ON THE LIFTS OF $F_a(5,1)$–STRUCTURE ON TANGENT AND COTANGENT BUNDLE

Fidan JABRAILZADE
Department of Algebra and Geometry, Baku State University, AZERBAIJAN

Abstract. This paper consist of three main sections. In the first part, we obtain the complete lifts of the $F_a(5,1)$–structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)$–structure. Later we get the conditions of to be the almost holomorphic vector field with respect to the complete lifts of $F_a(5,1)$–structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)$–structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)$–structure in tangent bundle $T^1(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)$–structure in cotangent bundle $T^*(M^n)$.

1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [17]. Also, the idea of $F$–structure manifold on a differentiable manifold developed by Yano [14], Ishihara and Yano [7], Goldberg [6] and among others. Moreover, Yano and Patterson [15,16] studied on the horizontal and complete lifts from a differentiable manifold $M^p$ of class $C^\infty$ to its cotangent bundles. Andreu has studied the structure defined by a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying $F^5 + F = 0$ [1]. Later Ram Nivas and C.S. Prasad [11] studied on more form $F_a(5,1)$–structure. This paper consist of three main sections. In the first part,
we obtain the complete lifts of the $F_a(5,1)$—structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)$—structure. Later we get the conditions of to be the almost holomorphic vector field with respect to the complete lifts of $F_a(5,1)$—structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)$—structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)$—structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)$—structure in cotangent bundle $T^*(M^n)$.

Let $M^n$ be an $n$—dimensional differentiable manifold of class $C^\infty$. Suppose there exist on $M^n$, a $(1,1)$ tensor field $F(\neq 0)$ satisfying
\begin{equation}
F^5 - a^2 F = 0,
\end{equation}
where $a$ is a complex number not equal to zero. If $a = i$ where $i = \sqrt{-1}$, our structure takes the form $F^5 + F = 0$ studied by Andreou [1].

Let us define on $M^n$, the operators $l$ and $m$ as follows:
\begin{equation}
l = (F^4/a^2) \text{ and } m = I - (F^4/a^2). \tag{2}
\end{equation}
$I$ being unit tensor field.

In view of equations (1) and (2), we have
\begin{equation}
l^2 = l, \quad m^2 = m \text{ and } l + m = I. \tag{3}
\end{equation}
For a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying (1) the operators $l$ and $m$ defined by (2), when applied to the tangent space of $M^n$ at a point, are complementary projection operators.

Thus there exist complementary distributions $L$ and $M$ corresponding to the projection operators $l$ and $m$ respectively. If the rank of $F$ is constant everywhere or equal to $r$, the dimensions of $L$ and $M$ are $r$ and $n - r$ respectively [10]. We call such a structure as $F_a(5,1)$—structure of rank $r$ [11].

For a tensor field $F(\neq 0)$ of type $(1,1)$ admitting $F_a(5,1)$—structure and for the projection operators $l$ and $m$ given by (2) we have
\begin{equation}
Fl = lF = F, \quad Fm = mF = 0. \tag{4}
\end{equation}
and
\begin{equation}
F^2l = lF^2 = F^2, \quad F^2m = mF^2 = 0. \tag{5}
\end{equation}
In the manifold $M^n$ endowed with $F_a(5,1)$—structure, the $(1,1)$ tensor field $\tilde{F}$ given by $\tilde{F} = l - m = (2F^4/a^2) - I$ gives an almost product structure [9].

1.1. Complete Lift of $F_a(5,1)$—Structure on Tangent Bundle. Let $M^n$ be an $n$—dimensional differentiable manifold of class $C^\infty$ and $T_p(M^n)$ the tangent space at a point $p$ of $M^n$ and
\begin{equation}
T(M^n) = \bigcup_{p \in M^n} T_p(M^n)
\end{equation}
is the tangent bundle over the manifold $M^n$.

