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Common fixed points for ψ-Geraghty-Jungck contraction type mappings in Branciari b-metric spaces

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Abstract

The main purpose of this paper is to define a class of contraction-type pair of mappings, called ψ-Geraghty-Jungck contraction pair, which consists in a Jungck pair of mappings satisfying the Geraghty condition and, furthermore, its contractive inequality is controlled by an altering distance function. For this class of mappings, we discuss the existence and uniqueness of its common fixed points under the weakly compatibility property. These mappings are defined in the setting of the so-called Branciari b-metric spaces.

Keywords: contraction-type mapping, common fixed point, weakly compatible mappings.
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1. Motivation and preliminary results

In 1922, S. Banach introduced his famous result, *the Banach Contraction Principle*, (in short (BCP)), in the metric fixed point theory. It is well know that the BCP has been generalized in different directions, some of them are: to modify the structure of usual metric space in order to pose this principle in more general spaces, and to extend the contractive inequality for include large classes of mappings. In this work we will follow these two directions of research by extending the BCP to pair of mappings and posing it in the setting of the so-called Branciari b-spaces, which are a combination of b-metric spaces.

More precisely, we want to discuss the existence and uniqueness of common fixed point for a pair of weakly compatible selfmaps that satisfying the ψ-Geraghty-Jungk contraction condition (see, inequality (2.2)).

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1.1. Branciari b-metric spaces

In this section we define some basic concepts, notions and properties of the so-called Branciari b-metric spaces, which are going to be used through this paper.

Definition 1.1 ([6]). *Let M be a non empty set and $s \geq 1$ be a given real number and let $\rho : M \times M \longrightarrow \mathbb{R}_+ := [0, \infty)$ be a function such that for all $x, y \in M$ and all distinct points $u, v \in M$ with $u, v \in M \setminus \{x, y\}$ the following conditions hold:*

Bb1.- $\rho(x, y) = 0$ *is and only if* $x = y$.
Bb2.- $\rho(x, y) = \rho(y, x)$.
Bb3.- $\rho(x, y) \leq s[\rho(x, u) + \rho(u, v) + \rho(v, y)]$.

Then, ρ is called a Branciari b-metric on M, and (M, ρ) is called a Branciari b-metric space, (in short BbMS) with coefficient $s \geq 1$.

Note that every usual metric space is a Branciari metric space ([3]) and every Branciari metric space is a Branciari b-metric space (with coefficient $s = 1$). Also, every metric space is a b-metric space ([2]) and every b-metric space is a Branciari b-metric space, not necessarily with the same coefficient. Moreover, the converse of all implications are not necessarily true, as it is shown in Examples 1.4, 1.5 and 1.7 in [6], proving in this way that this notion is a proper generalization.

Example 1.1 ([6]). *Let (M, δ) be a Branciari metric space and $p \geq 1$ be a real number. Let $\rho(x, y) = [\delta(x, y)]^p$. It is clear from the convexity of the function $f(t) = t^p$, for $t \geq 0$, and Jensen's inequality, that*

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p)$$

for nonnegative real numbers a, b, c. Therefore, for all $x, y \in M$ and all distinct points and $u, v \in M$ with $u, v \in M \setminus \{x, y\}$, we get

$$\begin{aligned} \rho(x, y) = [\delta(x, y)]^p &\leq (\delta(x, u) + \delta(u, v) + \delta(v, y))^p \\ &\leq 3^{p-1} [\delta(x, u)^p + \delta(u, v)^p + \delta(v, y)^p] \\ &\leq 3^{p-1} (\rho(x, u) + \rho(u, v) + \rho(v, y)). \end{aligned}$$

Thus, condition Bb3 in Definition 1.1 holds, then ρ is a Branciari b-metric with coefficient $s = 3^{p-1} > 1$. Therefore, (M, ρ) is a Branciari b-metric space.

In [14], T. Suzuki prove the following lemma which allows to construct Branciari metric spaces from bounded metric spaces. Then, from example above, we can construct Branciari b-metric spaces from bounded metric spaces by letting $\rho(x, y) := [\delta(x, y)]^p$, with $\delta(x, y)$ as in the next result.

