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Research Article

Some Laplace transforms and integral representations for parabolic cylinder functions and error functions

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Abstract

This paper uses the convolution theorem of the Laplace transform to derive new inverse Laplace transforms for the product of two parabolic cylinder functions in which the arguments may have opposite sign. These transforms are subsequently specialized for products of the error function and its complement thereby yielding new integral representations for products of the latter two functions. The transforms that are derived in this paper also allow to correct two inverse Laplace transforms that are widely reported in the literature and subsequently uses one of the corrected expressions to obtain two new definite integrals for the generalized hypergeometric function.

Mathematics Subject Classification (2020). 33B20, 33C05, 33C15, 33C20, 44A10, 44A35

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1. Introduction

The parabolic cylinder function is intensively used in various domains such as chemical physics [17], lattice field theory [8], astrophysics [30], finance [20], neurophysiology [5] and estimation theory [4]. Products of parabolic cylinder functions involving both positive and negative arguments arise in, for instance, problems of condensed matter physics [7,18] and the study of real zeros of parabolic cylinder functions [9–11]. The error function $\operatorname{erf}(x)$ and its complement $\operatorname{erfc}(x)$ emerge as special cases of the parabolic cylinder function and play a prominent role in, for instance, the conduction of heat [6], statistics and probability theory [15,23] and hydrology [2].

However, the extensive tables of inverse Laplace transforms [14,21,26] present relatively few expressions for products of parabolic cylinder functions especially when signs of the arguments differ. For example, [26] only specifies the following inverse Laplace transforms for such set–up

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$$D_\nu\left(a\sqrt{p+\sqrt{p^2+b^2}}\right)\left\{D_\nu\left(-a\sqrt{\sqrt{p^2+b^2}-p}\right)\pm D_\nu\left(a\sqrt{\sqrt{p^2+b^2}-p}\right)\right\}$$
$$D_\nu\left(a\sqrt{p+\sqrt{p^2-b^2}}\right)\left\{D_\nu\left(-a\sqrt{p-\sqrt{p^2-b^2}}\right)\pm D_\nu\left(a\sqrt{p-\sqrt{p^2-b^2}}\right)\right\}$$

see Equations (3.11.4.9) and (3.11.4.10).

This paper uses the convolution theorem of the Laplace transform to derive inverse Laplace transforms for

$$p^i\exp\left(\tfrac{1}{2}p\left(y-x\right)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)\left\{D_\nu\left(-2^{1/2}x^{1/2}p^{1/2}\right)\pm D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\}$$

with $i=0$ or $-\frac{1}{2}$, i.e. for expressions in which the arguments have opposite sign and differ, and where also the orders take on different values.

These results also offer inverse Laplace transforms for the product of (complementary) error functions as the parabolic cylinder function for order -1 specializes into the complementary error function. As a result, novel integral representations are obtained for products of the (complementary) error functions and, for instance, the integral representation for $1-\mathrm{erf}(a)^2$ in [19] can be generalized into $1-\mathrm{erf}(a)\mathrm{erf}(b)$.

The paper also corrects two inverse Laplace transforms that are reported in [14,21,26]. Combinations of one of the corrected results with the results derived in this paper are particularly interesting as they yield two definite integrals for the generalized hypergeometric function that are not reported in, for instance, the comprehensive overview in [16].

The remainder of this paper is organized as follows. Section 2 presents the relation between the parabolic cylinder function and the Kummer confluent hypergeometric function that is central to the subsequent derivations. Also, more detail is presented on the formulation of the convolution theorem for the Laplace transform given that the limits of integration in the integrals in the product differ. Section 3 presents the inverse Laplace transforms for products of the parabolic cylinder function and uses these results to obtain novel integral representations for products of (complementary) error functions. Section 4 corrects two widely-reported inverse Laplace transforms. Section 5 uses one of these corrected expressions together with the results of Section 3 to derive two novel definite integrals for the generalized hypergeometric function.

2. Notation and background

The parabolic cylinder function in the definition of Whittaker [29] is denoted by $D_\nu\left(z\right)$, where ν and z represent the order and the argument, respectively. Equation (4) on p. 117 in [13] defines the parabolic cylinder function as follows

$$\begin{aligned}D_\nu\left(z\right)=2^{\nu/2}\exp\left(-\tfrac{1}{4}z^2\right)&\left\{\frac{\Gamma\left[1/2\right]}{\Gamma\left[\left(1-\nu\right)/2\right]}\Phi\left(-\frac{\nu}{2};\frac{1}{2};\frac{1}{2}z^2\right)\right.\\&\left.+\frac{z}{2^{1/2}}\frac{\Gamma\left[-1/2\right]}{\Gamma\left[-\nu/2\right]}\Phi\left(\frac{1-\nu}{2};\frac{3}{2};\frac{1}{2}z^2\right)\right\}\end{aligned}\qquad(2.1)$$

where $\Phi\left(a;b;z\right)$ is Kummer's confluent hypergeometric function

$$\Phi\left(a;b;z\right)=\sum_{n=0}^{\infty}\frac{\left(a\right)_n}{\left(b\right)_n}\frac{z^n}{n!},$$

$\Gamma\left[z\right]$ denotes the gamma function

$$\Gamma\left[z\right]=\int_0^\infty t^{z-1}\exp\left(-t\right)dt$$

and $(z)_n$ denotes the Pochhammer symbol

$$(z)_n = \frac{\Gamma[z+n]}{\Gamma[z]},$$

see Equation (1) on p. 434 in [24], Equations (6.1.1) and (6.1.22) in [1], respectively.

Note that the definition (2.1) holds for z as well as $-z$ and adding the corresponding relation for $D_\nu(-z)$ to (2.1) then gives

$$D_\nu(-z) - D_\nu(z) = \frac{z2^{(\nu+3)/2}\sqrt{\pi}}{\Gamma[-\nu/2]} \exp\left(-\tfrac{1}{4}z^2\right) \Phi\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{1}{2}z^2\right) \tag{2.2}$$

$$D_\nu(-z) + D_\nu(z) = \frac{2^{(\nu+2)/2}\sqrt{\pi}}{\Gamma[(1-\nu)/2]} \exp\left(-\tfrac{1}{4}z^2\right) \Phi\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{1}{2}z^2\right) \tag{2.3}$$

see Equations (46:5:4) and (46:5:3) in [22].

The convolution theorem of the Laplace transform will be used to derive inverse Laplace transforms for products of two parabolic cylinder functions. The functions in the products are taken from inverse Laplace transforms for the parabolic cylinder function and the Kummer confluent hypergeometric function, respectively. The inverse Laplace transforms that will be used for $\Phi(a;b;z)$ and $D_\nu(z)$ are not both defined over the half–line $(0,\infty)$. As a result, the convolution theorem becomes somewhat more involved. The Laplace transforms of the original functions $f_1(t)$ and $f_2(t)$ are defined as

$$\overline{f}_1(p) = \int_{\alpha_1}^{\beta_1} \exp(-pt) f_1(t)\, dt \qquad \beta_1 > \alpha_1$$

$$\overline{f}_2(p) = \int_{\alpha_2}^{\beta_2} \exp(-pt) f_2(t)\, dt \qquad \beta_2 > \alpha_2$$

where $\operatorname{Re} p > 0$. The convolution theorem then can be specified, see [25], as

$$\overline{f}_1(p)\overline{f}_2(p) = \int_{\alpha_1+\alpha_2}^{\beta_1+\beta_2} \exp(-pt) f_1(t) * f_2(t)\, dt \tag{2.4}$$

where $f_1(t) * f_2(t)$ is the convolution of $f_1(t)$ and $f_2(t)$ that is to be obtained from

$$f_1(t) * f_2(t) = \int_{\max(\alpha_1;\, t-\beta_2)}^{\min(\beta_1;\, t-\alpha_2)} f_1(\tau) f_2(t-\tau)\, d\tau \tag{2.5}$$

3. Inverse Laplace transforms for products of parabolic cylinder functions

This section derives several inverse Laplace transforms for products of parabolic cylinder functions in which the sign of the arguments may differ and utilizes these results to obtain new integral representations for products of (complementary) error functions.

