Conference Proceeding Science and Technology, 3(1), 2020, 19–23

Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)

Linear Codes over the Ring $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

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Abstract: In this paper, we introduce the ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$ where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ over which the linear codes are studied. it's shown that the ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$ is a commutative, characteristic 8 ring with $u^2 = u$, $v^2 = v$, $uv = vu = 0$. Also, the ideals of $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$ are found. Moreover, we define the Lee distance and the Lee weight of an element of R and investigate the generator matrices of the linear code and its dual.

Keywords: Duality, Generator matrix, Lee weight, Linear codes over rings.

1 Introduction

In algebraic coding theory, the most important class of codes is the family of linear codes. A linear code of length n over $\mathbb{F}_q$ is a linear subspace of the vector space $\mathbb{F}_q^n$ where $\mathbb{F}_q$ is the finite field with q elements. A linear code of length n over a ring R is an R-submodule of R^n.

Codes over finite fields have been studied by many researchers. After the appearance of [1], a lot of researchers have considered codes over $\mathbb{Z}_4$. Later, these studies were mostly generalized to several new families of rings such as finite chain rings and rings of the form $\mathbb{F}_2/\langle u^m \rangle$ [2]. There is a very interesting connection between $\mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$. Both are commutative rings of size 4, they are both finite-chain rings. Some of the main differences between these two rings are that their characteristic is not the same, Gray images of $\mathbb{Z}_4^2$-codes are usually not linear while the Gray images of $\mathbb{F}_2 + u\mathbb{F}_2$-codes are linear.

Inspired by this similarity (and difference), in [3], Yildiz and Karadeniz considered linear self dual codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and proved the MacWilliams identities for the weight enumerators of the codes involved. The authors defined a linear Gray map from $\mathbb{Z}_4 + u\mathbb{Z}_4$ to $\mathbb{Z}_4^2$ and a non-linear Gray map from $\mathbb{Z}_4 + u\mathbb{Z}_4$ to $(\mathbb{F}_2 + u\mathbb{F}_2)^2$, and used them to successfully construct formally self-dual codes over $\mathbb{Z}_4$ and good non-linear codes over $\mathbb{F}_2 + u\mathbb{F}_2$.

In [4] the authors derived the certain lower and upper bounds on the minimum distances of the binary images in terms of the parameters of the $\mathbb{Z}_4 + u\mathbb{Z}_4$ codes. They performed same analogous procedure on the ring $\mathbb{Z}_8 + u\mathbb{Z}_8$, where $u^2 = 0$, which is a commutative local Frobenius non-chain ring of order 64. Then, the method was generalized to the class of rings $\mathbb{Z}_{2^r} + u\mathbb{Z}_{2^r}$, where $u^2 = 0$, for any positive integer r.

In [7] the linear codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4 + uv\mathbb{Z}_4$ where $u^2 = u$, $v^2 = v$, $uv = vu$ are introduced.

Motivated by the works in [4] and [7], in this paper, the ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$ where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ is introduced and the Lee distance and the Lee weight of an element of R are defined, and the generator matrices of the linear code and its dual are investigated.

2 The Ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

The ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$ is a commutative, characteristic 8 ring with $u^2 = u$, $v^2 = v$, $uv = vu = 0$. It can be also viewed as the quotient ring $\dfrac{\mathbb{Z}_8[u, v]}{\langle u^2 - u, v^2 - v, uv = vu \rangle}$. Let r be any element of R, which can be expressed uniquely as $r = a + ub + vc$, where $a, b, c \in \mathbb{Z}_8$.

Let $e_1 = 1 - u - v$, $e_2 = u$, $e_3 = v$, then e_1, e_2, e_3 are pairwise orthogonal non-zero idempotent elements over R, and the unit element 1 can be decomposed as $1 = e_1 + e_2 + e_3$. By the Chinese Remainder Theorem, we have $R = e_1R + e_2R + e_3R$, and r can be expressed uniquely as $r = e_1r_1 + e_2r_2 + e_3r_3$, where $r_1 = a$, $r_2 = a + b$, $r_3 = a + c$.