Let us denote by $T^r_s(M^n)$, the set of all tensor fields of class $C^\infty$ and of type $(r, s)$ in $M^n$ and $T(M^n)$ be the tangent bundle over $M^n$. The complete lift of $F^C$ of an element of $T^1_1(M^n)$ with local components $F^h_i$ has components of the form

$$F^C = \begin{bmatrix} F^h_i \\ 0 \\ \delta^h_i \\ F^h_i \end{bmatrix}.$$  \hfill (6)

Now we obtain the following results on the complete lift of $F$ satisfying $F^5 - a^2 F = 0$.

Let $F, G \in T^1_1(M^n)$. Then we have

$$(FG)^C = F^C G^C.$$ \hfill (7)

Replacing $G$ by $F$ in (7) we obtain

$$(FF)^C = F^C F^C \text{ or } (F^2)^C = (F^C)^2.$$ \hfill (8)

Now putting $G = F^4$ in (7) since $G$ is $(1, 1)$ tensor field therefore $F^4$ is also $(1, 1)$ so we obtain $(FF^4)^C = F^C (F^4)^C$ which in view of (8) becomes

$$(F^5)^C = (F^C)^5.$$ \hfill (9)

Taking complete lift on both sides of equation $F^5 - a^2 F = 0$ we get

$$(F^5)^C - (a^2 F)^C = 0$$

which in consequence of equation (9) gives

$$(F^C)^5 - a^2 F^C = 0.$$ \hfill (10)

Let $F$ satisfying $(1, 1)$ be an $F$–structure of rank $r$ in $M^n$. Then the complete lifts $l^C = (F^4)^C$ of $l$ and $m^C = I - (F^4)^C$ of $m$ are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions $L^C$ and $M^C$ determined by $l^C$ and $m^C$, respectively.

1.2. Horizontal Lift of $F^h_i(5, 1)$–Structure on Tangent Bundle. Let $F^h_i$ be the component of $F$ at $A$ in the coordinate neighbourhood $U$ of $M^n$. Then the horizontal lift $F^H$ of $F$ is also a tensor field of type $(1, 1)$ in $T(M^n)$ whose components $F^A_B$ in $\pi^{-1}(U)$ are given by

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F^h_i \\ -\Gamma^h_{ij} F^l_j + \Gamma^h_{ij} F^l_i \\ \Gamma^h_i \\ F^h_i \end{pmatrix}.$$ \hfill (11)

Let $F, G$ be two tensor fields of type $(1, 1)$ on the manifold $M$. If $F^H$ denotes the horizontal lift of $F$, we have

$$(FG)^H = F^H G^H.$$ \hfill (12)

Taking $F$ and $G$ identical, we get

$$(F^H)^2 = (F^2)^H.$$ \hfill (13)
Multiplying both sides by $F^H$ and making use of the same (12), we get
$$(F^H)^3 = (F^3)^H$$
and so on. Thus it follows that
$$(F^H)^4 = (F^4)^H, (F^H)^5 = (F^5)^H.$$  \hspace{1cm} (13)

Taking horizontal lift on both sides of equation $F^5 - a^2F = 0$ we get
$$(F^5)^H - (a^2F)^H = 0$$
view of (13), we can write
$$(F^H)^5 - a^2F^H = 0.$$  \hspace{1cm} (14)

2. Main Results

2.1. The Nijenhuis Tensor $N_{(F^5)^C(F^5)^C}(X^C,Y^C)$ of the Complete Lift $F^5$ on Tangent Bundle $T(M^n)$.

**Definition 1.** Let $F$ be a tensor field of type (1, 1) admitting $F_{\alpha}(5, 1)$–structure in $M^n$. The Nijenhuis tensor of a (1, 1) tensor field $F$ of $M^n$ is given by

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2 [X, Y]$$  \hspace{1cm} (15)

for any $X, Y \in \mathfrak{X}(M^n)$. The condition of $N_F(X, Y) = N(X, Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor $N_F$ is defined local coordinates by

$$N^k_{ij} = (F^i_j \partial_k F^k_j - F^j_i \partial_k F^k_i - \partial_k F^i_j F^k_i + \partial_k F^i_j F^k_i) \partial_k,$$

where $X = \partial_i, Y = \partial_j, F \in \mathfrak{X}(M^n)$.