Lemma 1.1 ([14]). *Let (M, d) be a bounded metric space and let K be a real number satisfying*

$$\sup\{d(x, y) \ : \ x, y \in M\} \leq K.$$

Let A and B two subsets of M with $M = A \cup B$ and $A \cap B = \emptyset$. Define a function δ from $M \times M$ into $\mathbb{R}_+$ by

$$\begin{aligned} \delta(x, x) =& 0 \\ \delta(x, y) =& \delta(y, x) = d(x, y), \quad \textit{if } x \in A, \ y \in B \\ \delta(x, y) =& K, \quad \textit{otherwise}. \end{aligned}$$

Then, (M, δ) is a Branciari metric space.

On a Branciari b-metric space (M, ρ) we define and denote an open ball of center $x \in M$ and radius $r > 0$ as

$$B(x, r) := \{y \in M : \ \rho(x, y) < r\}.$$

Let τ_ρ be the collection of all subsets $U \subset M$ with the following property: for each $y \in U$ there exists $r > 0$ such that $B(y, r) \subset U$. Then, τ_ρ defines a topology for the Branciari b-metric space which is not necessarily Hausdorff. Even more, the open balls are not always open sets as Example 1.2 shows.

The next definition gives the concepts of convergence of sequences, Cauchy sequences and completeness on Branciari b-metric spaces.

Definition 1.2. *Let (M, ρ) be a Branciari b-metric space with $s \geq 1$ and $(x_n)_n$ is a sequence in M.*

1. *The sequence $(x_n)_n$ is said to be BbMS-convergent to $x \in M$ if and only if for every $\epsilon > 0$, exists $n_0 \in \mathbb{N}$, such that for $n > n_0$ we have $\rho(x_n, x) < \epsilon$, or equivalently, if $\lim\limits_{n\to\infty} \rho(x_n, x) = 0$.*
2. *The sequence $(x_n)_n$ is called BbMS-Cauchy sequence if and only if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ and $p > 0$ we have $d(x_n, x_{n+p}) < \epsilon$, or equivalently, $\lim\limits_{n\to\infty} \rho(x_n, x_{n+p}) = 0$ for all $p > 0$.*
3. *A Branciari b-metric space (M, d) is called complete if every BbMS-Cauchy sequence in it is BbMS-convergent to some $x \in M$.*

The following example shows some properties of Branciari b-metric space which are not shared by the usual metric spaces. This example appears in [6], however, for the sake of the presentation, here we include all the missing computations to obtain the conclusions.

Example 1.2 ([6])**.** *Let $M = A \cup B$, where $A = \left\{\frac{1}{n} : \ n \in \mathbb{N}\right\}$ and $B = \mathbb{Z}_+$. Define $\rho : M \times M \longrightarrow \mathbb{R}_+$ such that $\rho(x, y) = \rho(y, x)$ for all $x, y \in M$ and*

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2\alpha & \text{if } x, y \in A \\ \dfrac{\alpha}{2n} & \text{if } x \in A \text{ and } y \in \{2, 3\} \\ \alpha & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is a constant. Notice that:

1. *(M, ρ) is not a Branciari metric space (hence it is not a usual metric space) since*
$$\rho\left(\frac{1}{2}, \frac{1}{3}\right) = 2\alpha > \frac{17}{12}\alpha = \rho\left(\frac{1}{2}, 4\right) + \rho\,(4, 3) + \rho\left(3, \frac{1}{3}\right).$$
Even more, $\rho\left(\frac{1}{n}, \frac{1}{m}\right) = 2\alpha > \frac{\alpha}{2n} + \alpha + \frac{\alpha}{2m} = \rho\left(\frac{1}{n}, u\right) + \rho(u, v) + \rho\left(v, \frac{1}{m}\right)$ for all $u \in \mathbb{Z}_+ \setminus \{2, 3\}$ and $v \in \{2, 3\}$. It is easy to check that in the remain cases the inequality $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y)$ holds. Thus, (M, ρ) becomes a BbMS if $2\alpha \leq s\left(\frac{\alpha}{2n} + \alpha + \frac{\alpha}{2m}\right)$, equivalently, if $s \geq \frac{4}{2+\frac{1}{n}+\frac{1}{m}}$. Since $\frac{8}{7} = \min_{n,m\in\mathbb{N}}\{\frac{4}{2+\frac{1}{n}+\frac{1}{m}}\}$, then (M, ρ) is a Branciari b-metric space with coefficient $s = \frac{8}{7} > 1$ (cf. [6]).
2. *(M, ρ) is not a b-metric space since for $x, y \in \{2, 3\}$ and $z \in A$, $\rho(x, y) = \alpha > \frac{\alpha}{n} = \rho(x, z) + \rho(z, y)$, $z = 1/n$, $(n > 1)$. In this situation, inequality Bb3 holds only if there exists $s > n$ for all $n \in \mathbb{N}$, which is not possible.*
3. *Taking $\alpha = 1$, we have that $B\left(\frac{1}{2}, \frac{1}{2}\right) = \{2, 3\}$ and there does not exists any open ball with center 2 and radius $r > 0$ contained in $B\left(\frac{1}{2}, \frac{1}{2}\right)$. In fact, notice that*
$$\rho(2, z) = \begin{cases} 1, & z \in B \\ \frac{1}{2n}, & z = \frac{1}{n} \in A. \end{cases}$$