The ring R has the following properties:

- The finite ring R is with 512 elements.
- Its units are given by

$$S = \left\{ a + ub + vc \mid a, \overline{a+b}, \overline{a+c} \in \{1, 3, 5, 7\} \right\}.$$

- It has a total of 64 ideals. Let $S_1 = \{1, 3, 5, 7\}$, $S_2 = \{2, 6\}$ and $S_3 = \{0, 2, 4, 6\}$. The trivial ideals are

$$\langle 0 \rangle = \{0\} \text{ and } \langle r \rangle, \text{ where } r \in S.$$

The other non-trivial ideals of R is given the last page of the paper.
- R is a principal ideal ring.
- R is not a finite chain ring.

Definition 1. *A linear code C of length n over the ring R is a R-submodule of R^n. A codeword is denoted as $\mathbf{c} = (c_1, c_2, \ldots, c_n)$.*

The Lee weights of $0, 1, 2, 3 \in \mathbb{Z}_4$ are defined by $w_L(0) = 0$, $w_L(1) = 1$, $w_L(2) = 2$ and $w_L(3) = 1$. In the case of $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, the Lee weight was defined in [5] as

$$w_L(d) = w_L(a, a+b, a+c)$$

where $a, b, c \in \mathbb{Z}_4$. A similar technique is adopted here.

The Lee weight of a vector $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \in (\mathbb{Z}_8)^n$ was defined as

$$\sum_{i=0}^{n-1} \min\{|v_i|, |8 - v_i|\}$$

in [6].

Let $r = a + ub + cv$ be an element of R, then we define the Lee weight of r as

$$w_L(r) = w_L\left(a, \overline{a+b}, \overline{a+c}\right)$$

where $a, b, c \in \mathbb{Z}_8$. The Lee weight of a vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in R^n$ to be the sum of Lee weights its components:

$$w_L(r) = w_L\left(a, \overline{a+b}, \overline{a+c}\right) = w_L(a) + w_L(\overline{a+b}) + w_L(\overline{a+c}).$$

For any elements $\mathbf{x}, \mathbf{y} \in R^n$, the Lee distance between $\mathbf{x}$ and $\mathbf{y}$ is given by

$$d_L(\mathbf{x} - \mathbf{y}) = w_L(\mathbf{x} - \mathbf{y}).$$

The minimum Lee distance defined as

$$d_L(C) = \min\{d_L(\mathbf{x} - \mathbf{y}) : \mathbf{x} \neq \mathbf{y}, \text{ for all } \mathbf{x}, \mathbf{y} \in C\}.$$

Example 1. *Let $r = 2 + 6u + v$ and $r' = 1 + u + 4v \in R$. The Lee weights of r and r' as follows*

$$w_L(r) = w_L\left(2, \overline{2+6}, \overline{2+1}\right) = w_L(2, 0, 3) = 5,$$

$$w_L(r') = w_L\left(1, \overline{1+1}, \overline{1+4}\right) = w_L(1, 2, 5) = 6.$$

The Lee distance between r and r' as follows

$$d_L(r - r') = w_L(r - r') = w_L(1 + 5u + 5v) = w_L\left(1, \overline{1+5}, \overline{1+5}\right) = 5.$$

Let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$, $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$ be two vectors in R^n. The inner product between $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1 + \ldots + x_{n-1} y_{n-1}$$

where the operation are performed in the ring R.

Definition 2. *Let C be a linear code over the ring R of length n, then we define the dual of C as*

$$C^\perp = \left\{\mathbf{y} \in R^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0, \textit{ for all } \mathbf{x} \in C\right\}$$

Note that from the definition of inner product, it is clear that $C^\perp$ is also a linear code over R^n. A code C is said to be self-orthogonal if $C \subseteq C^\perp$, and self-dual if $C = C^\perp$.

3 Linear Codes over $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

Let C be a linear code of length n over R, we denote C_i $(1 \leq i \leq 3)$ as:

$$\begin{aligned} C_1 &= \left\{\mathbf{a} \in \mathbb{Z}_8^n \mid \exists\, \mathbf{b}, \mathbf{c} \in \mathbb{Z}_8^n, (1 - u - v)\mathbf{a} + u\mathbf{b} + v\mathbf{c} \in C\right\} \\ C_2 &= \left\{\mathbf{b} \in \mathbb{Z}_8^n \mid \exists\, \mathbf{a}, \mathbf{c} \in \mathbb{Z}_8^n, (1 - u - v)\mathbf{a} + u\mathbf{b} + v\mathbf{c} \in C\right\} \\ C_3 &= \left\{\mathbf{c} \in \mathbb{Z}_8^n \mid \exists\, \mathbf{a}, \mathbf{d} \in \mathbb{Z}_8^n, (1 - u - v)\mathbf{a} + u\mathbf{b} + v\mathbf{c} \in C\right\} \end{aligned}$$