**Definition 2.** Let $X$ and $Y$ be any vector fields on a Riemannian manifold $(M^n, g)$, we have

$$[X^H, Y^H] = [X, Y]^H - (R(X, Y) u)^V,$$  \hspace{1cm} (16)

$$[X^H, Y^V] = (\nabla_X Y)^V,$$

$$[X^V, Y^V] = 0,$$

where $R$ is the Riemannian curvature tensor of $g$ defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$  \hspace{1cm} (17)

In particular, we have the vertical spray $u^V$ and the horizontal spray $u^H$ on $T(M^n)$ defined by

$$u^V = u^i (\partial_i)^V = u^i \partial_i, u^H = u^i (\partial_i)^H = u^i \delta_i,$$  \hspace{1cm} (18)

where $\delta_i = \partial_i - w^s j_s \partial_i$. $u^V$ is also called the canonical or Liouville vector field on $T(M^n)$. 
Theorem 3. The Nijenhuis tensor $N_{(F^5)^C(F^5)^C} (X^C,Y^C)$ of the complete lift of $F^5$ vanishes if the Nijenhuis tensor of the $F$ is zero.

Proof. In consequence of Definition 1 the Nijenhuis tensor of $(F^5)^C$ is given by

$$N_{(F^5)^C(F^5)^C} (X^C,Y^C) = \left[ (F^5)^C X^C, (F^5)^C Y^C \right] - (F^5)^C \left[ (F^5)^C X^C, Y^C \right]$$

$$= a^4 \left\{ [(FX)^C, (FY)^V] - (F)^C \left[ (FX)^C, Y^C \right] \right.$$

$$+ (F)^C \left[ X^C, (FY)^V \right] + \left( F^2 \left[ X, Y \right] \right)^V \}$$

$$= a^4 N(X,Y)^V$$

Theorem 4. The Nijenhuis tensor $N_{(F^5)^C(F^5)^C} (X^C,Y^V)$ of the complete lift of $F^5$ vanishes if the Nijenhuis tensor $F$ is zero.

Proof.

$$N_{(F^5)^C(F^5)^C} (X^C,Y^V) = \left[ (F^5)^C X^C, (F^5)^C Y^V \right] - (F^5)^C \left[ (F^5)^C X^C, Y^V \right]$$

$$= a^4 \left\{ [(FX)^C, (FY)^V] - (F)^C \left[ (FX)^C, Y^V \right] \right.$$

$$+ (F)^C \left[ X^C, (FY)^V \right] + \left( F^2 \left[ X, Y \right] \right)^V \}$$

$$= a^4 N(X,Y)^V$$

Theorem 5. The Nijenhuis tensor $N_{(F^5)^C(F^5)^C} (X^V,Y^V)$ of the complete lift of $F^5$ vanishes.

Proof. Thus $[X^V,Y^V] = 0$ for all $X,Y \in \mathfrak{g}(M^n)$, easily we get

$$N_{(F^5)^C(F^5)^C} (X^V,Y^V) = 0.$$
2.2. The Purity Conditions of Sasakian Metric with Respect to \((F^5)^C\) on \(T(M^n)\).

**Definition 6.** The Sasaki metric \(Sg\) is a (positive definite) Riemannian metric on the tangent bundle \(T(M^n)\) which is derived from the given Riemannian metric on \(M\) as follows:

\[
\begin{align*}
Sg(X^H, Y^H) &= g(X, Y), \\
Sg(X^H, Y^V) &= Sg(X^V, Y^H) = 0, \\
Sg(X^V, Y^V) &= g(X, Y)
\end{align*}
\]

for all \(X, Y \in \mathfrak{g}(M^n)\).

**Theorem 7.** The Sasaki metric \(Sg\) is pure with respect to \((F^5)^C\) if \(\nabla F = 0\) and \(F = a^2 I\), where \(I\) is the identity tensor field of type \((1,1)\).

