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Research Article

On the transfer of some t–locally properties

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Abstract

In this paper, we study the transfer of some t–locally properties which are stable under localization to t–flat overrings of an integral domain D. We show that D, $D[X]$, $D\langle X\rangle$, $D(X)$ and $D[X]_{N_v}$ are simultaneously t–locally PvMDs (resp., t–locally Krull, t–locally G-GCD, t–locally Noetherian, t–locally Strong Mori). A complete characterization of when a pullback is a t–locally PvMD (resp., t–locally GCD, t–locally G-GCD, t–locally Noetherian, t–locally Strong Mori, t–locally Mori) is given. As corollaries, we investigate the transfer of some t–locally properties among domains of the form $D + XK[X]$, $D + XK[[X]]$ and amalgamated algebras. A particular attention is devoted to the transfer of almost Krull and locally PvMD properties to integral closure of a domain having the same property.

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1. Introduction

It is convenient to begin by recalling some definitions and notation. Let D be an integral domain with quotient field K. For a nonzero fractional ideal I of D, we let $I^{-1} := \{x \in K|\ xI \subseteq D\}$. On D the *v–operation* is defined by $I_v = (I^{-1})^{-1}$; the *t–operation* is defined by $I_t := \bigcup J_v$, where J ranges over the set of all nonzero finitely generated ideals contained in I; and the *w–operation* is defined by $I_w := \{x \in K|\ xJ \subseteq I$ for some nonzero finitely generated ideal J of D with $J^{-1} = D\}$ for all nonzero fractional ideals I of D. A nonzero ideal I of D is *divisorial* (or *v–ideal*) (resp., *t–ideal*, *w–ideal*) if $I_v = I$ (resp., $I_t = I$, $I_w = I$). In general, for each nonzero fractional ideal I of D, $I \subseteq I_w \subseteq I_t \subseteq I_v$, and the inclusions may be strict (cf. [14, Proposition 1.2]). So, v–ideals are t–ideals and t–ideals are w–ideals. For $* = t$ or w, a $*$–ideal which is also prime is called a *$*$–prime* ideal, *$*$–maximal* ideal is an ideal that is maximal among the

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proper $*$–ideals and let $*$-Max(D) denote the set of all $*$–maximal ideals of D. Notice that w-Max$(D) = t$-Max(D) and each height-one prime is t–prime.

An integral domain D is said to be a *Prüfer v–multiplication domain* (for short, PvMD) (resp., *t–almost Dedekind domain*) if $D_{\mathfrak{m}}$ is a valuation domain (resp., a DVR) for each t–maximal ideal $\mathfrak{m}$ of D. Trivially, Krull domains and almost Dedekind domains are t–almost Dedekind domains and t–almost Dedekind domains are PvMDs. An integral domain D is a *Strong Mori domain* (for short, SM domain) (resp., *Mori domain*) if it satisfies the ascending chain condition (acc) on integral w–ideals (resp., v–ideals) of D. Clearly, Noetherian domains and Krull domains are SM and SM domains are Mori.

An integral domain D is a *GCD domain* (resp., *generalized GCD domain* (for short, G-GCD domain)) if the intersection of any two (integral) principal ideals (resp., invertible ideals) of D is still principal (resp., invertible). Notice that valuation domains are GCD domains, GCD domains are G-GCD domains and G-GCD domains form a subclass of PvMDs.

In this paper, we begin by the study of the transfer of some t–locally properties which are stable under localization to a t–flat overring of a domain D. Then we give several applications, namely for t–almost Dedekind domains. Among other results, we show that every t–linked overring of a t–almost Dedekind domain which is not a field is also a t–almost Dedekind domain. In our second major result we prove that for any integral domain D, the domains D, $D[X]$, $D\langle X\rangle$, $D(X)$ and $D[X]_{N_v}$ are simultaneously t–locally PvMDs (resp., t–locally Krull, t–locally G-GCD, t–locally Noetherian, t–locally SM). By the way, we treat a relevant case when D is a t–locally G-GCD domains. Next we establish necassary and sufficient conditions for a pullback construction to be t–locally PvMD (resp., t–locally GCD, t–locally G-GCD). As additional applications we recover the cases of domains of the form $D + XK[X]$, $D + XK[[X]]$ and amalgamated algebras. Then we extend [14, Theorem 3.11] to t–locally Noetherian (resp., t–locally SM) domains arising from pullback constructions. Finaly, while dealing with the integral closure of an integral domain, we show that the converse of [15, Theorem 2.13] holds with less hypotheses. Moreover, we investigate the transfer of the locally PvMD property to the integral closure $\overline{D}$ of an integrally closed domain D in an algebraic field extension of its quotient field, and we prove that D is a locally PvMD if and only if so is $\overline{D}$.