Therefore,

$$B(2,r) = \begin{cases} M, & r > 1 \\ A, & \frac{1}{2} < r \leq 1 \\ \{\frac{1}{n} : n \in \mathbb{N},\ n > \lceil \frac{1}{2r} \rceil\} \subset A, & 0 < r \leq \frac{1}{2}. \end{cases}$$

Thus $B\left(\frac{1}{2}, \frac{1}{2}\right)$ *is not an open set.*

4. *The sequence* $(x_n)_n = \left(\frac{1}{n}\right)_n$ *converges to* 2 *and* 3 *in the Branciari b-metric space, hence limit is not unique. Moreover,* $\rho\left(\frac{1}{n}, \frac{1}{n+p}\right) = 2\alpha \not\to 0$ *as* $n \to \infty$*, therefore,* $(x_n)_n = \left(\frac{1}{n}\right)_n$ *is not a BbMS-Cauchy sequence in* (M, ρ)*.*
5. *There are not exist any* $r_1, r_2 > 0$ *such that* $B(2, r_1) \cap B(3, r_2) = \emptyset$*. In fact,*

$$B(2,r) = B(3,r) = \begin{cases} M, & r > \alpha \\ \{\frac{1}{n} : n \in \mathbb{N},\ n > \lceil \frac{\alpha}{2r} \rceil\} \subset A, & 0 < r \leq \alpha. \end{cases}$$

Hence, (M, ρ) *is not Hausdorff.*

Example 1.3 ([13])**.** *Let* $A = \{0, 2\}$*,* $B = \left\{\frac{1}{n} :\ n \in \mathbb{N}\right\}$ *and* $M = A \cup B$*. Define* $\delta : M \times M \longrightarrow \mathbb{R}_+$ *as follows:*

$$\delta(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \text{ and } \{x,y\} \subset A \text{ or } \{x,y\} \subset B \\ y, & \text{if } x \in A \text{ and } y \in B \\ x, & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Then, (M, δ) *is a complete Branciari metric space, since if* $(x_n)_n$ *is a Cauchy sequence in* M*, for every* $\epsilon > 0$ *and for all* n *sufficiently large and* $p > 0$*,* $\delta(x_n, x_{n+p}) < \epsilon$*. By the definition of* δ *in* M*, this is possible only if* $x_n = x_{n+p}$ *for all* $p > 0$*, that is,* x_n *is a constant sequence, which converges. Thus,* (M, d) *is a complete Branciari space. Now, taking* $\rho(x, y) = (\delta(x, y))^2$*, according to Example 1.1, we obtain a Branciari b-metric space* (M, ρ) *with* $s = 3$*. Moreover, it can be proved that*

1. *The sequence* $(x_n)_n = \left(\frac{1}{n}\right)_n$ *converges to both* 0 *and* 2 *and it is not a Cauchy sequence, since*

$$\lim_{n\to\infty} d\left(\frac{1}{n}, \frac{1}{n+p}\right) = 1 \neq 0.$$

2. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$*, but* $1 = \lim\limits_{n\to\infty} \rho\left(\dfrac{1}{n}, \dfrac{1}{2}\right) \neq \rho\left(0, \dfrac{1}{2}\right) = \dfrac{1}{4}$*; hence* ρ *is not a continuous function.*

Regarding to the above facts about the Branciari b-metric spaces, the following results are useful in proving our main results. As example before shows, a sequence in a Branciari b-metric space may have two limits. However, there is a special situation where this is not possible.