Proof. \(S(\tilde{X}, \tilde{Y}) = Sg((F^5)^C \tilde{X}, \tilde{Y}) - Sg(\tilde{X}, (F^5)^C Y)\) if \(S(\tilde{X}, \tilde{Y}) = 0\) for all vector fields \(\tilde{X}\) and \(\tilde{Y}\) which are of the form \(X^V, Y^V\) or \(X^H, Y^H\) then \(S = 0\).

i) 
\[
S(X^V, Y^V) = Sg((F^5)^C X^V, Y^V) - Sg(X^V, (F^5)^C Y^V)
\]

\[
= a^2\{Sg(FX^V, Y^V) - Sg(X^V, FY^V)\}
\]

\[= a^2\{(g(FX, Y))^V - (g(X, FY))^V\}
\]

ii) 
\[
S(X^V, Y^H) = Sg((F^5)^C X^V, Y^H) - Sg(X^V, (F^5)^C Y^H)
\]

\[= -a^2 Sg(FX^V, (FY)^H + (\nabla \gamma F) Y^H)
\]

\[= -a^2 Sg(FX^V, ((\nabla F) u) Y)^V
\]

\[= -a^2\{g(FX, ((\nabla F) u) Y)^V\}
\]

iii) 
\[
S(X^H, Y^H) = Sg((F^5)^C X^H, Y^H) - Sg(X^H, (F^5)^C Y^H)
\]

\[= a^2 Sg(FX^H, (FY)^H + (\nabla \gamma F) X^H, Y^H)
\]

\[= a^2 Sg(FX^H, (FY)^H + (\nabla \gamma F) Y^H)
\]

\[= a^2\{(g(FX), Y)^V - g(X, (FY))^V\}
\]

\(\square\)
Theorem 11. Let $\varphi \in \mathfrak{S}_1(M^n)$, and $\mathfrak{S}(M^n) = \bigoplus_{r=s=0}^{\infty} \mathfrak{S}_r(M^n)$ be a tensor algebra over $R$. A map $\phi_\varphi \big|_{r+s=0} : \mathfrak{S}(M^n) \to \mathfrak{S}(M^n)$ is called as Tachibana operator or \phi operator on $M^n$ if

a) $\phi_\varphi$ is linear with respect to constant coefficient,

b) $\phi_\varphi : \mathfrak{S}(M^n) \to \mathfrak{S}_{r+1}(M^n)$ for all $r$ and $s$,

c) $\phi_\varphi(K \otimes L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in \mathfrak{S}(M^n)$,

d) $\phi_\varphi X = -(L_Y \varphi)X$ for all $X, Y \in \mathfrak{S}_0(M^n)$, where $L_Y$ is the Lie derivation with respect to $Y$ (see [3,5,8]),

e) $(\phi_\varphi X)Y = (d(vY))(\varphi X) - (d(i_Y(\varphi Y)))X + \eta(\varphi Y)X$

for all $\eta \in \mathfrak{S}_0(M^n)$ and $X, Y \in \mathfrak{S}_0(M^n)$, where $i_Y \eta = \eta(Y) = \eta \otimes Y, \mathfrak{S}_r(M^n)$ the module of all pure tensor fields of type $(r, s)$ on $M^n$ with respect to the symmetric field, $\otimes$ is a tensor product with a contraction $C$ (see [13] for applied to pure tensor field).

Remark 9. If $r = s = 0$, then from c), d) and e) of Definition 8 we have $\phi_\varphi X(i_Y \eta) = \phi X(i_Y \eta) - X(i_\varphi Y \eta)$ for $i_Y \eta \in \mathfrak{S}_0(M^n)$, which is not well-defined $\phi_\varphi$-operator. Different choices of $Y$ and $\eta$ leading to same function $f = i_Y \eta$ do get the same values. Consider $M^n = R^2$ with standard coordinates $x, y$. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Consider the function $f = 1$. This may be written in many different ways as $i_Y \eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial x}$ or $Y = x \frac{\partial}{\partial x}$. Now the right-hand side of $\phi_\varphi X(i_Y \eta) = \phi X(i_Y \eta) - X(i_\varphi Y \eta)$ is $\phi X(1) - 0 = 0$ in the first case, and $\phi X(1) - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$.