Theorem 3.1. *Let* ν *and* μ *be two complex numbers with* $\operatorname{Re}\nu < 1$ *and* $\operatorname{Re}\mu < \min[1-\operatorname{Re}\nu, 2+\operatorname{Re}\nu]$. *Then, the following inverse Laplace transform holds for* $\operatorname{Re} p > 0$, $x > 0$, $|\arg y| < \pi$, $y > 0$

$$\begin{aligned} &p^{-1/2} \exp\left(\tfrac{1}{2}p(y-x)\right) D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right) \left\{D_\nu\left(-2^{1/2}x^{1/2}p^{1/2}\right) - D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\} \\ &= \frac{2^{(\mu-\nu)/2}\sqrt{\pi}}{\Gamma[1+(\nu-\mu)/2]\,\Gamma[-\nu]} \int_0^x \exp(-pt)\, t^{(\nu-\mu)/2} (x-t)^{-(1+\nu)/2} \\ &\quad \times (y+t)^{\mu/2} {}_2F_1\left(-\frac{\mu}{2}, \frac{1+\nu}{2}; 1+\frac{\nu-\mu}{2}; \frac{t(x-y-t)}{(x-t)(y+t)}\right) dt \\ &\quad + \frac{2^{2+(\mu+\nu)/2}\sqrt{\pi} y^{1/2} x^{1/2}}{\Gamma[-\mu/2]\,\Gamma[-\nu/2]} \int_x^\infty \exp(-pt)\, t^{(\nu-1)/2} (t-x)^{-(1+\mu+\nu)/2} \\ &\quad \times (y-x+t)^{(\mu-1)/2} {}_2F_1\left(\frac{1-\mu}{2}, \frac{1-\nu}{2}; \frac{3}{2}; \frac{xy}{t(y-x+t)}\right) dt \end{aligned} \tag{3.1}$$

where ${}_2F_1(a,b;c;z)$ denotes the Gaussian hypergeometric function

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad |z| < 1, $$

see Equation (1) on p. 430 in [24].

Proof. The inverse Laplace transform in Equation (5) on p. 290 in [14] is

$$ \Gamma[\nu] \exp\left(\tfrac{1}{2}ap\right) D_{-2\nu}\left(2^{1/2}a^{1/2}p^{1/2}\right) = \int_0^{\infty} \exp(-pt)\, 2^{-\nu} a^{1/2} t^{\nu-1} (t+a)^{-\nu-1/2}\, dt \quad (3.2) $$
$$ [\operatorname{Re} p > 0, \operatorname{Re}\nu > 0, |\arg a| < \pi] $$

and the inverse Laplace transform in Equation (3.33.2.2) in [26] is

$$ \exp(-xp)\, \Phi(a;b;xp) = \frac{x^{1-b}\Gamma[b]}{\Gamma[b-a]\Gamma[a]} \int_0^x \exp(-pt)\, t^{b-a-1} (x-t)^{a-1}\, dt \quad (3.3) $$
$$ [\operatorname{Re} p > 0, \operatorname{Re} b > \operatorname{Re} a > 0, x > 0] $$

These two inverse Laplace transforms, in the notation of Theorem 3.1, are rewritten as

$$ \Gamma[-\mu/2] \exp\left(\tfrac{1}{2}yp\right) D_{\mu}\left(2^{1/2}y^{1/2}p^{1/2}\right) = \int_0^{\infty} \exp(-pt)\, 2^{\mu/2} y^{1/2} t^{-\mu/2-1} (t+y)^{(\mu-1)/2}\, dt $$
$$ [\operatorname{Re} p > 0, \operatorname{Re}\mu < 0, |\arg y| < \pi] \quad (3.4) $$

and

$$ x^{1/2} \frac{2}{\sqrt{\pi}} \Gamma[1+\nu/2]\Gamma[(1-\nu)/2] \exp(-xp)\, \Phi\left(\frac{1-\nu}{2}; \frac{3}{2}; px\right) $$
$$ = \int_0^x \exp(-pt)\, t^{\nu/2} (x-t)^{-(1+\nu)/2}\, dt $$
$$ [\operatorname{Re} p > 0, -2 < \operatorname{Re}\nu < 1, x > 0] \quad (3.5) $$

The original functions $f_1(t)$ and $f_2(t)$ are taken from the inverse Laplace transforms (3.4) and (3.5), respectively, with

$$ f_1(t) = 2^{\mu/2} y^{1/2} t^{-\mu/2-1} (t+y)^{(\mu-1)/2} \text{ and } f_2(t) = t^{\nu/2} (x-t)^{-(1+\nu)/2} $$

The integration limits in (2.4) and (2.5) are $\beta_1 = \infty, \beta_2 = x$ and $\alpha_1 = \alpha_2 = 0$ such that the convolution integral is given by

$$ \begin{aligned} f_1(t) * f_2(t) &= \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau \qquad t < x \\ &= \int_{t-x}^t f_1(\tau) f_2(t-\tau)\, d\tau \qquad t > x \end{aligned} \quad (3.6) $$

First, the convolution integral for $t < x$ is

$$ f_1(t) * f_2(t) = \int_0^t 2^{\mu/2} y^{1/2} \tau^{-\mu/2-1} (\tau+y)^{(\mu-1)/2} (t-\tau)^{\nu/2} (x-(t-\tau))^{-(1+\nu)/2}\, d\tau $$

The substitution $\tau = tu$ allows to rewrite the integral as

$$ f_1(t) * f_2(t) = 2^{\mu/2} t^{(\nu-\mu)/2} y^{\mu/2} (x-t)^{-(1+\nu)/2} $$
$$ \times \int_0^1 u^{-\mu/2-1} \left(1 + \frac{t}{y}u\right)^{(\mu-1)/2} (1-u)^{\nu/2} \left(1 - \frac{t}{t-x}u\right)^{-(1+\nu)/2} du $$

The integral in the latter equation will be expressed in terms of the Appell hypergeometric function $F_1(a, b_1, b_2; c; z_1, z_2)$, which is defined as

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{z_1^m z_2^n}{m!n!} \qquad \max\{|z_1|, |z_2|\} < 1,$$

see Equation (1) on p. 448 in [24]. In particular, the following integral representation of the Appell hypergeometric function $F_1(a, b_1, b_2; c; z_1, z_2)$ will be used

$$\frac{\Gamma[a]\Gamma[c-a]}{\Gamma[c]} F_1(a, b_1, b_2; c; z_1, z_2) = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-z_1u)^{-b_1}(1-z_2u)^{-b_2}\,du$$

for $\operatorname{Re} c > \operatorname{Re} a > 0$, see Equation (5) on p. 231 in [12]. This gives

$$f_1(t) * f_2(t) = 2^{\mu/2} t^{(\nu-\mu)/2} y^{\mu/2}(x-t)^{-(1+\nu)/2} \frac{\Gamma[-\mu/2]\Gamma[1+(\nu/2)]}{\Gamma[1+(\nu-\mu)/2]} \times F_1\left(-\frac{\mu}{2}, \frac{1+\nu}{2}, \frac{1-\mu}{2}; 1+\frac{\nu-\mu}{2}; \frac{t}{t-x}, -\frac{t}{y}\right)$$