Lemma 1.2 ([13])**.** *Let* (M, ρ) *be a Branciari b-metric space with* $s \geq 1$ *and let* $(x_n)_n$ *be a BbMS-Cauchy sequence in* M *such that* $x_n \neq x_m$ *for all* $n \neq m$*. Then,* $(x_n)_n$ *converges to at most one point.*

Lemma 1.3 ([5])**.** *Let* (M, ρ) *be a Branciari b-metric space with* $s \geq 1$ *and let* $(x_n)_n$ *be a sequence in* M *such that* $x_n \neq x_m$ *whenever* $n \neq m$ *and* $\lim\limits_{n\to\infty} \rho(x_n, x_{n+1}) = 0$ *and* $\lim\limits_{n\to\infty} \rho(x_n, x_{n+2}) = 0$*. If* $(x_n)_n$ *is not a BbMS-Cauchy sequences, there exists* $\epsilon > 0$ *and two sequences* $(n(k))_k$ *and* $(m(k))_k$ *of positive integers with* $n(k) > m(k) > k$ *such that*

$$\rho\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon, \text{ and } \rho\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon,$$

and

$$\begin{aligned}
\epsilon \leq & \lim_{k\to\infty} \sup \rho\left(x_{m(k)}, x_{n(k)}\right) \leq s\epsilon \\
\frac{\epsilon}{s} \leq & \lim_{k\to\infty} \sup \rho\left(x_{m(k)+1}, x_{n(k)-1}\right) \leq \epsilon \\
\frac{\epsilon}{s} \leq & \lim_{k\to\infty} \sup \rho\left(x_{m(k)}, x_{n(k)-2}\right) \leq s\epsilon \\
\frac{\epsilon}{s} \leq & \lim_{k\to\infty} \sup \rho\left(x_{m(k)+1}, x_{n(k)}\right) \leq s\epsilon \\
\frac{\epsilon}{s} \leq & \lim_{k\to\infty} \sup \rho\left(x_{m(k)-1}, x_{n(k)-1}\right) \leq s^2\epsilon.
\end{aligned}$$

2. ψ-Geraghty-Jungck contraction pair of mappings

In this section, we introduce a new pair of contractive-type mappings on b-Branciari metric spaces which generalize the Jungck pair of maps. To define it, we control the contractive inequality with altering distance functions and we introduce an extra function as Geraghty [7]. These extra functions allows to include large classes of mappings which we will prove they have common fixed points.

We recall that in 1984, M.S. Khan, M. Swalech and S. Sessa [10] introduced the notion of altering distance function as follows:

Definition 2.1. *A function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called an altering distance function if the following properties are satisfied:*

ψ1.- *ψ is monotonically non-decreasing.*
ψ2.- *ψ is a continuous mapping.*
ψ3.- *$\psi(t) = 0$ if and only if $t = 0$.*

In the sequel, we will denote by Ψ the set of all altering distance functions. The class of all functions $\beta : \mathbb{R}_+ \longrightarrow [0, 1/s]$, with $s > 1$, satisfying the following condition:

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \Longrightarrow \lim_{n\to\infty} t_n = 0, \tag{2.1}$$

for any $(t_n)_n \subset \mathbb{R}_+$, will be denoted by $\mathcal{B}_s$. Note that if we take $s = 1$ we obtain the Geraghty's condition ([7]).

Now, using the altering distance functions and condition (2.1), we introduce the class of pairs of ψ-Geraghty-Jungck contraction type mappings as follows:

Definition 2.2. *Let (M, ρ) be a Branciari b-metric space with $s \geq 1$. A pair of mappings $S, T : M \longrightarrow M$ is called a ψ-Geraghty-Jungck contraction pair if for all $x, y \in M$ there exists $\beta \in \mathcal{B}_s$ and $\psi \in \Psi$ such that*

$$\psi\left[s^2\rho(Sx, Sy)\right] \leq \beta\left[\psi(\rho(Tx, Ty))\right]\psi\left[\rho(Tx, Ty)\right]. \tag{2.2}$$

This class of mappings extend and generalize several classes of mappings of Jungck's and Geraghty's type. See, for instance, [4] and references therein.