Remark 10. From d) of Definition 8 we have

$\phi_\varphi X Y = [\varphi X, Y] - \varphi [X, Y]$.

By virtue of

$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$

for any $f, g \in \mathfrak{S}_0(M^n)$, we see that $\phi_\varphi X Y$ is linear in $X$, but not $Y$.

Theorem 11. Let $\phi_\varphi$ be the Tachibana operator and the structure $(F^5)^C - \alpha^2 F^C = 0$ defined by Definition 8 and [14], respectively. If $L_Y \varphi = 0$, then all results with respect to $(F^5)^C$ is zero, where $X, Y \in \mathfrak{S}_0(M)$, the complete lifts $X^C, Y^C \in \mathfrak{S}_0(T(M))$ and the vertical lift $X^V, Y^V \in \mathfrak{S}_0(T(M))$.

i) $\phi_{(F^5)^C} X^C Y^C = -\alpha^2 ((L_Y \varphi) X)^C$
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$$
\begin{align*}
\text{ii)} \phi_{(F3)c}^{Xc}Y^{V} & = -a^{2}((LYF)X)^{V} \\
\text{iii)} \phi_{(F3)c}^{XV}Y^{C} & = -a^{2}((LYF)X)^{V} \\
\text{iv)} \phi_{(F3)c}^{XV}Y^{V} & = 0
\end{align*}
$$

Proof. i)

$$
\begin{align*}
\phi_{(F3)c}^{Xc}Y^{C} & = -(LYc(F^{5})^{C})X^{C} \\
& = a^{2}\{-LYc(FX)^{C} + (F)^{C}LYcX^{C}\} \\
& = -a^{2}((LYF)X)^{C}
\end{align*}
$$

ii)

$$
\begin{align*}
\phi_{(F3)c}^{Xc}Y^{V} & = -(LYv(F^{5})^{C})X^{C} \\
& = -LYv(F^{5})^{C}X^{V} + (F^{5})^{C}LYvX^{C} \\
& = a^{2}\{-LYv(FX)^{V} + (F)^{C}LYvX^{C}\} \\
& = -a^{2}((LYF)X)^{V}
\end{align*}
$$

iii)

$$
\begin{align*}
\phi_{(F3)c}^{XV}Y^{C} & = -(LYc(F^{5})^{C})X^{V} \\
& = -LYc(F^{5})^{C}X^{V} + (F^{5})^{C}LYcX^{V} \\
& = a^{2}\{-LYc(FX)^{V} + (F)^{C}LYcX^{V}\} \\
& = -a^{2}((LYF)X)^{V}
\end{align*}
$$

iv)

$$
\begin{align*}
\phi_{(F3)c}^{XV}Y^{V} & = -(LYv(F^{5})^{C})X^{V} \\
& = -LYv(F^{5})^{C}X^{V} + (F^{5})^{C}LYvX^{V} \\
& = 0
\end{align*}
$$

\[\square\]

**Theorem 12.** If $LYF = 0$ for $Y \in M$, then its complete lift $Y^{C}$ to the tangent bundle is an almost holomorphic vector field with respect to the structure $(F^{5})^{C} - a^{2}F^{C} = 0$.

Proof. i)

$$
\begin{align*}
(LYc(F^{5})^{C})X^{C} & = LYc(F^{5})^{C}X^{C} - (F^{5})^{C}LYcX^{C} \\
& = a^{2}\{LYc(FX)^{C} - (F)^{C}LYcX^{C}\} \\
& = a^{2}((LYF)X)^{C}
\end{align*}
$$
\[ (L_Y\circ (F^5)^C)X^V = L_Y\circ (F^5)^C X^V - (F^5)^C L_Y\circ X^V \]
\[ = a^2 \{ L_Y\circ (FX)^V - (F)^C L_Y\circ X^V \} \]
\[ = a^2 \{(LYF)X)^V \}

\[ \square \]