The above Appell hypergeometric function can further be simplified into the Gaussian hypergeometric function given

$$F_1(a, b_1, b_2; b_1+b_2; z_1, z_2) = (1-z_2)^{-a}\,{}_2F_1\left(a, b_1; b_1+b_2; \frac{z_1-z_2}{1-z_2}\right)$$

see Equation (1) on p. 238 in [12]. The final expression for the convolution integral for $t < x$ then is

$$f_1(t) * f_2(t) = 2^{\mu/2} t^{(\nu-\mu)/2}(x-t)^{-(1+\nu)/2}(y+t)^{\mu/2} \frac{\Gamma[-\mu/2]\Gamma[1+(\nu/2)]}{\Gamma[1+(\nu-\mu)/2]} \times {}_2F_1\left(-\frac{\mu}{2}, \frac{1+\nu}{2}; 1+\frac{\nu-\mu}{2}; \frac{t(t+y-x)}{(t-x)(y+t)}\right) \qquad t < x \tag{3.7}$$

Second, the convolution integral for $t > x$ is given by

$$f_1(t) * f_2(t) = \int_{t-x}^{t} 2^{\mu/2} y^{1/2} \tau^{-\mu/2-1}(\tau+y)^{(\mu-1)/2}(t-\tau)^{\nu/2}(x-(t-\tau))^{-(1+\nu)/2}\,d\tau$$

The treatment of this convolution integral is similar to that of the integral for $t < x$ such that only the main steps are mentioned. The substitutions $\tau = s - x + t$ and $s = xu$ express the integral in terms of the Appell hypergeometric function $F_1(a, b_1, b_2; c; z_1, z_2)$ that again can be simplified into the Gaussian hypergeometric function. The convolution integral for $t > x$ then is given by

$$f_1(t) * f_2(t) = \frac{1}{\sqrt{\pi}} x^{1/2} y^{1/2} 2^{1+(\mu/2)}(t-x)^{-(1+\mu+\nu)/2}(y+t-x)^{(\mu-1)/2} t^{(\nu-1)/2} \times \Gamma[(1-\nu)/2]\Gamma[1+(\nu/2)]\,{}_2F_1\left(\frac{1-\mu}{2}, \frac{1-\nu}{2}; \frac{3}{2}; \frac{xy}{t(t+y-x)}\right) \qquad t > x \tag{3.8}$$

of which the derivation also used the following linear transformation formula

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

see Equation (15.3.4) in [1].

Plugging (3.7) and (3.8) into the convolution integral (3.6) then gives

$$\exp\left(\tfrac{1}{2}py - px\right) D_\mu\left(2^{1/2} y^{1/2} p^{1/2}\right) \Phi\left(\frac{1-\nu}{2}; \frac{3}{2}; px\right) \tag{3.9}$$

$$
\begin{aligned}
&= \frac{2^{(\mu/2)-1}\sqrt{\pi}x^{-1/2}}{\Gamma\left[1+(\nu-\mu)/2\right]\Gamma\left[(1-\nu)/2\right]}\int_0^x \exp(-pt)\, t^{(\nu-\mu)/2}(x-t)^{-(1+\nu)/2} \\
&\quad\times (y+t)^{\mu/2}\, {}_2F_1\left(-\frac{\mu}{2},\frac{1+\nu}{2};1+\frac{\nu-\mu}{2};\frac{t(x-y-t)}{(x-t)(y+t)}\right)dt \\
&\quad+\frac{2^{\mu/2}y^{1/2}}{\Gamma\left[-\mu/2\right]}\int_x^\infty \exp(-pt)\, t^{(\nu-1)/2}(t-x)^{-(1+\mu+\nu)/2} \\
&\quad\times (y-x+t)^{(\mu-1)/2}\, {}_2F_1\left(\frac{1-\mu}{2},\frac{1-\nu}{2};\frac{3}{2};\frac{xy}{t(y-x+t)}\right)dt
\end{aligned}
$$

in which the recurrence and duplication formulas of the gamma function were employed to simplify expressions given that

$$
\Gamma\left[1+z\right]=z\Gamma\left[z\right],\quad \Gamma\left[2z\right]=\frac{1}{\sqrt{2\pi}}2^{2z-\frac{1}{2}}\Gamma\left[z\right]\Gamma\left[z+\tfrac{1}{2}\right],
$$

see Equations (6.1.15) and (6.1.18) in [1].

Finally, plugging the definition (2.2) into (3.9) and simplifying gives the inverse Laplace transform (3.1). □

The parabolic cylinder function specializes into the complementary error function when its order is at -1. The inverse Laplace transform (3.1) thus can be used to obtain an integral representation for the product of complementary error functions. However, this result will not be shown here as its integrand contains an inverse trigonometric function rather than the rational functions that are typical for existing integral representations, see for instance [16, 19]. Instead, the term $p^{-1/2}$ in inverse Laplace transforms such as (3.1) will be removed given that the resulting relations yield integrands in which such rational functions emerge. This will be illustrated in Theorem 3.2 and Corollary 3.3.

Theorem 3.2. *Let* ν *and* μ *be two complex numbers with* $\operatorname{Re}\nu<1$ *and* $\operatorname{Re}\mu<\min\left[1-\operatorname{Re}\nu,\ 2+\operatorname{Re}\nu\right]$. *Then, the following inverse Laplace transform holds for* $\operatorname{Re}p>0$, $x>0$, $|\arg y|<\pi$, $y>0$

$$
\begin{aligned}
&\exp\left(\tfrac{1}{2}p(y-x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)\left\{D_\nu\left(-2^{1/2}x^{1/2}p^{1/2}\right)-D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\} \qquad (3.10)\\
&= \frac{2^{(\mu-\nu)/2}\sqrt{\pi}y^{-1/2}}{\Gamma\left[(1-\mu+\nu)/2\right]\Gamma\left[-\nu\right]}\int_0^x \exp(-pt)\, t^{-(1+\mu-\nu)/2}(x-t)^{-(1+\nu)/2} \\
&\quad\times (y+t)^{(1+\mu)/2}\left\{{}_2F_1\left(-\frac{1+\mu}{2},\frac{1+\nu}{2};\frac{1-\mu+\nu}{2};\frac{t(x-y-t)}{(x-t)(y+t)}\right)\right. \\
&\quad\left.+\frac{\mu t}{(1-\mu+\nu)(y+t)}{}_2F_1\left(\frac{1-\mu}{2},\frac{1+\nu}{2};\frac{3-\mu+\nu}{2};\frac{t(x-y-t)}{(x-t)(y+t)}\right)\right\}dt \\
&\quad+\frac{2^{(4+\mu+\nu)/2}\sqrt{\pi}x^{1/2}}{\Gamma\left[-(1+\mu)/2\right]\Gamma\left[-\nu/2\right]}\int_x^\infty \exp(-pt)\, t^{(\nu-1)/2}(t-x)^{-(2+\mu+\nu)/2} \\
&\quad\times (y-x+t)^{\mu/2}\left\{{}_2F_1\left(-\frac{\mu}{2},\frac{1-\nu}{2};\frac{3}{2};\frac{xy}{t(y-x+t)}\right)\right. \\
&\quad\left.-\frac{\mu(t-x)}{(1+\mu)(y-x+t)}{}_2F_1\left(\frac{2-\mu}{2},\frac{1-\nu}{2};\frac{3}{2};\frac{xy}{t(y-x+t)}\right)\right\}dt
\end{aligned}
$$

