $\vec{\Lambda} \cdot \hat{n} = 0$ and $lap_S(-2\Omega) = div_S grad_S(-2\Omega) = div_S \vec{\Lambda}$.

4. Distributional Derivatives of the Regular Components

Theorem 4.1. *The spatial derivatives of the regular components of scalar/vector distributions* $\{V(\vec{r};t)\},\left\{\vec{A}(\vec{r};t)\right\}\in E$ *have the general form*

$$\begin{aligned}\frac{\partial}{\partial x_i}\{V(\vec{r};t)\}&=\left\{\frac{\partial}{\partial x_i}V(\vec{r};t)\right\}+\hat{x}_i\cdot\hat{n}\;\Delta[V]\delta(S),\\ \frac{\partial}{\partial x_i}\left\{\vec{A}(\vec{r};t)\right\}&=\left\{\frac{\partial}{\partial x_i}\vec{A}(\vec{r};t)\right\}+\hat{x}_i\cdot\hat{n}\;\Delta[\vec{A}]\delta(S)\end{aligned}\tag{4.1}$$

where

$$\Delta[V]\triangleq V(\vec{r}_S^{+};t)-V(\vec{r}_S^{-};t),\Delta[\vec{A}]\triangleq\vec{A}(\vec{r}_S^{+};t)-\vec{A}(\vec{r}_S^{-};t).\tag{4.2}$$

A proof is available in [2, Sec.5.5]. The smooth behaviors of $\Delta[V]$ and $\Delta[\vec{A}]$ are guaranteed in E.

Corollary 4.1. *If* $\{V(\vec{r};t)\},\left\{\vec{A}(\vec{r};t)\right\}\in E$,

$$grad\,\{V(\vec{r};t)\}=\{gradV(\vec{r};t)\}+\hat{n}\;\Delta[V]H[S]\delta(\bar{S})\tag{4.3}$$

$$div\left\{\vec{A}(\vec{r};t)\right\}=\left\{div\vec{A}(\vec{r};t)\right\}+\hat{n}\cdot\Delta[\vec{A}]H[S]\delta(\bar{S})\tag{4.4}$$

$$curl\left\{\vec{A}(\vec{r};t)\right\}=\left\{curl\vec{A}(\vec{r};t)\right\}+\hat{n}\times\Delta[\vec{A}]H[S]\delta(\bar{S})\tag{4.5}$$

The results obtained so far can serve to calculate the scalar and vector Laplacian operators in E as follows.

Theorem 4.2. *If* $\{V(\vec{r};t)\},\left\{\vec{A}(\vec{r};t)\right\}\in E$,

$$lap\,\{V(\vec{r};t)\}=\{lapV(\vec{r};t)\}+\left(\Delta\left[\frac{dV}{dn}\right]-2\Omega\Delta[V]\right)H[S]\delta(\bar{S})+\Delta[V]H[S]\delta^{(1)}(\bar{S})\tag{4.6}$$

$$\begin{aligned}&lap\left\{\vec{A}(\vec{r};t)\right\}=\left\{lap\vec{A}(\vec{r};t)\right\}\\&+\left[\hat{n}\Delta[div\vec{A}]-\hat{n}\times\Delta[curl\vec{A}]+grad_S\left(\hat{n}\cdot\Delta[\vec{A}]\right)-curl_S\left(\hat{n}\times\Delta[\vec{A}]\right)\right]H[S]\delta(\bar{S})\\&+\Delta[\vec{A}]H[S]\delta^{(1)}(\bar{S})-\left[\hat{v}\left(\hat{n}\cdot\Delta[\vec{A}]\right)+\hat{n}\left(\hat{v}\cdot\Delta[\vec{A}]\right)\right]\delta(\bar{C})\end{aligned}\tag{4.7}$$

(4.6) also holds with the end result [2, Sec.5.6, eq.(8)]. The action of vector operators on arbitrary scalar/vector field quantities expressed by (2.5) is obtained as follows:

Corollary 4.2.