where C_1, C_2 and C_3 are linear codes over $\mathbb{Z}_8^n$ of length n. And C can be uniquely expressed as

$$C = (1 - u - v)C_1 + uC_2 + vC_3.$$

According to the direct sum decomposition in above, we have $|C| = |C_1|\,|C_2|\,|C_3|$.

Theorem 1. *Let C be a linear code of length n over R, then*

1. $C=(1-u-v)C_1+uC_2+vC_3$, *where C_i $(1\le i\le 3)$ is a linear code of length n over $\mathbb{Z}_8$, and the direct sum decomposition is unique.*
2. $C^{\perp}=(1-u-v)C_1^{\perp}+uC_2^{\perp}+vC_3^{\perp}$, *where $C_i^{\perp}$ is the dual code of C_i $(1\le i\le 3)$.*
3. *C is a self-orthogonal code if and only if C_i $(1\le i\le 3)$ is a self-orthogonal code over $\mathbb{Z}_8$. Furthermore, C is a self-dual code if and only if C_i $(1\le i\le 3)$ is a self-dual code over $\mathbb{Z}_8$.*

Proof: 1. Let $\mathbf{r}=\left(r^{(0)},r^{(1)},\ldots,r^{(n-1)}\right)\in R^n$, where $r^{(i)}=(1-u-v)r_{i1}+ur_{i2}+vr_{i3}$ and $i=0,1,\ldots,n-1$. It is clear that $1-u-v$, u and v are pairwise orthogonal non-zero idempotent elements over R, then $\mathbf{r}$ can be uniquely expressed as $\mathbf{r}=(1-u-v)\mathbf{r}_1+u\mathbf{r}_2+v\mathbf{r}_3$, where $\mathbf{r}_j=\left(r_{0j},r_{1j},\ldots,r_{n-1,j}\right)\in\mathbb{Z}_8^n$ and $j=1,2,3$. Since a linear code C over R is a subgroup of R^n, then C can be uniquely expressed as $C=(1-u-v)C_1+uC_2+vC_3$.
2. Let $D=(1-u-v)C_1^{\perp}+uC_2^{\perp}+vC_3^{\perp}$, for any $\mathbf{d}=(1-u-v)\mathbf{a}+u\mathbf{b}+v\mathbf{c}\in C$, $\mathbf{d}'=(1-u-v)\mathbf{a}'+u\mathbf{b}'+v\mathbf{c}'\in D$, where $\mathbf{a},\mathbf{b},\mathbf{c}\in C$ and $\mathbf{a}',\mathbf{b}',\mathbf{c}'\in D$. Then we have

$$\mathbf{d}\cdot\mathbf{d}'=(1-u-v)\mathbf{aa}'+u\mathbf{bb}'+v\mathbf{cc}'.$$

Hence, $\mathbf{d}\cdot\mathbf{d}'=0$, so we have $D\subseteq C^{\perp}$. Moreover, the ring R is Frobenius ring [8], so $|C|\left|C^{\perp}\right|=|R|^n$ [8]. Thus

$$|D|=\left|C_1^{\perp}\right|\left|C_2^{\perp}\right|\left|C_3^{\perp}\right|=\frac{8^n}{|C_1|}\frac{8^n}{|C_2|}\frac{8^n}{|C_3|}=\frac{R^n}{|C|}=\left|C^{\perp}\right|,$$

therefore we have $D=C^{\perp}$.
3. According to (1) and (2), we have $C\subseteq C^{\perp}$ if and only if $C_i\subseteq C_i^{\perp}$ $(1\le i\le 3)$ is a self-orthogonal code over $\mathbb{Z}_8$. Similarly, C is a self-dual code if and only if $C_i\subseteq C_i^{\perp}$ $(1\le i\le 3)$ is a self-dual code over $\mathbb{Z}_8$. □