Proof. The recurrence relation of the parabolic cylinder function is given by

$$
zD_\mu(z)=D_{\mu+1}(z)+\mu D_{\mu-1}(z)
$$

see Equation (14) on p. 119 in [13]. Replacing z by $2^{1/2}y^{1/2}p^{1/2}$ and multiplying by $p^{-1/2}\exp\left(\frac{1}{2}p(y-x)\right)\left\{D_\nu\left(-2^{1/2}x^{1/2}p^{1/2}\right)-D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\}$ gives

$$
2^{1/2}y^{1/2}\exp\left(\tfrac{1}{2}p(y-x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)\left\{D_\nu\left(-2^{1/2}x^{1/2}p^{1/2}\right)-D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\}
$$

$$= p^{-1/2}\exp\left(\tfrac{1}{2}p\left(y-x\right)\right)D_{\mu+1}\left(2^{1/2}y^{1/2}p^{1/2}\right)\left\{D_{\nu}\left(-2^{1/2}x^{1/2}p^{1/2}\right)\right. \tag{3.11}$$
$$\left.-D_{\nu}\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\}+\mu p^{-1/2}\exp\left(\tfrac{1}{2}p\left(y-x\right)\right)D_{\mu-1}\left(2^{1/2}y^{1/2}p^{1/2}\right)$$
$$\times\left\{D_{\nu}\left(-2^{1/2}x^{1/2}p^{1/2}\right)-D_{\nu}\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\}$$

Plugging the transform (3.1) into (3.11) and simplifying gives (3.10). □

Corollary 3.3. *The relation between the parabolic cylinder function and the complementary error function is given by*

$$D_{-1}\left(z\right)=\sqrt{\frac{\pi}{2}}\exp\left(\frac{z^{2}}{4}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)$$

see Equation (9.254.1) in [16] in which erfc(z) denotes the complementary error function. Equations (E.3c) and (E.3d) in [3] specify the following relations between the error function and its complement

$$\operatorname{erfc}\left(z\right)+\operatorname{erf}\left(z\right)=1$$
$$\operatorname{erfc}\left(-z\right)=1+\operatorname{erf}\left(z\right)$$

and thus

$$\operatorname{erfc}\left(-z\right)-\operatorname{erfc}\left(z\right)=2\operatorname{erf}\left(z\right) \tag{3.12}$$

where erf(z) denotes the error function. The below derivations also use the following properties of the Gaussian hypergeometric function

$${}_2F_1\left(0,b;c;z\right)={}_2F_1\left(a,0;c;z\right)=1$$
$${}_2F_1\left(1,\frac{3}{2};\frac{3}{2};z\right)=\frac{1}{1-z}$$

see Equations (15.1.1) and (15.1.8) in [1]. Plugging the transform (3.1) into (3.11), using $\mu=\nu=-1$ and (3.12) gives the following inverse Laplace transform for the product of two (complementary) error functions

$$\exp\left(py\right)\ \operatorname{erfc}\left(y^{1/2}p^{1/2}\right)\ \operatorname{erf}\left(x^{1/2}p^{1/2}\right)= \tag{3.13}$$
$$\frac{1}{\pi}\int_0^x\exp\left(-pt\right)\frac{\sqrt{y}}{\sqrt{t}\left(y+t\right)}dt-\frac{1}{\pi}\int_x^\infty\exp\left(-pt\right)\frac{\sqrt{x}}{\sqrt{y-x+t}\left(y+t\right)}dt$$
$$\left[\operatorname{Re}p>0,\left|\arg y\right|<\pi,y>0,\left|\arg x\right|<\pi,x\geqslant 0\right]$$

Using $p=1$ and setting a and b at $y^{1/2}$ and $x^{1/2}$, respectively, then gives the following integral representation

$$\operatorname{erfc}\left(a\right)\ \operatorname{erf}\left(b\right)= \tag{3.14}$$
$$\frac{a\exp\left(-a^{2}\right)}{\pi}\int_0^{b^{2}}\frac{\exp\left(-t\right)}{\left(t+a^{2}\right)\sqrt{t}}dt-\frac{b\exp\left(-\left(a^{2}+b^{2}\right)\right)}{\pi}\int_0^\infty\frac{\exp\left(-t\right)}{\left(t+a^{2}+b^{2}\right)\sqrt{t+a^{2}}}dt$$
$$\left[\operatorname{Re}a>0,\operatorname{Re}b\geqslant 0\right]$$

which is not present in, for instance, the extensive overview in [19].

Theorem 3.4. *Let ν and μ be two complex numbers with $\operatorname{Re}\nu<1$ and $\operatorname{Re}\mu<\min\left[1-\operatorname{Re}\nu,\ 2+\operatorname{Re}\nu\right]$. Then, the following inverse Laplace transform holds for $\operatorname{Re}p>0$, $\left|\arg x\right|<\pi$, $x\geqslant 0$, $\left|\arg y\right|<\pi$, $y>0$*

$$p^{-1/2}\exp\left(\tfrac{1}{2}p\left(y-x\right)\right)D_{\mu}\left(2^{1/2}y^{1/2}p^{1/2}\right)\left\{D_{\nu}\left(-2^{1/2}x^{1/2}p^{1/2}\right)+D_{\nu}\left(2^{1/2}x^{1/2}p^{1/2}\right)\right\}$$
$$=\frac{2^{(\mu-\nu)/2}\sqrt{\pi}}{\Gamma\left[1+\left(\nu-\mu\right)/2\right]\Gamma\left[-\nu\right]}\int_0^x\exp\left(-pt\right)t^{(\nu-\mu)/2}\left(x-t\right)^{-(1+\nu)/2} \tag{3.15}$$

$$\times (y+t)^{\mu/2} {}_2F_1\left(-\frac{\mu}{2}, \frac{1+\nu}{2}; 1+\frac{\nu-\mu}{2}; \frac{t(x-y-t)}{(x-t)(y+t)}\right) dt$$
$$+ \frac{2^{1+(\mu+\nu)/2}\sqrt{\pi}}{\Gamma[(1-\nu)/2]\Gamma[(1-\mu)/2]} \int_x^\infty \exp(-pt)\, t^{\nu/2} (t-x)^{-(1+\mu+\nu)/2}$$
$$\times (y-x+t)^{\mu/2} {}_2F_1\left(-\frac{\mu}{2}, -\frac{\nu}{2}; \frac{1}{2}; \frac{xy}{t(y-x+t)}\right) dt$$

Proof. The inverse Laplace transform in Equation (6) on p. 290 in [14] is

$$\Gamma[\nu]\, p^{-1/2} \exp\left(\tfrac{1}{2}ap\right) D_{1-2\nu}\left(2^{1/2}a^{1/2}p^{1/2}\right) = \int_0^\infty \exp(-pt)\, 2^{1/2-\nu} t^{\nu-1} (t+a)^{1/2-\nu}\, dt$$
$$[\operatorname{Re} p > 0, \operatorname{Re} \nu > 0, |\arg a| < \pi]$$

which in the notation of Theorem 3.3 gives

$$\Gamma[(1-\mu)/2]\, p^{-1/2} \exp\left(\tfrac{1}{2}yp\right) D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)$$
$$= \int_0^\infty \exp(-pt)\, 2^{\mu/2} t^{-(\mu+1)/2} (t+y)^{\mu/2}\, dt$$
$$[\operatorname{Re} p > 0, \operatorname{Re} \mu < 1, |\arg y| < \pi] \tag{3.16}$$

The inverse Laplace transform (3.3) is specialized for $a = -\frac{\nu}{2}$ and $b = \frac{1}{2}$ and gives

$$\frac{x^{-1/2}}{\sqrt{\pi}} \Gamma[(1+\nu)/2]\, \Gamma[-\nu/2] \exp(-xp)\, \Phi\left(-\frac{\nu}{2}; \frac{1}{2}; xp\right)$$
$$= \int_0^x \exp(-pt)\, t^{(\nu-1)/2} (x-t)^{-(\nu/2)-1}\, dt$$
$$[\operatorname{Re} p > 0, -1 < \operatorname{Re} \nu < 0, x > 0] \tag{3.17}$$