$$\begin{aligned}&grad\left(\{V(\vec{r};t)\}+H[S]\sum_{k=0}^{N}V_k(\vec{r}_S;t)\delta^{(k)}(\bar{S})\right)=\\&\{gradV(\vec{r};t)\}+\left(\hat{n}\Delta[V]+grad_SV_0\right)H[S]\delta(\bar{S})\\&+H[S]\sum_{k=1}^{N}\left(grad_SV_k+\hat{n}V_{k-1}\right)\delta^{(k)}(\bar{S})+\hat{n}V_NH[S]\delta^{(N+1)}(\bar{S})-\hat{v}\sum_{k=0}^{N}V_k\delta_n^{(k)}(\bar{C})\end{aligned}\tag{4.8}$$

$$div\left(\left\{\vec{A}(\vec{r};t)\right\}+H[S]\sum_{k=0}^{N}\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(\bar{S})\right)=\left\{div\vec{A}(\vec{r};t)\right\}$$
$$+\left(\hat{n}\cdot\Delta[\vec{A}]+div_S\vec{A}_0\right)H[S]\delta(\bar{S})+H[S]\sum_{k=1}^{N}\left(div_S\vec{A}_k+\hat{n}\cdot\vec{A}_{k-1}\right)\delta^{(k)}(\bar{S}) \tag{4.9}$$
$$+\hat{n}\cdot\vec{A}_N H[S]\delta^{(N+1)}(\bar{S})-\hat{v}\cdot\sum_{k=0}^{N}\vec{A}_k\delta_n^{(k)}(\bar{C})$$

$$curl\left(\left\{\vec{A}(\vec{r};t)\right\}+H[S]\sum_{k=0}^{N}\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(\bar{S})\right)=\left\{curl\vec{A}(\vec{r};t)\right\}$$
$$+\left(\hat{n}\times\Delta[\vec{A}]+curl_S\vec{A}_0\right)H[S]\delta(\bar{S})+H[S]\sum_{k=1}^{N}\left(curl_S\vec{A}_k+\hat{n}\times\vec{A}_{k-1}\right)\delta^{(k)}(\bar{S}) \tag{4.10}$$
$$+\hat{n}\times\vec{A}_N H[S]\delta^{(N+1)}(\bar{S})-\hat{v}\times\sum_{k=0}^{N}\vec{A}_k\delta_n^{(k)}(\bar{C})$$

In virtue of (3.36), similar results for the field quantities expressed in terms of the other set of distributions $d_n^k\delta(\bar{S})$ can also be derived.

5. Boundary Relations in Presence of Second Order Singularities

For certain practical results to be employed in the distributional investigation of the equations of mathematical physics including the Maxwell equations in Sec.6, we outline the jump and compatibility relations (based on Theorem 2.4 and Corollary 2.1) for the five basic field equations in Table 2 for two alternative sets of sources given by

$$\begin{aligned}Set1:\vec{f}&=\left\{\vec{f}\right\}+\vec{f}_0H[S]\delta(\bar{S})+\vec{f}_1H[S]\delta^{(1)}(\bar{S})+\vec{\alpha}_0\delta(\bar{C}),\\ g&=\{g\}+g_0H[S]\delta(\bar{S})+g_1H[S]\delta^{(1)}(\bar{S})+\beta_0\delta(\bar{C})\end{aligned} \tag{5.1}$$

$$\begin{aligned}Set2:\vec{f}&=\left\{\vec{f}\right\}+\vec{f}_0H[S]\delta(\bar{S})+\vec{f}_1H[S]d_n\delta(\bar{S})+\vec{\alpha}_0\delta(\bar{C}),\\ g&=\{g\}+g_0H[S]\delta(\bar{S})+g_1H[S]d_n\delta(\bar{S})+\beta_0\delta(\bar{C})\end{aligned} \tag{5.2}$$

In virtue of Theorem 2.1, the corresponding field quantities for both sets of sources have the same general form

$$\begin{aligned}&F=\{F\}+F_0H[S]\delta(\bar{S}),\vec{A}=\left\{\vec{A}\right\}+\vec{A}_0H[S]\delta(\bar{S}),\\ &\vec{B}=\left\{\vec{B}\right\}+\vec{B}_0H[S]\delta(\bar{S}),G=\{G\},\vec{C}=\left\{\vec{C}\right\}.\end{aligned} \tag{5.3}$$