We recall that for two self mappings S, T on M, a point $x \in M$ is called a *coincidence point* of S and T if $Sx = Tx$. A point $w \in M$ is called a *point of coincidence* (in short, POC) of S and T if there exists a coincidence point $x \in M$ of S and T such that $Sx = Tx = w$.

The next result will help us to show the existence of a POC for the class of ψ-Geraghty-Jungck contraction pairs.

Proposition 2.1. *Let (M, ρ) be a Branciari b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two mappings with $SM \subset TM$. If the pair (S, T) satisfies condition (2.2), then for any $x_0 \in M$, the sequence defined by $y_n = Sx_n = Tx_{n+1}$, with $n = 0, 1, \ldots,$ satisfies:*

1. $\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0$ *and* $\lim_{n\to\infty} \rho(y_n, y_{n+2}) = 0$,
2. $(y_n)_n$ *is a BbMS-Cauchy sequence in* M.

Proof. For any arbitrary point $x_0 \in M$, from condition $SM \subset TM$, we choose sequences $(x_n)_n$ and $(y_n)_n$ in M as

$$y_n = Sx_n = Tx_{n+1}, \quad n = 0, 1, \ldots$$

If $y_k = y_{k+1}$ for some $k \in \mathbb{N}$, then

$$Sx_{k+1} = y_{k+1} = y_k = Tx_{k+1}.$$

That is, S and T have a POC. Therefore, we suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = y_{n+1}$ and $y = y_n$ in (2.2) we obtain

$$\begin{aligned}
\psi\left[\rho(y_n, y_{n+1})\right] &\leq \psi\left[s^2\rho(y_n, y_{n+1})\right] = \psi\left[s^2\rho(Sx_n, Sx_{n+1})\right] \\
&\leq \beta\left[\psi\left(\rho(Tx_n, Tx_{n+1})\right)\right]\psi(\rho(Tx_n, Tx_{n+1})) \\
&= \beta\left[\psi\left(\rho(y_{n-1}, y_n)\right)\right]\psi(\rho(y_{n-1}, y_n)) \\
&< \frac{1}{s}\psi(\rho(y_{n-1}, y_n)) < \psi(\rho(y_{n-1}, y_n)).
\end{aligned}$$

Since $\psi \in \Psi$, this implies

$$\rho(y_n, y_{n+1}) < \rho(y_{n-1}, y_n).$$

It follows that $z_n = \left(\rho(y_n, y_{n+1})\right)_n$ is a monotone non increasing sequence of positive real numbers, consequently there exists $L \geq 0$ such that

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = L.$$

Notice that, if $L > 0$, then

$$\begin{aligned}
0 \leq \psi(L) &\leq \psi(s^2 L) = \lim_{n\to\infty} \psi\left(s^2\rho(y_n, y_{n+1})\right) \\
&\leq \lim_{n\to\infty} \sup \beta\left[\psi(\rho(y_{n-1}, y_n))\right] \lim_{n\to\infty} \sup \psi(\rho(y_{n-1}, y_n)) \\
&< \frac{1}{s}\psi(L) < \psi(L)
\end{aligned}$$

which is a contradiction, therefore, $L = 0$. Thus,

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

In a similar way we can prove:

$$\lim_{n\to\infty} \rho(y_n, y_{n+2}) = 0.$$

Suppose now that $y_n = y_m$ for some $n > m$, hence $y_{n+k} = y_{m+k}$ for $k \in \mathbb{N}$. Then, from (2.2) we have

$$\begin{aligned}
\psi\left[\rho(y_m, y_{m+1})\right] &\leq \psi\left[s^2\rho(y_m, y_{m+1})\right] = \psi\left[s^2\rho(y_n, y_{n+1})\right] = \psi\left[s^2\rho(Sx_n, Sx_{n+1})\right] \\
&\leq \beta\left[\psi(\rho(Tx_n, Tx_{n+1}))\right]\psi(\rho(Tx_n, Tx_{n+1})) \\
&= \beta\left[\psi(\rho(y_{n-1}, y_n))\right]\psi(\rho(y_{n-1}, y_n)) \\
&< \psi(\rho(y_{n-1}, y_n)) \leq \psi(s^2\rho(y_{n-1}, y_n)) \\
&\vdots \\
&\leq \beta\left[\psi(\rho(y_m, y_{m+1}))\right]\psi(\rho(y_m, y_{m+1})) < \psi(\rho(y_m, y_{m+1}))
\end{aligned}$$

which is a contradiction. Thus, in what follows, we assume that $y_n \neq y_m$ for $n \neq m$.