Corollary 1. *There are self-dual codes of arbitrary lengths over R.*

Proof: From Theorem 1, there exists a self-dual code over R if and only if there exists a self-dual code over $\mathbb{Z}_8$. Clearly, there exists a self-dual code over $\mathbb{Z}_8$ generated by

$$\begin{pmatrix} 4 & & \\ & \ddots & \\ & & 4 \end{pmatrix}_{n\times n}.$$

□

We give the generator matrix of the linear codes over R. Let $C=(1-u-v)C_1+uC_2+vC_3$, for C_i $(1\le i\le 3)$ is a linear code over $\mathbb{Z}_8$, then C_i is permutation-equivalent to a code generated by

$$G_i=\begin{pmatrix} I_{k_{i0}} & A_i & B_i & T_i \\ 0 & 2I_{k_{i1}} & 2D_i & 2E_i \\ 0 & 0 & 4I_{k_{i2}} & 4F_i \end{pmatrix}[9].$$

Thus, C is permutation-equivalent to a linear code generated by

$$G=\begin{pmatrix} (1-u-v)G_1 \\ uG_2 \\ vG_3 \end{pmatrix}.$$

The dual code $C_i^{\perp}$ of the $\mathbb{Z}_8$-linear code C_i has the generator matrix

$$H_i=\begin{pmatrix} -T_i^t+E_i^tA_i^t+F_i^tB_i^t-F_i^tD_i^tA_i^t & -E_i^t+F_i^tD_i^t & -F_i^t & I_{n-k_{i0}-k_{i1}-k_{i2}} \\ -2B_i^t+2D_i^tA_i^t & -2D_i^t & 2I_{k_{i2}} & 0 \\ -4A_i^t & 4I_{k_{i1}} & 0 & 0 \end{pmatrix}.$$

[9]. Then $C^{\perp}$ is permutation-equivalent to a linear code generated by

$$H=\begin{pmatrix} (1-u-v)H_1 \\ uH_2 \\ vH_3 \end{pmatrix}.$$

H is called the party-check matrix of C.

Example 2. *Let $C=(1-u-v)C_1+uC_2+vC_3$, where C_1,C_2 and C_3 are linear codes over $\mathbb{Z}_8^2$ generated by*

$$G_1=\begin{pmatrix} 4 & 0 \end{pmatrix},\ G_2=\begin{pmatrix} 4 & 4 \end{pmatrix},\ G_3=\begin{pmatrix} 4 & 0 \end{pmatrix}.$$

Then C is generated by

$$G=\begin{pmatrix} 4-4u-4v & 0 \\ 4u & 4v \\ 4v & 0 \end{pmatrix}.$$

The dual codes C_1,C_2 and C_3 have the generator matrix

$$H_1=\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},\ H_2=\begin{pmatrix} 7 & 1 \\ 2 & 0 \end{pmatrix},\ H_3=\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Then the dual code $C^{\perp}$ is generated by

$$H=\begin{pmatrix} 0 & 1-u-v \\ 2-2u-2v & 0 \\ 7u & u \\ 2u & 0 \\ 0 & v \\ 2v & 0 \end{pmatrix}.$$