The original functions $f_1(t)$ and $f_2(t)$ are taken from the inverse Laplace transforms (3.16) and (3.17), respectively

$$f_1(t) = 2^{\mu/2} t^{-(\mu+1)/2} (t+y)^{\mu/2} \text{ and } f_2(t) = t^{(\nu-1)/2} (x-t)^{-(\nu/2)-1}$$

Using steps akin to those used in the proof of Theorem 3.1 then yields

$$p^{-1/2} \exp\left(\tfrac{1}{2}py - px\right) D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right) \Phi\left(-\frac{\nu}{2}; \frac{1}{2}; px\right) \tag{3.18}$$
$$= \frac{2^{\mu/2}\sqrt{\pi}x^{1/2}y^{1/2}}{\Gamma[1+(\nu-\mu)/2]\,\Gamma[-\nu/2]} \int_0^x \exp(-pt)\, t^{(\nu-\mu)/2} (x-t)^{-1-(\nu/2)}$$
$$\times (y+t)^{(\mu-1)/2} {}_2F_1\left(\frac{1-\mu}{2}, 1+\frac{\nu}{2}; 1+\frac{\nu-\mu}{2}; \frac{t(x-y-t)}{(x-t)(y+t)}\right) dt$$
$$+ \frac{2^{\mu/2}}{\Gamma[(1-\mu)/2]} \int_x^\infty \exp(-pt)\, t^{\nu/2} (t-x)^{-(1+\mu+\nu)/2}$$
$$\times (y-x+t)^{\mu/2} {}_2F_1\left(-\frac{\mu}{2}, -\frac{\nu}{2}; \frac{1}{2}; \frac{xy}{t(y-x+t)}\right) dt$$

The first integral in (3.18) can be rewritten via the following linear transformation formula for the Gaussian hypergeometric function

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z) \tag{3.19}$$

see Equation (15.3.3) in [1]. Combining the resulting expression for the transform (3.18) with the definition (2.3) then gives the inverse Laplace transform (3.15). □

Theorem 3.5 specifies the inverse Laplace transform for the product of two parabolic cylinder functions of which the arguments have opposite sign and Corollary 3.6 specializes this expression for a single parabolic cylinder function with negative sign in the argument.

Theorem 3.5. *Let ν and μ be two complex numbers with $\operatorname{Re}\nu < 1$ and $\operatorname{Re}\mu < \min[1-\operatorname{Re}\nu, 2+\operatorname{Re}\nu]$. Then, the following inverse Laplace transform holds for $\operatorname{Re}p > 0$, $x > 0$, $|\arg y| < \pi$, $y > 0$*

$$p^{-1/2}\exp\left(\tfrac{1}{2}p\,(y-x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_\nu\left(-2^{1/2}x^{1/2}p^{1/2}\right) \tag{3.20}$$
$$=\frac{2^{(\mu-\nu)/2}\sqrt{\pi}}{\Gamma[1+(\nu-\mu)/2]\,\Gamma[-\nu]}\int_0^x \exp(-pt)\,t^{(\nu-\mu)/2}(x-t)^{-(1+\nu)/2}$$
$$\times(y+t)^{\mu/2}\,{}_2F_1\left(-\frac{\mu}{2},\frac{1+\nu}{2};1+\frac{\nu-\mu}{2};\frac{t\,(x-y-t)}{(x-t)\,(y+t)}\right)dt$$
$$+\frac{2^{1+(\mu+\nu)/2}\sqrt{\pi}x^{1/2}y^{1/2}}{\Gamma[-\mu/2]\,\Gamma[-\nu/2]}\int_x^\infty \exp(-pt)\,t^{(\nu-1)/2}(t-x)^{-(1+\mu+\nu)/2}$$
$$\times(y-x+t)^{(\mu-1)/2}\left\{{}_2F_1\left(\frac{1-\mu}{2},\frac{1-\nu}{2};\frac{3}{2};\frac{xy}{t\,(y-x+t)}\right)\right.$$
$$\left.+\frac{\Gamma[-\mu/2]\,\Gamma[-\nu/2]}{\Gamma[(1-\mu)/2]\,\Gamma[(1-\nu)/2]}\left(\frac{t\,(y-x+t)}{4xy}\right)^{1/2}{}_2F_1\left(-\frac{\mu}{2},-\frac{\nu}{2};\frac{1}{2};\frac{xy}{t\,(y-x+t)}\right)\right\}dt$$

Proof. The transform (3.20) is obtained by adding the inverse Laplace transforms (3.1) and (3.15) and simplifying the resulting expression. □

Corollary 3.6. *Using $y = 0$, the properties*

$$D_\mu(0)=\frac{2^{\mu/2}\sqrt{\pi}}{\Gamma[(1-\mu)/2]}$$
$${}_2F_1(a,b;c;1)=\frac{\Gamma[c]\,\Gamma[c-a-b]}{\Gamma[c-a]\,\Gamma[c-b]}$$

see Equations (46:7:1) in [22] and (15.1.20) in [1], and $\mu = 0$ gives

$$p^{-1/2}\exp\left(-\tfrac{1}{2}px\right)D_v\left(-2^{1/2}x^{1/2}p^{1/2}\right)= \tag{3.21}$$
$$\frac{2^{-\nu/2}\sqrt{\pi}}{\Gamma[-\nu]\,\Gamma[1+\nu/2]}\int_0^x \exp(-pt)\,t^{\nu/2}(x-t)^{-(1+\nu)/2}\,dt$$
$$+\frac{2^{\nu/2}}{\Gamma[(1-\nu)/2]}\int_x^\infty \exp(-pt)\,t^{\nu/2}(t-x)^{-(1+\nu)/2}\,dt$$
$$[\operatorname{Re}p>0,\operatorname{Re}\nu<1,x>0]$$

Theorem 3.7. *Let ν and μ be two complex numbers with $\operatorname{Re}(\nu+\mu) < 1$. Then, the following inverse Laplace transform holds for $\operatorname{Re}p > 0$, $|\arg x| < \pi$, $x \geqslant 0$, $|\arg y| < \pi$, $y \geqslant 0$, $|\arg x + \arg y| < \pi$*

$$p^{-1/2}\exp\left(\tfrac{1}{2}p\,(y+x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)= \tag{3.22}$$
$$\frac{2^{(\mu+\nu)/2}}{\Gamma[(1-\mu-\nu)/2]}\int_0^\infty \exp(-pt)\,t^{-(1+\mu+\nu)/2}(y+t)^{\mu/2}(x+t)^{\nu/2}$$
$$\times{}_2F_1\left(-\frac{\mu}{2},-\frac{\nu}{2};\frac{1-\mu-\nu}{2};\frac{t\,(x+y+t)}{(x+t)\,(y+t)}\right)dt$$

which is identical to the transform in Equation (2.1) in [28].