6. Some Applications in Electromagnetic Theory

The macroscopic electromagnetic phenomena of stationary media are governed by the Maxwell equations

$$curl\vec{E}(\vec{r};t)+\frac{\partial}{\partial t}\vec{B}(\vec{r};t)=\vec{0},curl\vec{H}(\vec{r};t)-\frac{\partial}{\partial t}\vec{D}(\vec{r};t)=\vec{J}_C(\vec{r};t) \tag{6.1}$$

$$div\vec{D}(\vec{r};t)=\rho_f(\vec{r};t),div\vec{B}(\vec{r};t)=0 \tag{6.2}$$

TABLE 2. Boundary relations for fundamental field equations in presence of second order singularities.

Field Equation	Boundary Relations for 1st Set of Sources	Boundary Relations for 2nd Set of Sources
$gradF = \vec{f}$	$\hat{n}\Delta[F] + grad_S F_0 = \vec{f}_0$ $\hat{n}F_0 = \vec{f}_1,\ \hat{\nu}F_0 + \vec{\alpha}_0\vert_{\bar{C}} = 0$	$\hat{n}\Delta[F] + grad_S F_0 + 2\Omega\hat{n}F_0 = \vec{f}_0$ $\hat{n}F_0 = \vec{f}_1,\ \hat{\nu}F_0 + \vec{\alpha}_0\vert_{\bar{C}} = 0$
$div\vec{A} = g$	$\hat{n}\cdot\Delta[\vec{A}] + div_S\vec{A}_0 = g_0$ $\hat{n}\cdot\vec{A}_0 = g_1,\ \hat{\upsilon}\cdot\vec{A}_0 + \beta_0\Big\vert_{\bar{C}} = 0$	$\hat{n}\cdot\Delta[\vec{A}] + div_S\vec{A}_0 + 2\Omega\hat{n}\cdot\vec{A}_0 = g_0$ $\hat{n}\cdot\vec{A}_0 = g_1,\ \hat{\upsilon}\cdot\vec{A}_0 + \beta_0\Big\vert_{\bar{C}} = 0$
$curl\vec{B} = \vec{f}$	$\hat{n}\times\Delta[\vec{B}] + curl_S\vec{B}_0 = \vec{f}_0$ $\hat{n}\times\vec{B}_0 = \vec{f}_1,\ \hat{\upsilon}\times\vec{B}_0 + \vec{\alpha}_0\Big\vert_{\bar{C}} = \vec{0}$	$\hat{n}\times\Delta[\vec{B}] + curl_S\vec{B}_0 + 2\Omega\hat{n}\times\vec{B}_0 = \vec{f}_0$ $\hat{n}\times\vec{B}_0 = \vec{f}_1,\ \hat{\upsilon}\times\vec{B}_0 + \vec{\alpha}_0\Big\vert_{\bar{C}} = \vec{0}$
$lapG = g$	$\Delta\left[\frac{dG}{dn}\right] - 2\Omega\Delta[G] = g_0,$ $\Delta[G] = g_1, \beta_0 = 0$	$\Delta\left[\frac{dG}{dn}\right] = g_0, \Delta[G] = g_1, \beta_0 = 0$
$lap\vec{C} = \vec{f}$	$\hat{n}\Delta[div\vec{C}] - \hat{n}\times\Delta[curl\vec{C}]$ $+grad_S\left(\hat{n}\cdot\Delta[\vec{C}]\right)$ $-curl_S\left(\hat{n}\times\Delta[\vec{C}]\right) = \vec{f}_0$	$\hat{n}\Delta[div\vec{C}] - \hat{n}\times\Delta[curl\vec{C}]$ $+grad_S\left(\hat{n}\cdot\Delta[\vec{C}]\right)$ $-curl_S\left(\hat{n}\times\Delta[\vec{C}]\right)$ $+2\Omega\Delta[\vec{C}] = \vec{f}_0, \Delta[\vec{C}] = \vec{f}_1,$ $\hat{\upsilon}\left(\hat{n}\cdot\Delta[\vec{A}]\right) + \hat{n}\left(\hat{\upsilon}\cdot\Delta[\vec{A}]\right) + \vec{\alpha}_0\Big\vert_{\bar{C}} = \vec{0}$