ideals with 2 elements	$\{a+ub+vc \mid a=b=0, c=4\}$ $\{a+ub+vc \mid b=c=0, a=4\}$ $\{a+ub+vc \mid a=c=4, b=0\}$	$\{a+ub+vc \mid a=c=0, b=4\}$ $\{a+ub+vc \mid a=b=4, c=0\}$ $\{a+ub+vc \mid b=c=4, a=0\}$
ideals with 4 elements	$\{a+ub+vc \mid a=b=0, c\in S_2\}$ $\{a+ub+vc \mid b=c=0, a\in S_2\}$	$\{a+ub+vc \mid a=c=0, b\in S_2\}$
ideals with 8 elements	$\{a+ub+vc \mid a=b=0, c\in S_1\}$ $\{a+ub+vc \mid b=c=0, a\in S_1\}$ $\{a+ub+vc \mid a=0, c=4, b\in S_2\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=4, \overline{a+c}=0\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}=0, \overline{a+c}\in S_2\right\}$	$\{a+ub+vc \mid a=c=0, b\in S_1\}$ $\{a+ub+vc \mid a=0, b=4, c\in S_2\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=0, \overline{a+c}=4\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}\in S_2, \overline{a+c}=0\right\}$ $\left\{a+ub+vc \mid a=\overline{a+b}=\overline{a+c}=4\right\}$
ideals with 16 elements	$\{a+ub+vc \mid a=0, b\in S_1, c=4\}$ $\{a+ub+vc \mid a=0, b, c\in S_2\}$ $\{a+ub+vc \mid a=\overline{a+c}=4, b\in S_2\}$ $\left\{a+ub+vc \mid a=\overline{a+b}\in S_2, \overline{a+c}=0,\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}\in S_1, \overline{a+c}=0\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}=4, \overline{a+c}=0\right\}$	$\{a+ub+vc \mid a=0, b=4, c\in S_1\}$ $\left\{a+ub+vc \mid a=\overline{a+b}=4, c\in S_2\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=\overline{a+c}=4,\right\}$ $\left\{a+ub+vc \mid a=\overline{a+c}\in S_2, \overline{a+b}=0\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}=0, \overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}=0, \overline{a+c}=4\right\}$
ideals with 32 elements	$\left\{a+ub+vc \mid a=0, \overline{a+b}\in S_1, \overline{a+c}\in S_2\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}\in S_2, \overline{a+c}=0\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}=\overline{a+c}=4\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}\in S_1, \overline{a+c}=0\right\}$ $\left\{a+ub+vc \mid a=\overline{a+c}\in S_2, \overline{a+b}=4\right\}$ $\left\{a+ub+vc \mid a=\overline{a+b}=4, \overline{a+c}\in S_1,\right\}$	$\left\{a+ub+vc \mid a=0, \overline{a+b}\in S_2, \overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}=0, \overline{a+c}\in S_2\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=0, \overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a=\overline{a+b}\in S_2, \overline{a+c}=4\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}=\overline{a+c}\in S_2,\right\}$ $\left\{a+ub+vc \mid a=\overline{a+c}=4, \overline{a+b}\in S_1,\right\}$
ideals with 64 elements	$\left\{a+ub+vc \mid a=0, \overline{a+b}=\overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}\in S_2, \overline{a+c}=4\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}\in S_1, \overline{a+c}\in S_2\right\}$ $\left\{a+ub+vc \mid a=\overline{a+b}=\overline{a+c}\in S_2\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}=4, \overline{a+c}\in S_2\right\}$	$\left\{a+ub+vc \mid a=\overline{a+b}\in S_1, \overline{a+c}=0\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}\in S_1, \overline{a+c}=4\right\}$ $\left\{a+ub+vc \mid a=4, \overline{a+b}\in S_2, \overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a=\overline{a+c}\in S_1, \overline{a+b}=0\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=4, \overline{a+c}\in S_1\right\}$
ideals with 128 elements	$\left\{a+ub+vc \mid a=4, \overline{a+b}=\overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a=\overline{a+c}\in S_1, \overline{a+b}=4\right\}$ $\left\{a+ub+vc \mid a=\overline{a+c}\in S_2, \overline{a+b}\in S_1\right\}$	$\left\{a+ub+vc \mid a=\overline{a+b}\in S_1, \overline{a+c}=4\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=\overline{a+c}\in S_1\right\}$ $\left\{a+ub+vc \mid a\in S_1, \overline{a+b}=\overline{a+c}\in S_2\right\}$
ideals with 256 elements	$\left\{a+ub+vc \mid a=\overline{a+b}\in S_1, \overline{a+c}\in S_3\right\}$ $\left\{a+ub+vc \mid a\in S_2, \overline{a+b}=\overline{a+c}\in S_1\right\}$	$\left\{a+ub+vc \mid a=\overline{a+c}\in S_1, \overline{a+b}\in S_3\right\}$

Table 1 Ideals of R

4 Conclusion

In this work, it's shown that the ring $R=\mathbb{Z}_8+u\mathbb{Z}_8+v\mathbb{Z}_8$ is a commutative, characteristic 8 ring with $u^2=u$, $v^2=v$, $uv=vu=0$. Moreover, the ideals of $\mathbb{Z}_8+u\mathbb{Z}_8+v\mathbb{Z}_8$ are found and the Lee weight is defined on $\mathbb{Z}_8+u\mathbb{Z}_8+v\mathbb{Z}_8$. In the last part the generator matrices of the linear code and its dual are obtained.

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