2. Main results

Let $(\mathcal{P})$ denote a property of integral domains. An integral domain D is said to be *locally* $(\mathcal{P})$ (resp., *t–locally* $(\mathcal{P})$) if $D_{\mathfrak{m}}$ is $(\mathcal{P})$ for each maximal ideal (resp., t–maximal ideal) $\mathfrak{m}$ of D. Notice that in domains that are Prüfer or of dimension one, t–locally $(\mathcal{P})$ coincides with locally $(\mathcal{P})$.

By an *overring* of D we mean any domain R between D and the quotient field of D. Recall from [12] that an overring R of D is said to be *t–flat* over D if $R_{\mathfrak{m}} = D_{\mathfrak{m}\cap D}$, for each t–maximal ideal $\mathfrak{m}$ of R, or equivalently $R_{\mathfrak{p}} = D_{\mathfrak{p}\cap D}$, for each t–prime ideal $\mathfrak{p}$ of R (cf. [5, Theorem 2.6]).

Proposition 2.1. *Let $(\mathcal{P})$ be a property of integral domains which is stable under localization. Then, for any integral domain D, the following statements are equivalent:*

(1) *D is t–locally $(\mathcal{P})$;*
(2) *$D_{\mathfrak{p}}$ is $(\mathcal{P})$ for each t–prime ideal $\mathfrak{p}$ of D;*
(3) *Each t–flat overring of D is also t–locally $(\mathcal{P})$.*

Proof. (1) $\Rightarrow$ (2) Assume that D is t–locally $(\mathcal{P})$ and let $\mathfrak{p}$ be a t–prime ideal of D. Then there exists a t–maximal ideal $\mathfrak{m}$ of D such that $\mathfrak{p} \subseteq \mathfrak{m}$. It follows from [2, Lemma 1] that $D_{\mathfrak{p}} = (D_{\mathfrak{m}})_{\mathfrak{p}D_{\mathfrak{m}}}$. Hence, $D_{\mathfrak{p}}$ is a $(\mathcal{P})$ domain as a localization of the $(\mathcal{P})$ domain $D_{\mathfrak{m}}$.

(2) ⇒ (3) Let R be a (proper) t–flat overring of D and $\mathfrak{q}$ be a t–maximal ideal of R. Then, by [5, Lemma 1.2], $\mathfrak{p} := \mathfrak{q} \cap D$ is a t–prime ideal of D, and hence $D_{\mathfrak{p}} = R_{\mathfrak{q}}$ is a $(\mathcal{P})$ domain. Thus, R is t–locally $(\mathcal{P})$.

(3) ⇒ (1) Straightforward. □

Similarly, using [16, Theorem 2], it easy to prove an analogue of the previous result when dealing with flat overrings of a locally $(\mathcal{P})$ domain.

Corollary 2.2. *Let $(\mathcal{P})$ denote one of the following properties: GCD, Krull, PvMD, G-GCD, Noetherian, SM or Mori. Then, D is a t–locally $(\mathcal{P})$ domain if and only if every t–flat overring of D is also t–locally $(\mathcal{P})$.*

For t–almost Dedekind domains, we get a more interesting result.

Recall that an overring R of D is *t–linked* over D if, for each nonzero finitely generated ideal I of D such that $I^{-1} = D$, we have $(IR)^{-1} = R$. Notice that every t–flat overring is t–linked.