as well as the continuity equation

$$div\vec{J}_C(\vec{r};t) + \frac{\partial}{\partial t}\rho_f(\vec{r};t) = 0 \tag{6.3}$$

and the Lorentz potentials described by

$$\vec{B}(\vec{r};t) = curl\vec{A}(\vec{r};t), \vec{E}(\vec{r};t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r};t) - gradV(\vec{r};t) \tag{6.4}$$

Without losing generality, let us assume the ambient medium as free space where the field equations are accompanied by the constitutive relations

$$\vec{D}(\vec{r};t) = \varepsilon_0\vec{E}(\vec{r};t), \vec{B}(\vec{r};t) = \mu_0\vec{H}(\vec{r};t) \tag{6.5}$$

as well as the Lorentz gauge and the wave equations

$$div\vec{A}(\vec{r};t) + \varepsilon_0\mu_0\frac{\partial}{\partial t}V(\vec{r};t) = 0 \tag{6.6}$$

$$\left(lap - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)V(\vec{r};t) = -\left(1/\varepsilon_0\right)\rho_f(\vec{r};t), \left(lap - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r};t) = -\mu_0\vec{J}_C(\vec{r};t). \tag{6.7}$$

The Poynting theorem in point form is given by

$$div\vec{P} + \vec{E}\cdot\vec{J}_d^e + \vec{H}\cdot\vec{J}_d^m + \vec{E}\cdot\vec{J}_C = 0 \tag{6.8}$$

where

$$\vec{P}(\vec{r};t) = \vec{E}(\vec{r};t)\times\vec{H}(\vec{r};t) \tag{6.9}$$

is the usual Poynting vector and

$$\vec{J}_d^e(\vec{r};t) = \frac{\partial}{\partial t}\vec{D}(\vec{r};t), \vec{J}_d^m(\vec{r};t) = \frac{\partial}{\partial t}\vec{B}(\vec{r};t) \tag{6.10}$$

stand for the electric and magnetic displacement current densities.

6.1. **A Single Layer Supporting Dynamic Current.** Let us assume the surface S with enclosure $\bar{C}$ supports first order sources with free charge and conduction current density functions represented by

$$\rho_f(\vec{r};t) = \rho_S(\vec{r}_S;t)H[S]\delta(\bar{S}), \vec{J}_C(\vec{r};t) = \vec{J}_S(\vec{r}_S;t)H[S]\delta(\bar{S}). \tag{6.11}$$

Then the distributional form of the continuity equation (6.3) reads

$$div_S\vec{J}_S + \frac{\partial \rho_S}{\partial t} = 0 \tag{6.12}$$

$$\hat{n}\cdot\vec{J}_S = 0, \, \hat{v}\cdot\vec{J}_S\Big|_{\bar{C}} = 0. \tag{6.13}$$

(6.12) describes the conservation of current flowing on the surface, while (6.13) signify that the current cannot support a component flowing normal to the surface as well as its $\hat{v}$-component directed outward from the surface is zero on the boundary points.

The distributional form of the rest of the field equations yield the well known jump conditions

$$\hat{n}\times\Delta[\vec{E}] = \vec{0}, \hat{n}\cdot\Delta[\vec{D}] = \rho_S, \hat{n}\times\Delta[\vec{H}] = \vec{J}_S, \hat{n}\cdot\Delta[\vec{B}] = 0 \tag{6.14}$$

$$\Delta[V] = 0, \Delta[\vec{A}] = \vec{0} \tag{6.15}$$

The distributional form of Poynting's theorem reads

$$-\hat{n}\cdot\Delta[\vec{P}] = \vec{J}_S\cdot\vec{E} = \vec{J}_S\cdot\left(-\frac{\partial\vec{A}}{\partial t} - gradV\right) \tag{6.16}$$

which can be interpreted as "*the total instantaneous electromagnetic power density (generated by external sources) entering into any point on the surface is converted to heat at that point*".