Now, we want to show that $(y_n)_n$ is a BbMS-Cauchy sequence in M. Suppose the contrary. Then, there exists an $\epsilon > 0$ and sequences $(n(k))_k$ and $(m(k))_k$ of positive integers such that $n(k) > m(k) > k$ satisfying

$$\rho(y_{m(k)}, y_{n(k)}) \geq \epsilon$$

and

$$\rho(y_{m(k)}, y_{n(k)-1}) < \epsilon.$$

Now, we substitute $x = y_{m(k)}$ and $y = y_{n(k)}$ in (2.2). We have,

$$\begin{aligned}
\psi\left[s^2\rho(y_{m(k)}, y_{n(k)})\right] &= \psi\left[s^2\rho(Sx_{m(k)}, Sx_{n(k)})\right] \\
&\leq \beta\left[\psi\left(\rho(Tx_{m(k)}, Tx_{n(k)})\right)\right]\psi\left(\rho(Tx_{m(k)}, Tx_{n(k)})\right) \\
&= \beta\left[\psi\left(\rho(y_{m(k)-1}, Tx_{n(k)-1})\right)\right]\psi\left(\rho(y_{m(k)-1}, Tx_{n(k)-1})\right).
\end{aligned}$$

Taking lim sup as $k \to \infty$ and using Lemma 1.3, we obtain

$$\begin{aligned}
\psi(s^2\epsilon) &\leq \limsup_{k\to\infty} \psi\left[s^2\rho(y_{m(k)}, y_{n(k)})\right] \\
&\leq \limsup_{k\to\infty} \beta\left[\psi\left(\rho(y_{m(k)-1}, Tx_{n(k)-1})\right)\right] \limsup_{k\to\infty} \psi\left(\rho(y_{m(k)-1}, Tx_{n(k)-1})\right) \\
&< \frac{1}{s}\psi(s^2\epsilon) < \psi(s^2\epsilon),
\end{aligned}$$

which is a contradiction. Therefore, $(y_n)_n$ is a BbMS-Cauchy sequence in M. □

Proposition 2.2. *Let S and T be two self maps on a Branciari-b-metric space (M, ρ) with $s \geq 1$. Let us assume that the pair (S, T) satisfies condition (2.2). If S and T have a POC in M, then it is unique.*

Proof. Let z and w be two POC of S and T. Thus, there exist some $x, y \in M$ such that

$$w = Sx = Tx \text{ and } z = Sy = Ty.$$

By (2.2), we have

$$\begin{aligned}
\psi(\rho(w, z)) &\leq \psi(s^2\rho(w, z)) = \psi(s^2\rho(Sx, Sy)) \\
&\leq \beta\left[\psi(\rho(Tx, Ty))\right]\psi(\rho(Tx, Ty)) \\
&= \beta\left[\psi(\rho(w, z))\right]\psi(\rho(w, z)) \\
&< \frac{1}{s}\psi(\rho(w, z)) < \psi(\rho(w, z))
\end{aligned}$$

which is a contradiction. Thus, we conclude that $w = z$ and the POC is unique. □

3. On the existence and uniqueness of common fixed points

In this section we prove our main results concerning to the existence and uniqueness of common fixed points for ψ-Geraghty-Jungck contraction type mappings defined on Branciari b-metric spaces, without assuming continuity requirements. The classical assumptions in this line of research, as the commutativity property, in this case is reduced to the existence of points of coincidence, and the completeness of the space is reduced to natural conditions. Even more, we prove that the Jungck-Picard iterative scheme ([8]) converges to the unique common fixed point of a ψ-Geraghty-Jungck contraction pair.

We recall that S and T are said to be *weakly compatible* if S and T commute at their coincidence points, that is, if $Sx = Tx$ then $STx = TSx$ ([9]). We would like to point out that weakly compatible is a minimal requirement for the existence of common fixed points for contractive pair of mappings. For a discussion in the subject see, e.g., [11, 12].