Corollary 2.3. *Let D be a t–almost Dedekind domain which is not a field. Then, each t–linked overring of D is also t–almost Dedekind.*

Proof. Let R be a (proper) t–linked overring of D. Since any t–almost Dedekind domain is a PvMD, it follows from [12, Proposition 2.10] that R is t–flat over D and then, by Proposition 2.1, R is a t–almost Dedekind. □

Corollary 2.4. *Let D be a t–almost Dedekind domain. We have:*

(1) *If $R = \bigcap_\alpha D_\alpha$, with each D_α is a t–linked overring of D, then R is a t–almost Dedekind domain.*

(2) *If T is an overring of D and $\mathfrak{p}$ is a t–prime ideal of D, then $T_{D\setminus\mathfrak{p}}$ is a t–almost Dedekind domain.*

(3) *The complete integral closure of D is a t–almost Dedekind domain.*

Proof. Follows from Corollary 2.3 and [4, Propositions 2.2(b), 2.9, and Corollary 2.3]. □

Now, let X be an indeterminate over an integral domain D. For each polynomial $f \in D[X]$, we denote by $c(f)$ the *content* of f, that is, the ideal of D generated by the coefficients of f. The sets $U = \{f \in D[X] | \ f \text{ is monic}\}$, $S = \{f \in D[X] | \ c(f) = D\}$ and $N_v = \{f \in D[X] | \ c(f)_v = D\}$ are multiplicatively closed subsets of $D[X]$. The localization $D\langle X\rangle := D[X]_U$ (resp., $D(X) := D[X]_S$, $D[X]_{N_v}$) is called the *Serre conjecture* (resp., the *Nagata*, the *t–Nagata*) *ring* of D. Note that $D[X] \subseteq D\langle X\rangle \subseteq D(X) \subseteq D[X]_{N_v}$.

Theorem 2.5. *Let $(\mathcal{P})$ denote one of the following properties: PvMD, Krull, G-GCD, Noetherian or SM. Then, for any integral domain D, the following statements are equivalent:*

(1) *D is a t–locally $(\mathcal{P})$ domain;*
(2) *$D[X]$ is a t–locally $(\mathcal{P})$ domain;*
(3) *$D\langle X\rangle$ is a t–locally $(\mathcal{P})$ domain;*
(4) *$D(X)$ is a t–locally $(\mathcal{P})$ domain;*
(5) *$D[X]_{N_v}$ is a t–locally $(\mathcal{P})$ domain;*
(6) *$D[X]_{N_v}$ is a locally $(\mathcal{P})$ domain.*

The proof of this theorem requires the following preparatory lemmas.

Lemma 2.6. *Let D be an integral domain. Then:*

(1) $\operatorname{Max}(D[X]_{N_v}) = t\text{-}\operatorname{Max}(D[X]_{N_v}) = \{\mathfrak{m}[X]_{N_v} | \ \mathfrak{m} \in t\text{-}\operatorname{Max}(D)\}$.

(2) *For each t–maximal ideal $\mathfrak{m}$ of D, we have: $D[X]_{\mathfrak{m}[X]} = (D[X]_{N_v})_{\mathfrak{m}[X]_{N_v}} = D(X)_{\mathfrak{m}D(X)} = D_{\mathfrak{m}}(X)$.*

(3) *For each t–maximal ideal Q of $D[X]$, we have either: $Q \cap D = (0)$, or $Q \cap D$ is a t–maximal ideal of D and $Q = (Q \cap D)[X]$.*

Proof. (1) [11, Propositions 2.1 and 2.2].

(2) [2, Lemma 2].

(3) [7, Proposition 2.2]. □

Lemma 2.7. *Let D be an integral domain with quotient field K. Then, $D(X)$ is a PvMD (resp., Krull, G-GCD, Noetherian, SM) if and only if D has the same property.*

Proof. It is well known that D is a Krull (resp., G-GCD) domain if and only if $D(X)$ has the same property [1, Theorem 5.2(1)] (resp., [1, Theorem 5.1(1)]) .