Since the relation (6.8) does not involve a point type singularity, on an arbitrary point on the surface one also has

$$\lim_{\Delta\Sigma\to 0}\oint_{\Delta\Sigma}\left\{\vec{P}(\vec{r};t)\right\}\cdot d\vec{\Sigma} = 0 \tag{6.17}$$

where $\Delta\Sigma$ signifies the enclosure of an infinitely small region $\Delta\vartheta$ that shrinks onto an arbitrary point on S. When the point is taken on $\bar{C}$, the condition (6.17) is known as the "edge condition" in electromagnetic theory, which addresses the asymptotic field behavior at edge points of a single layer based on the principle of power conservation. The results (6.16) and (6.17) can be visualized in Fig.2.

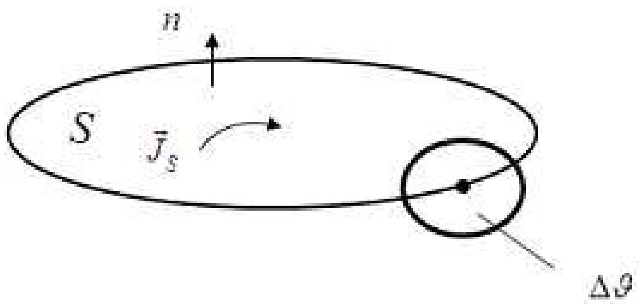

Figure 2. An illustration of the distributional interpretation of the Poynting theorem on a single layer

6.2. **A Double Layer Supporting Dynamic Currents.** A dynamic double layer current source is as an isolated system of two identical surfaces S, S_h separated by a fixed infinitesimal normal distance h in free space supporting nonuniform and opposite directed surface current and charge densities $-\vec{J}_S$,$+\vec{J}_{S_h}$ and $-\rho_S$,$+\rho_{S_h}$, respectively (see Fig.3).

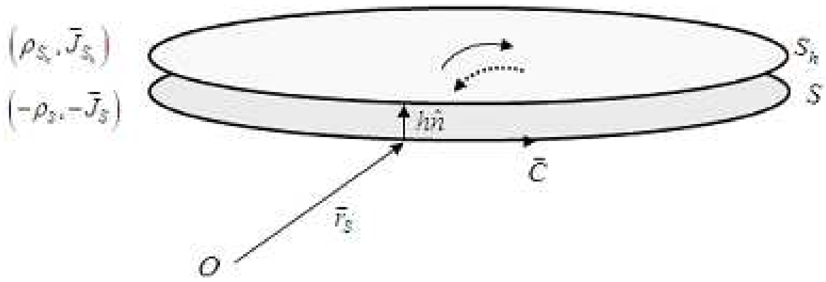

Figure 3. A Double Layer Current Configuration

While the volume charge and conduction current densities of the actual physical system are written as

$$\begin{aligned}\rho_f(\vec{r}) &= -\rho_S H[S]\delta\left(\vec{r}-\vec{r}_S\right)+\rho_{S_h}H[S_h]\delta\left(\vec{r}-(\vec{r}_S+h\hat{n})\right), \vec{J}_C(\vec{r}) \\ &= -\vec{J}_S\delta\left(\vec{r}-\vec{r}_S\right)+\vec{J}_{S_h}\delta\left(\vec{r}-(\vec{r}_S+h\hat{n})\right),\end{aligned} \tag{6.18}$$

the hypothetical double layer source is described by multiplying (6.18) by h/h and taking the limit as $\rho_S, \rho_{S_h} \to \infty$,$J_S, J_{S_h} \to \infty$ and $h \to 0$ to get