Lemma 3.1 ([9])**.** *Let S and T be weakly compatible self mappings on a non empty set M. If S and T have a unique POC, $w = Sx = Tx$, then w is the unique common fixed point of S and T.*

Theorem 3.1. *Let (M, ρ) be a Branciari b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ two self maps such that*

i. $SM \subset TM$.

ii. $TM \subset M$ is a complete subspace.

iii. Suppose that the pair (S,T) satisfies the condition (2.2).

Then,

1. *S and T have a unique POC.*
2. *If S and T are weakly compatible self mappings, then S and T have a unique common fixed point and the Jungck-Picard iterative scheme*

$$y_n = Sx_n = Tx_{n+1}, \quad x_0 \in M, \quad n = 0, 1, \ldots \tag{3.1}$$

converges to the unique common fixed point of (S,T).

Proof. Let $x_0 \in M$ be an arbitrary point, using the condition $SM \subset TM$ we choose sequences $(x_n)_n$ and $(y_n)_n$ in M such that $y_n = Sx_n = Tx_{n+1}$, for $n = 0, 1, \ldots$ From Proposition 2.1, we have the following conclusions:

1. $$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0 \text{ and } \lim_{n\to\infty} \rho(y_n, y_{n+2}) = 0. \tag{3.2}$$
2. $(y_n)_n$ is BbMS-Cauchy sequence in M.

Since $(Tx_{n+1})_n \subset TM$ and $TM \subset M$ is a closed subspace, there exists $z \in TM$ such that

$$\lim_{n\to\infty} Tx_{n+1} = z \tag{3.3}$$

that is, $z = Tu$ for some $u \in M$. Now, we prove that $z = Su = Tu$.

Suppose that $Su \neq Tu$. Then by Lemma 1.2 it follows that y_n differs from Su and Tu for n sufficiently large. Hence, applying the quadrilateral inequality, we obtain

$$\rho(Su, Tu) \leq s[\rho(Su, y_{n-1}) + \rho(y_{n-1}, y_n) + \rho(y_n, Tu)].$$

Letting $n \to \infty$, and applying (3.2) and (3.3) in the above inequality, we get

$$\rho(Su, Tu) = 0, \text{ so } z = Su = Tu.$$

From Proposition 2.2 we get that z is unique POC of S and T. Finally, if S and T are weakly compatible, by Lemma 3.1 we conclude that z is the unique common fixed point of S and T in M which is the limit of the Jungck-Picard iterative scheme (3.1). □

Example 3.1. *Let $M = \{0, 1, 2, 3\}$ and define $\rho : M \times M \longrightarrow \mathbb{R}_+$ as follows:*

1. *$\rho(x, y) = 0$ if and only if $x = y$,*
2. *$\rho(x, y) = \rho(y, x)$ for all $x, y \in M$ and*
3. *$\rho(0, 3) = \rho(2, 3) = \rho(0, 2) = 1$, $\rho(1, 3) = 3$, $\rho(0, 1) = 6$, $\rho(1, 2) = 5$.*

The authors in [1] proved that (M, ρ) is a Branciari b-metric space. We define $S, T : M \longrightarrow M$ as follows:

1. *$T0 = 0$, $T1 = 2$, $T2 = 3$, $T3 = 1$, $TM = \{0, 1, 2, 3\}$.*
2. *$S0 = S1 = S2 = 0$, $S3 = 2$, $SM = \{0, 2\} \subset TM$.*

Hence,

$$\frac{6}{5} = \psi\left(s^2\rho(S0, S3)\right) \leq \beta\left[\psi(\rho(T0, T3))\right]\psi(\rho(T0, T3)) = \frac{6}{5}\sqrt{6}.$$

$$\frac{6}{5} = \psi\left(s^2\rho(S1, S3)\right) \leq \beta\left[\psi(\rho(T1, T3))\right]\psi(\rho(T1, T3)) = \frac{6}{5}\sqrt{5}.$$

$$\frac{6}{5} = \psi\left(s^2\rho(S2, S3)\right) \leq \beta\left[\psi(\rho(T2, T3))\right]\psi(\rho(T2, T3)) = \frac{6}{5}\sqrt{3}.$$

Therefore, S and T satisfy inequality (2.2). It is clear that $SM \subset TM$ and $TM \subset M$ is complete, also S and T are weakly compatible, thus we can apply Theorem 3.1 to conclude that $z = 0$ is the unique common fixed point of S and T.

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