Now, if D is a PvMD (resp., a Noetherian domain, an SM domain), then so is $D[X]$ and hence its localization $D(X)$ has the same property. Conversely, assume that $D(X)$ is a PvMD and let $\mathfrak{m}$ be a t–maximal ideal of D. By [11, Corollary 2.3(2)], $\mathfrak{m}D(X)$ is a t–prime ideal of $D(X)$, and then $D(X)_{\mathfrak{m}D(X)} = D[X]_{\mathfrak{m}[X]} = D_{\mathfrak{m}}(X)$ is a valuation domain. Thus, $D_{\mathfrak{m}}$ is a valuation domain since $D_{\mathfrak{m}} = D_{\mathfrak{m}}(X) \cap K$. Therefore, D is a PvMD. Next, assume that $D(X)$ is a Noetherian domain and let I be an ideal of D. Then, $ID(X)$ is finitely generated and so is I [1, Theorem 2.2(2)]. Lastly, assume that $D(X)$ is an SM domain and let $\mathfrak{m}$ be a t–maximal ideal of D. By [11, Corollary 2.3(2)], $\mathfrak{m}D(X)$ is a t–prime ideal of $D(X)$, and then $D(X)_{\mathfrak{m}D(X)} = D[X]_{\mathfrak{m}[X]} = D_{\mathfrak{m}}(X)$ is a Noetherian domain. Hence, $D_{\mathfrak{m}}$ is a Noetherian domain and thus D is t–locally Noetherian. On the other hand, since $D = D(X) \cap K$, D is a Mori domain as an intersection of two Mori domains. Therefore, D is an SM domain. □

Proof of Theorem 2.5. (1) ⇒ (2) Let Q be a t–maximal ideal of $D[X]$ and set $P = Q \cap D$. If $P = (0)$ then $D[X]_Q = K[X]_{QK[X]}$ is a DVR, where K is the quotient field of D. If $P \neq (0)$ then, by Lemma 2.6(3), $Q = P[X]$ and P is a t–maximal ideal of D. Hence, D_P is a $(\mathcal{P})$ domain, so by Lemmas 2.6(2) and 2.7, $D[X]_Q = D[X]_{P[X]} = D_P(X)$ is a $(\mathcal{P})$ domain. Therefore, $D[X]$ is a t–locally $(\mathcal{P})$ domain.

(2) ⇒ (3) and (3) ⇒ (4) follows from Corollary 2.1, since $D\langle X\rangle = D[X]_U$ is a localization of $D[X]$ and $D(X)$ is a localization of $D\langle X\rangle$.

(4) ⇒ (1) Let $\mathfrak{m}$ be a t–maximal ideal of D. Then, by [11, Corollary 2.3(2)], $\mathfrak{m}D(X)$ is a t–prime ideal of $D(X)$. Hence, $D(X)_{\mathfrak{m}D(X)} = D[X]_{\mathfrak{m}[X]} = D_{\mathfrak{m}}(X)$ is a $(\mathcal{P})$ domain. Thus, by Lemma 2.7, $D_{\mathfrak{m}}$ is a $(\mathcal{P})$ domain. Therefore, D is a t–locally $(\mathcal{P})$ domain.

(1) ⇒ (6) Let Q be a maximal ideal of $D[X]_{N_v}$. By Lemma 2.6(1), $Q = \mathfrak{m}[X]_{N_v}$ for some t–maximal ideal $\mathfrak{m}$ of D. As $(D[X]_{N_v})_{\mathfrak{m}[X]_{N_v}} = D[X]_{\mathfrak{m}[X]} = D_{\mathfrak{m}}(X)$ and $D_{\mathfrak{m}}$ is a $(\mathcal{P})$ domain, it follows from Lemma 2.7 that $(D[X]_{N_v})_{\mathfrak{m}[X]_{N_v}}$ is a $(\mathcal{P})$ domain and hence, $D[X]_{N_v}$ is locally $(\mathcal{P})$.

(6) ⇔ (5) This equivalence follows from Lemma 2.6(1).

(5) ⇒ (1) Let $\mathfrak{m}$ be a t–maximal ideal of D. Then, $\mathfrak{m}[X]_{N_v}$ is a t–maximal ideal of $D[X]_{N_v}$, and hence $(D[X]_{N_v})_{\mathfrak{m}[X]_{N_v}} = D[X]_{\mathfrak{m}[X]} = D_{\mathfrak{m}}(X)$ is a $(\mathcal{P})$ domain. Thus, by Lemma 2.7, $D_{\mathfrak{m}}$ is a $(\mathcal{P})$ domain and hence, D is t–locally $(\mathcal{P})$. □

For the case of G-GCD domains we have a more precise result.