$$\rho_f(\vec{r};t) = -\vec{P}_S(\vec{r}_S;t)H[S]\,\delta^{(1)}(\bar{S}), \vec{J}_C(\vec{r};t) = -\vec{C}_S(\vec{r}_S;t)H[S]\,\delta^{(1)}(\bar{S}) \tag{6.19}$$

with

$$\vec{P}_S(\vec{r}_S;t) = P_S(\vec{r}_S;t)\,\hat{n} \tag{6.20}$$

and the limits

$$\vec{C}_S(\vec{r}_S;t) \;=\; \lim_{\substack{J_s\to\infty \\ h\to 0}} \vec{J}_S\,(\vec{r}_S;t)h, P_S(\vec{r}_S;t) \;=\; \lim_{\substack{\rho_s\to\infty \\ h\to 0}} \rho_S(\vec{r}_S;t)\,h \tag{6.21}$$

describing the dipole moment densities are assumed to remain finite. Such a mathematical idealization is only suitable for an investigation of far field patterns and bound to corrupt when an observer approaches close enough to the system. Therefore the distributional investigation of the field equations should not be expected to demonstrate the actual physical picture *on* such systems in general.

Based on Theorem 2.1 the electromagnetic fields on the source may possess first order singular components

$$\left[\vec{E}(\vec{r};t)\right]_S = \vec{E}_0(\vec{r}_S;t)\,H[S]\,\delta(\bar{S}), \left[\vec{D}(\vec{r};t)\right]_S = \vec{D}_0(\vec{r}_S;t)\,H[S]\,\delta(\bar{S}) \tag{6.22}$$

$$\left[\vec{H}(\vec{r};t)\right]_S = \vec{H}_0(\vec{r}_S;t)\,H[S]\,\delta(\bar{S}), \left[\vec{B}(\vec{r};t)\right]_S = \vec{B}_0(\vec{r}_S;t)\,H[S]\,\delta(\bar{S}) \tag{6.23}$$

with

$$\vec{D}_0 = \varepsilon_0\vec{E}_0, \vec{B}_0 = \mu_0\vec{H}_0 \tag{6.24}$$

while

$$\left[V(\vec{r};t)\right]_S = 0, \left[\vec{A}(\vec{r};t)\right]_S = \vec{0}. \tag{6.25}$$

The distributional investigation of the field equations yields the jump and compatibility relations

$$\hat{n}\times\Delta[\vec{E}] = -curl_S\vec{E}_0 - \frac{\partial\vec{B}_0}{\partial t}, \hat{n}\times\vec{E}_0 = \vec{0}, \hat{v}\times\vec{E}_0\Big|_{\bar{C}} = \vec{0} \tag{6.26}$$

$$\hat{n}\times\Delta[\vec{H}] = -curl_S\vec{H}_0 + \frac{\partial\vec{D}_0}{\partial t}, \hat{n}\times\vec{H}_0 = \vec{0},\, \hat{v}\times\vec{H}_0\Big|_{\bar{C}} = \vec{0} \tag{6.27}$$

$$\hat{n}\cdot\Delta[\vec{D}] = -div_S\vec{D}_0, \hat{n}\cdot\vec{D}_0 = -P_S\ ,\ \hat{v}\cdot\vec{D}_0\Big|_{\bar{C}} = 0 \tag{6.28}$$

$$\hat{n}\cdot\Delta[\vec{B}] = -div_S\vec{B}_0, \hat{n}\cdot\vec{B}_0 = 0,\, \hat{v}\cdot\vec{B}_0\Big|_{\bar{C}} = 0 \tag{6.29}$$

$$div_S\vec{C}_S + \frac{\partial P_S}{\partial t} = 0, \hat{n}\cdot\vec{C}_S = 0,\, \hat{v}\cdot\vec{C}_S\Big|_{\bar{C}} = 0 \tag{6.30}$$

$$\Delta\left[\frac{dV}{dn}\right] - 2\Omega\Delta[V] = 0, \Delta[V] = P_S/\varepsilon_0 \tag{6.31}$$

$$\hat{n}\times\Delta[\vec{A}] = \vec{B}_0, \hat{n}\cdot\Delta[\vec{A}] = 0 \tag{6.32}$$