Proof. Subtracting the inverse Laplace transform (3.1) from (3.10) gives

$$p^{-1/2}\exp\left(\tfrac{1}{2}p\,(y-x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)=$$
$$+\frac{2^{(\mu+\nu)/2}}{\Gamma[(1-\mu-\nu)/2]}\int_x^\infty \exp(-pt)\,t^{\nu/2}(t-x)^{-(1+\mu+\nu)/2}$$

$$\times (y-x+t)^{\mu/2}\left\{\frac{\sqrt{\pi}\Gamma\left[(1-\mu-\nu)/2\right]}{\Gamma\left[(1-\mu)/2\right]\Gamma\left[(1-\nu)/2\right]}{}_2F_1\left(-\frac{\mu}{2},-\frac{\nu}{2};\frac{1}{2};\frac{xy}{t(y-x+t)}\right)\right.$$
$$\left.-\frac{\sqrt{\pi}\Gamma\left[(1-\mu-\nu)/2\right]}{\Gamma\left[-\mu/2\right]\Gamma\left[-\nu/2\right]}\left(\frac{4xy}{t(y-x+t)}\right)^{1/2}{}_2F_1\left(\frac{1-\mu}{2},\frac{1-\nu}{2};\frac{3}{2};\frac{xy}{t(y-x+t)}\right)\right\}dt$$

in which the linear transformation formula (3.19) was used. Subsequently, using the linear transformation formula

$${}_2F_1(a,b;c;z)=\frac{\Gamma[c]\Gamma[c-a-b]}{\Gamma[c-a]\Gamma[c-b]}{}_2F_1(a,b;a+b-c+1;1-z)$$
$$+(1-z)^{c-a-b}\frac{\Gamma[c]\Gamma[a+b-c]}{\Gamma[a]\Gamma[b]}{}_2F_1(c-a,c-b;c-a-b+1;1-z)$$

in Equation (15.3.6) in [1] gives

$$p^{-1/2}\exp\left(\tfrac{1}{2}p(y-x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)=$$
$$\frac{2^{(\nu+\mu)/2}}{\Gamma\left[(1-\mu-\nu]/2\right)}\int_x^\infty\exp(-pt)\,t^{\nu/2}(t-x)^{-(1+\mu+\nu)/2}(y-x+t)^{\mu/2}$$
$$\times{}_2F_1\left(-\frac{\mu}{2},-\frac{\nu}{2};\frac{1-\mu-\nu}{2};\frac{(t-x)(y+t)}{t(y-x+t)}\right)dt$$

Multiplying both sides by $\exp(px)$, using the substitution $s=t-x$ and subsequently re-introducing t then gives (3.22). □

As noted earlier, removing the term $p^{-1/2}$ from transforms such as (3.22) allows obtaining integral representations for (complementary) error functions in which the integrand contains rational functions. This is illustrated in Theorem 3.8 and Corollary 3.9 in which the integral representation for $1-\mathrm{erf}(a)^2$ in [19] is generalized into $1-\mathrm{erf}(a)\mathrm{erf}(b)$.

Theorem 3.8. *Let ν and μ be two complex numbers with $\mathrm{Re}(\nu+\mu)<1$. Then, the following inverse Laplace transform holds for $\mathrm{Re}\,p>0$, $|\arg x|<\pi$, $x>0$, $|\arg y|<\pi$, $y>0$, $|\arg x+\arg y|<\pi$*

$$\exp\left(\tfrac{1}{2}p(y+x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)= \tag{3.23}$$
$$\frac{2^{(\mu+\nu)/2}x^{-1/2}}{\Gamma\left[-(\mu+\nu)/2\right]}\int_0^\infty\exp(-pt)\,t^{-1-(\nu+\mu)/2}(y+t)^{\mu/2}$$
$$\times(x+t)^{(1+\nu)/2}\left\{{}_2F_1\left(-\frac{\mu}{2},-\frac{1+\nu}{2};-\frac{\mu+\nu}{2};\frac{t(x+y+t)}{(x+t)(y+t)}\right)\right.$$
$$\left.-\frac{\nu t}{(\mu+\nu)(x+t)}{}_2F_1\left(-\frac{\mu}{2},\frac{1-\nu}{2};1-\frac{\mu+\nu}{2};\frac{t(x+y+t)}{(x+t)(y+t)}\right)\right\}dt$$

Proof. The inverse Laplace transform (3.23) is obtained via the above recurrence relation of the parabolic cylinder function. Replacing z by $2^{1/2}x^{1/2}p^{1/2}$ in the recurrence relation and multiplying by $p^{-1/2}\exp\left(\frac{1}{2}p(y+x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)$ gives

$$\exp\left(\tfrac{1}{2}p(y+x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)=$$
$$2^{-1/2}x^{-1/2}p^{-1/2}\exp\left(\tfrac{1}{2}p(y+x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_{\nu+1}\left(2^{1/2}x^{1/2}p^{1/2}\right)$$
$$+\nu 2^{-1/2}x^{-1/2}p^{-1/2}\exp\left(\tfrac{1}{2}p(y+x)\right)D_\mu\left(2^{1/2}y^{1/2}p^{1/2}\right)D_{\nu-1}\left(2^{1/2}x^{1/2}p^{1/2}\right)$$

Plugging the transform (3.22) into the latter expression and simplifying the result via the linear transformation formula (3.19) gives (3.23). □

Corollary 3.9. *The below derivations employ the following property of the Gaussian hypergeometric function*

$$ {}_2F_1\left(1,\frac{1}{2};2;z\right) = {}_2F_1\left(\frac{1}{2},1;2;z\right) = \frac{2}{1+\sqrt{1-z}} $$

see Equation (84) on p. 473 in [24]*. Using* $\mu = \nu = -1$ *in (3.23) gives the following inverse Laplace transform for the product of two complementary error functions*

$$ \exp\left(p\left(x+y\right)\right)\ \operatorname{erfc}\left(y^{1/2}p^{1/2}\right)\ \operatorname{erfc}\left(x^{1/2}p^{1/2}\right) = \tag{3.24} $$

$$ \frac{1}{\pi}\int_0^\infty \exp\left(-pt\right)\frac{\sqrt{x}\sqrt{x+t}+\sqrt{y}\sqrt{y+t}}{\left(x+y+t\right)\sqrt{\left(x+t\right)\left(y+t\right)}}dt $$

$$ [\operatorname{Re}p>0, |\arg y|<\pi, y\geqslant 0, |\arg x|<\pi, x\geqslant 0, |\arg x+\arg y|<\pi] $$

Using $p=1$*,* $y^{1/2}=a$ *and* $x^{1/2}=b$ *then gives the following integral representation for the product of two complementary error functions*

$$ \operatorname{erfc}\left(a\right)\ \operatorname{erfc}\left(b\right) = \tag{3.25} $$

$$ \frac{1}{\pi}\exp\left(-\left(a^2+b^2\right)\right)\int_0^\infty \exp\left(-t\right)\frac{a\sqrt{t+a^2}+b\sqrt{t+b^2}}{\left(t+a^2+b^2\right)\sqrt{\left(t+a^2\right)\left(t+b^2\right)}}dt $$

$$ [\operatorname{Re}a>0, \operatorname{Re}b>0] $$

which gives an alternative to the representation given on p. 70 in [27]*. Using* $a=0$ *and* $\operatorname{erfc}(0)=1$*, see Equation (40:7) in* [22]*, gives*

$$ \operatorname{erfc}\left(b\right) = \frac{b}{\pi}\exp\left(-b^2\right)\int_0^\infty \frac{\exp\left(-t\right)}{\left(t+b^2\right)\sqrt{t}}dt $$

$$ [\operatorname{Re}b>0] $$

$$ \operatorname{erf}\left(b\right) = 1-\frac{b}{\pi}\exp\left(-b^2\right)\int_0^\infty \frac{\exp\left(-t\right)}{\left(t+b^2\right)\sqrt{t}}dt \tag{3.26} $$

$$ [\operatorname{Re}b>0] $$

The definition of the complementary error function gives $\operatorname{erf}(a)\operatorname{erf}(b) = \operatorname{erf}(b) - \operatorname{erfc}(a)\operatorname{erf}(b)$ *such that plugging (3.26) and (3.14) into the latter relation gives*

$$ 1-\operatorname{erf}(a)\operatorname{erf}(b) = \tag{3.27} $$

$$ \frac{b}{\pi}\exp\left(-b^2\right)\int_0^\infty \exp\left(-t\right)\left\{\frac{1}{\left(t+b^2\right)\sqrt{t}} - \frac{\exp\left(-a^2\right)}{\left(t+a^2+b^2\right)\sqrt{t+a^2}}\right\}dt $$

$$ +\frac{a}{\pi}\exp\left(-a^2\right)\int_0^{b^2}\frac{\exp\left(-t\right)}{\left(t+a^2\right)\sqrt{t}}dt $$

$$ [\operatorname{Re}a>0, \operatorname{Re}b>0] $$

which generalizes the expression for $1-\operatorname{erf}(a)^2$ *in Equation (8) on p. 4 in* [19] *to differing arguments. Note that the representation in* [19] *can easily be obtained from (3.27) by using* $a=b$ *which gives*

$$ 1-\operatorname{erf}(a)^2 = \frac{2a}{\pi}\exp\left(-a^2\right)\int_0^{a^2}\frac{\exp\left(-t\right)}{\left(t+a^2\right)\sqrt{t}}dt $$