Proposition 2.8. *For any integral domain D, the following statements are equivalent:*

(1) *D is a t–locally G-GCD domain;*
(2) *$D[X]$ is a t–locally G-GCD domain;*
(3) *$D(X)$ is a locally GCD domain;*
(4) *$D[X]_{N_v}$ is a locally GCD domain.*

Proof. The proof is similar to the proof of the above theorem by using the fact that D is a G-GCD domain if and only if $D(X)$ is a GCD domain (cf. [1, Theorem 5.1(1)]). □

To avoid unnecessary repetition, let us fix some notation for the remainder of this paper.

Let T be an integral domain, $\mathfrak{M}$ a maximal ideal of T, K the residue field $T/\mathfrak{M}$, $\varphi : T \to K$ is the natural projection, D a proper subring of K. Let $R := \varphi^{-1}(D)$ be the pullback arising from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K. \end{array}$$

We shall refer to this as a pullback diagram of type ($\square$).

Proposition 2.9. *Let* ($\mathcal{P}$) *denote one of the following properties: PvMD, GCD or G-GCD. Then, for a pullback diagram of type* ($\square$)*, R is a t–locally* ($\mathcal{P}$) *domain if and only if $qf(D) = K$, D and T are t–locally* ($\mathcal{P}$) *domains and $T_{\mathfrak{M}}$ is valuation.*

Proof. Recall that R is a ($\mathcal{P}$) domain if and only if $qf(D) = K$, D and T are ($\mathcal{P}$) domains and $T_{\mathfrak{M}}$ is valuation (cf. [6, Theorems 4.1 and 4.2(a-b)]).
Assume that R is a t–locally ($\mathcal{P}$) domain. Since $\mathfrak{M}$ is a t–prime ideal of R, $R_{\mathfrak{M}}$ is a ($\mathcal{P}$) domain.

Let Q be a t–maximal ideal of T. If $Q = \mathfrak{M}$, then we localize the previous diagram at $\mathfrak{M}$ to obtain the following pullback:

$$\begin{array}{ccc} R_{\mathfrak{M}} & \longrightarrow & D_{\varphi(\mathfrak{M})} \\ \downarrow & & \downarrow \\ T_{\mathfrak{M}} & \xrightarrow{\varphi} & K. \end{array}$$

It follows from [6, Theorems 4.1 and 4.2(a-b)] that $qf(D) = K$ and $T_{\mathfrak{M}}$ is a ($\mathcal{P}$) domain. If $Q \neq \mathfrak{M}$, then $P := Q \cap R$ is a t–maximal ideal of R and hence $T_Q = R_P$ is a ($\mathcal{P}$) domain. Thus, T is a t–locally ($\mathcal{P}$) domain.
Let P be a t–maximal ideal of D and set $Q := \varphi^{-1}(P)$. Considering the following pullback:

$$\begin{array}{ccc} R_Q & \longrightarrow & D_P \\ \downarrow & & \downarrow \\ T_{\mathfrak{M}} & \xrightarrow{\varphi} & K. \end{array}$$

By [6, Theorems 4.1 and 4.2(a-b)], D_P is a ($\mathcal{P}$) domain and $T_{\mathfrak{M}}$ is a valuation domain.

Conversely, let Q be a t–maximal ideal of R. If $Q = \mathfrak{M}$, then, by [6, Theorems 4.1 and 4.2(a-b)], $R_{\mathfrak{M}}$ is a ($\mathcal{P}$) domain. If $Q \neq \mathfrak{M}$, then there is only one t–maximal ideal P of T such that $P \cap R = Q$ (cf. [9, Theorem 2.6(1)]), and hence $R_Q = T_P$ is a ($\mathcal{P}$) domain since T is a t–locally ($\mathcal{P}$) domain. Thus, R is a t–locally ($\mathcal{P}$) domain. $\square$

From [3] we introduce the definition of amalgamated algebras along an ideal as follows:
Let A and B be two rings, $f : A \to B$ a ring homomorphism and J an ideal of B. The following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) |\ a \in A \text{ and } j \in J\},$$

is called the *amalgamation* of A with B along J with respect to f.