$$\Delta[\vec{A}] = \mu_0\vec{C}_S,\, \hat{n}\cdot\Delta[\vec{A}]\Big|_{\bar{C}} = 0,\, \hat{v}\cdot\Delta[\vec{A}]\Big|_{\bar{C}} = 0 \tag{6.33}$$

from which the density (aka contact) fields are obtained as

$$\vec{E}_0 = -\vec{P}_S\Big/\varepsilon_0, \vec{D}_0 = -\vec{P}_S, \vec{H}_0 = \hat{n}\times\vec{C}_S, \vec{B}_0 = \mu_0\hat{n}\times\vec{C}_S \tag{6.34}$$

while they vanish on the boundary:

$$\vec{E}_0\Big|_{\bar{C}} = \vec{0},\, \vec{D}_0\Big|_{\bar{C}} = \vec{0},\, \vec{H}_0\Big|_{\bar{C}} = \vec{0},\, \vec{B}_0\Big|_{\bar{C}} = \vec{0}. \tag{6.35}$$

Next we substitute the density fields (6.34) into the jump relations to reach to their resultant forms

$$\hat{n}\times\Delta[\vec{E}] = -\frac{1}{\varepsilon_0}curl_S\,\vec{P}_S - \mu_0\hat{n}\times\frac{\partial\vec{C}_S}{\partial t} = \frac{1}{\varepsilon_0}\left(grad_S P_S\right)\times\hat{n} - \mu_0\hat{n}\times\frac{\partial\vec{C}_S}{\partial t} \tag{6.36}$$

$$\hat{n}\cdot\Delta[\vec{D}] = -2\Omega P_S \tag{6.37}$$

$$\begin{aligned}\hat{n}\times\Delta[\vec{H}] &= curl_S\left(\vec{C}_S\times\hat{n}\right) - \frac{\partial\vec{P}_S}{\partial t} = \hat{n}\cdot grad_S\vec{C}_S + \vec{C}_S\left(div_S\hat{n}\right)\\ &- \vec{C}_S\cdot grad_S\hat{n} - \hat{n}\left(div_S\vec{C}_S\right) - \frac{\partial\vec{P}_S}{\partial t}\\ &= -2\Omega\vec{C}_S - \vec{C}_S\cdot grad_S\hat{n}\end{aligned} \tag{6.38}$$

$$\hat{n}\cdot\Delta[\vec{B}] = \mu_0\,div_S\left(\vec{C}_S\times\hat{n}\right) = \mu_0\,\hat{n}\cdot curl_S\vec{C}_S. \tag{6.39}$$

Applying partial time differentiation in (6.36) and incorporating (6.30), the end relation

$$\hat{n}\times\Delta\left[\frac{\partial\vec{E}}{\partial t}\right] = -\frac{1}{\varepsilon_0}\left(grad_S div_S\vec{C}_S\right)\times\hat{n} - \mu_0\hat{n}\times\frac{\partial^2\vec{C}_S}{\partial t^2}$$

can be verified with [12, (A.37)] for monochromatic case $\frac{\partial}{\partial t}\to j\omega$.

7. Conclusion

The present work is a tutorial review of the theory of generalized derivatives of surface distributions of arbitrary order in a Schwartz-Sobolev space setting based on the pioneering works of Estrada and Kanwal through 1980's on the propagation of wavefronts and multilayers. The essential contributions of the present investigation to literature can be considered as Theorem 3.4, Theorem 3.5, Corollary 3.6 and equations (4.7)-(4.10) by which we introduce an explicit representation of the first and second distributional derivatives of multilayers as well as an extension of the available results for vector operators from *closed* to *open* surfaces featuring boundary distributions of *arbitrary* order. Our results also conform with those obtained by Betounes [10] using distributional tensor analysis for the special case of first order boundary distributions including Corollary 3.2, Table 2 as well as the boundary relations on the enclosure of a double layer. While the applications to single and double layers in Sec.6 utilize only certain of the analytical tools devised in the manuscript, it is expected that these tools find important applications in many branches of mathematical physics.

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