2.3. The Structure \((F^5)^H - a^2 F^H = 0\) on Tangent Bundle \(T(M^n)\).

**Theorem 13.** The Nijenhuis tensor \(N_{(F^5)^H,(F^5)^H}(X^H,Y^H)\) of the horizontal lift of \(F^5\) vanishes if the Nijenhuis tensor of the \(F\) is zero and \((-\hat{R}(FX,FY)u) + (F(\hat{R}(FX,Y)u)) + (F(R(X,FY)u)) - ((F)^2(\hat{R}(X,Y)u))V = 0.\)

**Proof.**

\[ N_{(F^5)^H,(F^5)^H}(X^H,Y^H) = \{(F)^H X^H, (F^5)^H Y^H\} - (F^5)^H \{(F)^H X^H, Y^H\} \]
\[ = (F^5)^H [X^H, (F^5)^H Y^H] + (F^5)^H (F^5)^H [X^H, Y^H] \]
\[ = a^4 \{(FX,FY) - (F)[FX,Y] - (F)[X,FY] - (F)[X,Y]\} \]
\[ = a^4 \{(FX,FY) - (\hat{R}(FX,FY)u) + (F(\hat{R}(FX,Y)u))V \}
\[ = a^4 \{((N_{FF}(X,Y))^H - (\hat{R}(FX,FY)u)V + (F(\hat{R}(FX,Y)u)V) \}
\[ = a^4 \{(\hat{R}(FX,FY)u)V + (F(\hat{R}(FX,Y)u)V \}
\[ = a^4 \{(\hat{R}(X,Y)u)V \} \].

\[ \square \]

If \(N_{FF}(X,Y) = 0\) and \((-\hat{R}(FX,FY)u + (F(\hat{R}(FX,Y)u)) - (F(\hat{R}(X,Y)u)V = 0\), then we get \(N_{(F^5)^H,(F^5)^H}(X^H,Y^H) = 0.\) The theorem is proved.

Where \(\hat{R}\) denotes the curvature tensor of the affine connection \(\hat{\nabla}\) defined by \(\hat{\nabla}_X Y = \nabla_Y X + [X,Y]\) (see [17] p.88-89).

**Theorem 14.** The Nijenhuis tensor \(N_{(F^5)^H,(F^5)^H}(X^H,Y^V)\) of the horizontal lift of \(F^5\) vanishes if the Nijenhuis tensor of the \(F\) is zero and \(\nabla F = 0.\)

**Proof.**

\[ N_{(F^5)^H,(F^5)^H}(X^H,Y^V) = \{(F)^H X^H, (F^5)^H Y^V\} - (F^5)^H \{(F)^H X^H, Y^V\} \]
\[ = (F^5)^H [X^H, (F^5)^H Y^V] + (F^5)^H (F^5)^H [X^H, Y^V] \]
\[ = a^4 \{(FX,FY)^V - (F)[FX,Y]^V - (F)[X,FY]^V \}
\[ = a^4 \{(FX,FY)^V + (\nabla_{FY}FX)^V - (F(\nabla_Y FX))^V \}. \]
Theorem 15. The Nijenhuis tensor $N_{(F^5)^H,(F^5)^H}(X^V,Y^V)$ of the horizontal lift of $F^5$ vanishes.

Proof. Because of $[X^V,Y^V] = 0$ for $X,Y \in M$, easily we get

$$N_{(F^5)^H,(F^5)^H}(X^V,Y^V) = 0.$$ 

Theorem 16. The Sasakian metric $Sg$ is pure with respect to $(F^5)^H$ if $F = a^2I$, where $I$ = identity tensor field of type $(1,1)$.

Proof. $S(\tilde{X},\tilde{Y}) = Sg((F^5)^H \tilde{X},\tilde{Y}) - Sg(\tilde{X},(F^5)^H \tilde{Y})$ if $S(\tilde{X},\tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form $X^V,Y^V$ or $X^H,Y^H$ then $S = 0$.