Corollary 2.10. *Let A and B be two integral domains, J a maximal ideal of B and $f : A \to B$ a ring homomorphism such that $f^{-1}(J) = \{0\}$. Let* ($\mathcal{P}$) *denote one of the following properties: PvMD, GCD or G-GCD. Then, $A \bowtie^f J$ is a t–locally* ($\mathcal{P}$) *domain if and only if $qf(A) = B/J$, A and B are t–locally* ($\mathcal{P}$) *domains and B_J is valuation.*

Proof. By [3, Proposition 4.2], we have the following pullback:

$$\begin{array}{ccc} A \bowtie^f J & \longrightarrow & A \\ \downarrow & & \downarrow \tilde{f} \\ B & \xrightarrow{\varphi} & B/J, \end{array}$$

where $\tilde{f} = \varphi \circ f$. The result follows immediately by applying Proposition 2.9. $\square$

Now, we recover the case of simple amalgamation.

Corollary 2.11. *Let A be an integral domain, I a maximal ideal of A and let $(\mathcal{P})$ denote one of the following properties: PvMD, GCD or G-GCD. Then, $A \bowtie I$ is a t–locally $(\mathcal{P})$ domain if and only if A is a field.*

Proof. Take $B = A$ and $f = \mathrm{Id}_A$ in Corollary 2.10. □

Corollary 2.12. *Let K be a field, X an indeterminate over K, D a subring of K and let $(\mathcal{P})$ denote one of the following properties: PvMD, GCD or G-GCD. If R is an integral domain of the form $D + XK[X]$ or $D + XK[[X]]$, then R is a t–locally $(\mathcal{P})$ domain if and only if so is D and $qf(D) = K$.*

Proof. Let $T = K[X]$ (resp., $T = K[[X]]$) and $\mathfrak{M} = XK[X]$ (resp., $\mathfrak{M} = XK[[X]]$). Then, T is a PID with $T/\mathfrak{M} \cong K$ and $T_{\mathfrak{M}}$ is a DVR. Thus the conclusion follows from Proposition 2.9. □

We now study the transfer of the t–locally Noetherian (resp., the t–locally SM) notion to pullbacks. In fact, we extend [14, Theorem 3.11] to t–locally Noetherian (resp., t–locally SM) domains.

Proposition 2.13. *For a pullback diagram of type $(\square)$, R is a t–locally Noetherian domain (resp., a t–locally SM domain) if and only if T is a t–locally Noetherian domain (resp., a t–locally SM domain), $T_{\mathfrak{M}}$ is Noetherian, $D = k$ is a field, and $[K : k]$ is finite. In particular, if T is local, then R is a t–locally Noetherian domain (resp., a t–locally SM domain) if and only if R is Noetherian.*

Proof. Assume that R is t–locally Noetherian. If D is not a field, then D has a nonzero t–maximal ideal P. Set $Q = \varphi^{-1}(P)$. Then Q is a t–maximal ideal of R and so R_Q is Noetherian. Now consider the following pullback:

$$\begin{array}{ccc} R_Q & \longrightarrow & D_P \\ \downarrow & & \downarrow \\ T_S & \longrightarrow & K, \end{array}$$

where $S = R \setminus Q$. Necessarily $D_P = k$ is a field, which is absurd. Thus $D = k$ is a field. Now that $D = k$ is a field implies that $\mathfrak{M}$ is a maximal ideal of R which is divisorial and so it is a t–maximal ideal. Localizing at $\mathfrak{M}$, we obtain the follwing pullback:

$$\begin{array}{ccc} R_{\mathfrak{M}} & \longrightarrow & k \\ \downarrow & & \downarrow \\ T_{\mathfrak{M}} & \longrightarrow & K. \end{array}$$

So that $R_{\mathfrak{M}}$ is Noetherian implies that $T_{\mathfrak{M}}$ is Noetherian and $[K : k]$ is finite.