The substitution $t=a^2s^2$ *then gives*

$$ 1-\operatorname{erf}(a)^2 = \frac{4}{\pi}\exp\left(-a^2\right)\int_0^1\frac{\exp\left(-a^2s^2\right)}{\left(s^2+1\right)}ds $$

which is the integral representation in [19].

4. Correcting two inverse Laplace transforms

This Section utilizes the above results to correct two inverse Laplace transforms that are frequently found.

4.1. First correction

The following inverse Laplace transform is specified in Equation (3.11.4.3) in [26]

$$D_\nu\left(a\sqrt{p}\right) D_{-\nu-1}\left(a\sqrt{p}\right) = \int_a^\infty \exp\left(-pt\right) \frac{\left(t^2-a^2\right)^{-1/2}}{\sqrt{2t}} \cos\left[\left(\nu+\frac{1}{2}\right)\arccos\left[\frac{a^2}{2t}\right]\right] dt \quad **$$

where ∗∗ indicates that the expression is not correct. The corrected expression, however, can easily be obtained from the results in Section 3.

Theorem 4.1. *Let ν be a complex number. Then, the following inverse Laplace transform holds for* $\operatorname{Re} p > 0$ *and* $\operatorname{Re} a > 0$

$$D_\nu\left(a\sqrt{p}\right) D_{-\nu-1}\left(a\sqrt{p}\right) = \int_{\frac{1}{2}a^2}^\infty \exp\left(-pt\right) \frac{a\left(t^2-\frac{a^4}{4}\right)^{-1/2}}{\sqrt{2\pi t}} \cos\left[\left(2\nu+1\right)\arcsin\left[\sqrt{\frac{2t-a^2}{4t}}\right]\right] dt \tag{4.1}$$

Proof. Using $a = 2^{1/2}x^{1/2} = 2^{1/2}y^{1/2}$ and $\mu = -\nu - 1$ allows to rewrite (3.23) as follows

$$\exp\left(\tfrac{1}{2}a^2p\right) D_\nu\left(a\sqrt{p}\right) D_{-\nu-1}\left(a\sqrt{p}\right) = \frac{1}{a\sqrt{\pi}}\int_0^\infty \exp\left(-pt\right) t^{-1/2} \left\{ {}_2F_1\left(-\frac{1+\nu}{2}, \frac{1+\nu}{2}; \frac{1}{2}; \frac{4t\left(a^2+t\right)}{\left(a^2+2t\right)^2}\right) + \frac{2\nu t}{a^2+2t} {}_2F_1\left(\frac{1-\nu}{2}, \frac{1+\nu}{2}; \frac{3}{2}; \frac{4t\left(a^2+t\right)}{\left(a^2+2t\right)^2}\right)\right\} dt$$

Multiplying both sides by $\exp\left(-\frac{1}{2}a^2p\right)$, using the substitution $s = t + \frac{1}{2}a^2$ and subsequently re-introducing t gives

$$D_\nu\left(a\sqrt{p}\right) D_{-\nu-1}\left(a\sqrt{p}\right) = \frac{2^{1/2}}{a\sqrt{\pi}}\int_{\frac{1}{2}a^2}^\infty \exp\left(-pt\right) \left(2t-a^2\right)^{-1/2} \left\{ {}_2F_1\left(-\frac{1+\nu}{2}, \frac{1+\nu}{2}; \frac{1}{2}; \frac{4t^2-a^4}{4t^2}\right) + \frac{\nu\left(2t-a^2\right)}{2t} {}_2F_1\left(\frac{1-\nu}{2}, \frac{1+\nu}{2}; \frac{3}{2}; \frac{4t^2-a^4}{4t^2}\right)\right\} dt$$

The quadratic transformation formula in Equation (15.3.22) in [1] states

$${}_2F_1\left(a, b; a+b+\frac{1}{2}; z\right) = {}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; \frac{1}{2} - \frac{1}{2}\sqrt{1-z}\right)$$

Using the latter relation gives

$$D_\nu\left(a\sqrt{p}\right) D_{-\nu-1}\left(a\sqrt{p}\right) = \frac{2^{1/2}}{a\sqrt{\pi}}\int_{\frac{1}{2}a^2}^\infty \exp\left(-pt\right) \left(2t-a^2\right)^{-1/2} \left\{ {}_2F_1\left(-1-\nu, 1+\nu; \frac{1}{2}; \frac{2t-a^2}{4t}\right) + \frac{\nu\left(2t-a^2\right)}{2t} {}_2F_1\left(1-\nu, 1+\nu; \frac{3}{2}; \frac{2t-a^2}{4t}\right)\right\} dt$$

The latter result can be simplified on the basis of the relations (15.2.10) and (15.2.20) in [1], respectively

$$(c-a)\,{}_2F_1\left(a-1,b;c;z\right)+\left(2a-c-az+bz\right){}_2F_1\left(a,b;c;z\right)$$
$$+\,a\left(z-1\right){}_2F_1\left(a+1,b;c;z\right)=0$$
$$c\left(1-z\right){}_2F_1\left(a,b;c;z\right)-c{}_2F_1\left(a-1,b;c;z\right)+\left(c-b\right)z{}_2F_1\left(a,b;c+1;z\right)=0$$

The latter two relations can be combined into

$$\left(ac-c^2\right){}_2F_1\left(a-1,b;c;z\right)+\left(c^2-ac+c\left(a-b\right)z\right){}_2F_1\left(a,b;c;z\right)$$
$$+\,a\left(b-c\right)z{}_2F_1\left(a+1,b;c+1;z\right)=0$$

which gives

$$\frac{a^2}{2t}{}_2F_1\left(1+\nu,-\nu;\frac{1}{2};\frac{2t-a^2}{4t}\right)={}_2F_1\left(-1-\nu,1+\nu;\frac{1}{2};\frac{2t-a^2}{4t}\right)$$
$$+\,\frac{\nu\left(2t-a^2\right)}{2t}{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{2t-a^2}{4t}\right)$$

This allows to rewrite the inverse Laplace transform as

$$D_\nu\left(a\sqrt{p}\right)D_{-\nu-1}\left(a\sqrt{p}\right)=$$
$$\frac{a}{\sqrt{2\pi}}\int_{\frac{1}{2}a^2}^{\infty}\exp\left(-pt\right)\frac{\left(2t-a^2\right)^{-1/2}}{t}{}_2F_1\left(1+\nu,-\nu;\frac{1}{2};\frac{2t-a^2}{4t}\right)dt$$