Conversely, let Q be a t–maximal ideal of R. Then, we distinguish the following two possible cases:

Case 1: $Q = \mathfrak{M}$. Since $T_{\mathfrak{M}}$ is Noetherian, $D = k$ is a field, and $[K : k]$ is finite, it follows from [8, Theorem 4.12] that $R_{\mathfrak{M}}$ is Noetherian.

Case 2: $Q \neq \mathfrak{M}$. Then there is a unique t–maximal ideal P of T such that $P \cap R = Q$ and hence $R_Q = T_P$ is a Noetherian domain because T is a t–locally Noetherian domain.

Therefore, R is a t–locally Noetherian domain.

The case of t–locally SM domains is similar to the previous case by using [14, Theorem 3.11].

For the particular case, we have $T = T_{\mathfrak{M}}$ and so the conclusion follows from [8, Theorem 4.12]. □

Corollary 2.14. *Let A and B be two integral domains, J a maximal ideal of B and $f : A \to B$ a ring homomorphism such that $f^{-1}(J) = \{0\}$. Then, $A \bowtie^f J$ is a t–locally Noetherian domain (resp., a t–locally SM domain) if and only if B is a t–locally Noetherian domain (resp., a t–locally SM domain), B_J is Noetherian, A is a field, and $[B/J : A]$ is finite.*

Proof. It follows from [3, Proposition 4.2] and Proposition 2.13. □

Corollary 2.15. *Let K be a field, X an indeterminate over K, D a subring of K. If R is an integral domain of the form $D+XK[X]$ or $D+XK[[X]]$, then the following statements are equivalent.*

(1) *R is a t–locally Noetherian domain;*
(2) *R is a t–locally SM domain;*
(3) *$D = k$ is a field and $[K : k]$ is finite;*
(4) *R is Noetherian.*

By adapting the proof of Proposition 2.13 and using [8, Theorem 4.18], we get the following:

Proposition 2.16. *For a pullback diagram of type (□), R is a t–locally Mori domain if and only if T is a t–locally Mori domain, $T_{\mathfrak{M}}$ is Mori, and $D = k$ is a field.*

In [15, Theorem 2.13], Pirtle established that if D is an almost Krull domain, i.e., a locally Krull domain, with quotient field K, then the integral closure of D in a finite field extension of K is also almost Krull. Next, we show that the converse holds without the finitness condition.

Proposition 2.17. *Let D be an integrally closed domain with quotient field K, let L be an algebraic field extension of K and let $\overline{D}$ be the integral closure of D in L. If $\overline{D}$ is an almost Krull domain then so is D.*

Proof. Let $\mathfrak{p}$ be a prime ideal of D and $\mathfrak{q}$ be a prime ideal of $\overline{D}$ lying over $\mathfrak{p}$. Since $\overline{D}$ is almost Krull, $\overline{D}_{\mathfrak{q}}$ is a Krull domain and then, by [10, Theorem 1], $\overline{D}_{\mathfrak{q}} \cap K = D_{\mathfrak{p}}$ is also a Krull domain. That is, D is an almost Krull domain. □

In the case of locally PvMDs we get a stronger result.

Proposition 2.18. *Let D be an integrally closed domain with quotient field K, let L be an algebraic field extension of K and let $\overline{D}$ be the integral closure of D in L. Then, D is a locally PvMD if and only if so is $\overline{D}$.*

Proof. Assume that D is a locally PvMD and let M be a maximal ideal of $\overline{D}$. Set $P = M \cap D$ and $S = D \setminus P$. Then, D_P is a PvMD. Since $\overline{D}$ is the integral closure of D in L, $\overline{D}_S$ is the integral closure of D_P in L and hence it follows from [13, Theorems 4.4 and 4.6] that $\overline{D}_S$ is a PvMD. Thus we deduce from the equality $\overline{D}_M = (\overline{D}_S)_{M\overline{D}_S}$ that $\overline{D}_M$ is a PvMD. Therefore, $\overline{D}$ is a locally PvMD. The converse is similar to that of the proof of the previous proposition. □

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