Equation (90) on p. 460 in [24] states

$${}_2F_1\left(a,1-a;\frac{1}{2};z\right)={}_2F_1\left(1-a,a;\frac{1}{2};z\right)=\frac{1}{\sqrt{1-z}}\cos\left[\left(2a-1\right)\arcsin\left[\sqrt{z}\right]\right]$$

Employing the latter property then gives (4.1). □

4.2. Second correction

The following inverse Laplace transform can be found in Equation (11) on p. 218 in [14], in Equation (16.7) on p. 379 in [21] and in Equation (3.11.5.1) in [26]

$$\exp\left(\tfrac{1}{4}a^2p^2\right)D_\mu\left(ap\right)D_\nu\left(ap\right)=$$
$$\frac{1}{\Gamma\left[-\mu-\nu\right]}\int_0^\infty\exp\left(-pt\right)a^{\mu+\nu}t^{-(1+\mu+\nu)}\exp\left(-\frac{t^2}{2a^2}\right)$$
$$\times\,{}_2F_2\left(-\mu,-\nu;-\frac{\mu+\nu}{2},\frac{1-\mu-\nu}{2};\frac{t^2}{4a^2}\right)dt\qquad **$$

Theorem 4.2. *Let ν and μ be two complex numbers with* $\operatorname{Re}\left(\mu+\nu\right)<0$. *Then, the following inverse Laplace transform holds for* $\operatorname{Re}p>0$ *and* $\operatorname{Re}a>0$

$$\exp\left(\tfrac{1}{2}a^2p^2\right)D_\mu\left(ap\right)D_\nu\left(ap\right)=\tag{4.2}$$
$$\frac{1}{\Gamma\left[-\mu-\nu\right]}\int_0^\infty\exp\left(-pt\right)a^{\mu+\nu}t^{-(1+\mu+\nu)}\exp\left(-\frac{t^2}{2a^2}\right)$$
$$\times\,{}_2F_2\left(-\mu,-\nu;-\frac{\mu+\nu}{2},\frac{1-\mu-\nu}{2};\frac{t^2}{4a^2}\right)dt$$

Proof. From the specification of, for instance, the inverse Laplace transform (3.23), it is clear that the left-hand side of the expression in [14, 21, 26] contains a misprint as the exponential term should be $\exp\left(\frac{1}{2}a^2p^2\right)$ rather than $\exp\left(\frac{1}{4}a^2p^2\right)$. □

5. Two new definite integrals for the generalized hypergeometric function

The below definite integrals for the generalized hypergeometric function are derived from the inverse Laplace transform (4.2) in combination with two results from Section 3.

5.1. First integral

Using $a = 2^{1/2}x^{1/2}$ in (4.2) gives

$$\exp\left(p^2 x\right) D_\mu\left(2^{1/2}x^{1/2}p\right) D_\nu\left(2^{1/2}x^{1/2}p\right) = \frac{(2x)^{(\mu+\nu)/2}}{\Gamma[-\mu-\nu]}\int_0^\infty \exp(-pt)\, t^{-(1+\mu+\nu)} \exp\left(-\frac{t^2}{4x}\right) \times {}_2F_2\left(-\mu,-\nu;-\frac{\mu+\nu}{2},\frac{1-\mu-\nu}{2};\frac{t^2}{8x}\right) dt \tag{5.1}$$

and the inverse Laplace transform (3.23) for $y = x$ is

$$\exp(px) D_\mu\left(2^{1/2}x^{1/2}p^{1/2}\right) D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right) = \frac{2^{(\mu+\nu)/2}x^{-1/2}}{\Gamma[-(\mu+\nu)/2]}\int_0^\infty \exp(-pt)\, t^{-1-(\nu+\mu)/2}(x+t)^{(1+\mu+\nu)/2} \times\left\{{}_2F_1\left(-\frac{\mu}{2},-\frac{1+\nu}{2};-\frac{\mu+\nu}{2};\frac{t(2x+t)}{(x+t)^2}\right) - \frac{\nu t}{(\mu+\nu)(x+t)}{}_2F_1\left(-\frac{\mu}{2},\frac{1-\nu}{2};1-\frac{\mu+\nu}{2};\frac{t(2x+t)}{(x+t)^2}\right)\right\} dt \tag{5.2}$$

Let $f(t)$ be the original function in the Laplace transform (5.1) and $F(p)$ be the corresponding image function. Equation (26) on p. 4 of [26] states that the original function of the image function $F\left(p^{1/2}\right)$ then is related to $f(t)$ as follows

$$\frac{1}{2\sqrt{\pi t^3}}\int_0^\infty \tau \exp\left(-\frac{\tau^2}{4t}\right) f(\tau)\, d\tau \tag{5.3}$$

Hence, plugging the original function for the inverse Laplace transform (5.1) into the expression (5.3) gives the original function of expression (5.2). Straightforward simplifications and redefinitions of variables then give the following definite integral for the generalized hypergeometric function

$$\int_0^\infty t^{-(\mu+\nu)} \exp\left(-\frac{x+y}{4xy}t^2\right) {}_2F_2\left(-\mu,-\nu;-\frac{\mu+\nu}{2},\frac{1-\mu-\nu}{2};\frac{t^2}{8x}\right) dt = 2^{-(\mu+\nu)}\Gamma\left[\frac{1-\mu-\nu}{2}\right] y\left(\frac{x+y}{xy}\right)^{(1+\mu+\nu)/2}\left\{{}_2F_1\left(-\frac{\mu}{2},-\frac{1+\nu}{2};-\frac{\mu+\nu}{2};\frac{y(2x+y)}{(x+y)^2}\right) - \frac{\nu y}{(\mu+\nu)(x+y)}{}_2F_1\left(-\frac{\mu}{2},\frac{1-\nu}{2};1-\frac{\mu+\nu}{2};\frac{y(2x+y)}{(x+y)^2}\right)\right\} \tag{5.4}$$

$$[\operatorname{Re}(\mu+\nu)<1, \operatorname{Re}x>0, \operatorname{Re}y>0]$$

5.2. Second integral

Again, let $f(t)$ be the original function in the Laplace transform (5.1) and $F(p)$ be the corresponding image function. Equation (29) on p. 5 of [26] states that the original function of the image function $p^{-1/2}F\left(p^{1/2}\right)$ is given by

$$\frac{1}{\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) f(\tau)\,d\tau \tag{5.5}$$

The property in (5.5) establishes a relation between the inverse Laplace transforms for $\exp\left(p^2 x\right) D_\mu\left(2^{1/2}x^{1/2}p\right) D_\nu\left(2^{1/2}x^{1/2}p\right)$ and
$p^{-1/2}\exp(px)\, D_\mu\left(2^{1/2}x^{1/2}p^{1/2}\right) D_\nu\left(2^{1/2}x^{1/2}p^{1/2}\right)$. Equation (5.5) then allows us to obtain the following indefinite integral

$$\int_0^\infty t^{-(1+\mu+\nu)}\exp\left(-\frac{x+y}{4xy}t^2\right) {}_2F_2\left(-\mu,-\nu;-\frac{\mu+\nu}{2},\frac{1-\mu-\nu}{2};\frac{t^2}{8x}\right)dt =$$
$$2^{-(1+\mu+\nu)}\Gamma\left[-\frac{\mu+\nu}{2}\right]\left(\frac{x+y}{xy}\right)^{(\mu+\nu)/2} {}_2F_1\left(-\frac{\mu}{2},-\frac{\nu}{2};\frac{1-\mu-\nu}{2};\frac{y(2x+y)}{(x+y)^2}\right) \tag{5.6}$$
$$[\operatorname{Re}(\mu+\nu)<0, \operatorname{Re}x\geqslant 0, \operatorname{Re}y\geqslant 0]$$

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