i) \[ S(X^V,Y^V) = Sg((F^5)^H X^V,Y^V) - Sg(X^V,(F^5)^H Y^V) \]

\[ = a^2\{Sg((FX)^V,Y^V) - Sg(X^V,(FY)^V)\} \]

\[ = a^2\{(g(FX),Y)^V - (g(X,FY))^V\} \]

ii) \[ S(X^V,Y^H) = Sg((F^5)^H X^V,Y^H) - Sg(X^V,(F^5)^H Y^H) \]

\[ = -a^2Sg(X^V,(FY)^H) \]

\[ = 0 \]

iii) \[ S(X^H,Y^H) = Sg((F^5)^H X^H,Y^H) - Sg(X^H,(F^5)^H Y^H) \]

\[ = a^2\{Sg((FX)^H,Y^H) - Sg(X^H,(FY)^H)\} \]

\[ = a^2\{(g(FX),Y)^V - (g(X,(FY)^H))^V\} \]

Theorem 17. Let $\phi_{\mu}$ be the Tachibana operator and the structure $(F^5)^H - a^2F^H = 0$ defined by Definition $\mathbb{S}$ and $\mathbb{I}$, respectively. If $L_YF = 0$ and $F = a^2I$, then all results with respect to $(F^5)^H$ is zero, where $X,Y \in \mathbb{S}_{(0)}(M)$, the horizontal lifts $X^H,Y^H \in \mathbb{S}_{(0)}(T(M^n))$ and the vertical lift $X^V,Y^V \in \mathbb{S}_{(0)}(T(M^n))$

i) $\phi_{(F^5)^H}X^H = a^2\{(-((L_YF)X)^H + (\hat{R}(Y,FX)u)^V - (F(\hat{R}(Y,X)u))^V\}$

ii) $\phi_{(F^5)^H}Y^V = a^2\{(-((L_YF)X)^V + ((\nabla_YF)X)^V\}$,
iii) \( \phi_{(F^5)H} X V Y^H = a^2 \{ - ((L_Y F) X)^V - (\nabla_{FX} Y)^V + (F (\nabla_X Y))^V \} \),

iv) \( \phi_{(F^5)H} X V Y^V = 0 \).

Proof. i)

\[
\begin{align*}
\phi_{(F^5)H} X V Y^H &= -(L_{YH} (F^5)^H) X^H \\
&= -L_{YV} (F^5)^H X^H + (F^5)^H L_{YH} X^H \\
&= -a^2 [Y, FX]^H + a^2 \gamma \tilde{R}[Y, FX] \\
&+ a^2 (F [Y, X])^H - a^2 (F)^H (\tilde{R}(Y, X) u)^V \\
&= -a^2 \{ - ((L_Y F) X)^H + (\tilde{R}(Y, FX) u)^V \\
&- (F(\tilde{R}(Y, X) u))^V \} \\
&= -L_{YV} (F^5)^H X^H + (F^5)^H L_{YH} X^H \end{align*}
\]

ii)

\[
\begin{align*}
\phi_{(F^5)H} X H Y^V &= -(L_{YV} (F^5)^H) X^H \\
&= -L_{YV} (F^5 X)^H + (F^5)^H L_{YV} X^H \\
&= -a^2 [Y, FX]^V + a^2 (\nabla_Y FX)^V \\
&+ a^2 (F [Y, X])^V - a^2 (F (\nabla_Y X))^V \\
&= a^2 \{ - ((L_Y F) X)^V + (F (\nabla_Y X))^V \} \\
\end{align*}
\]

iii)

\[
\begin{align*}
\phi_{(F^5)H} X V Y^H &= -(L_{YH} (F^5)^H) X^V \\
&= -L_{YH} (F^5 X)^V + (F^5)^H L_{YH} X^V \\
&= a^2 [Y, FX]^V - a^2 (\nabla_{FX} Y)^V \\
&+ a^2 (F [Y, X])^H + a^2 (F (\nabla_X Y))^V \\
&= a^2 \{ - ((L_Y F) X)^V - (\nabla_{FX} Y)^V + (F (\nabla_X Y))^V \} \\
\end{align*}
\]

iv)

\[
\begin{align*}
\phi_{(F^5)H} X V Y^V &= -(L_{YV} (F^5)^H) X^V \\
&= -a^2 L_{YV} (FX)^V + a^2 (F)^H L_{YV} X^V \\
&= 0 \]
\]

2.4. The Structure \((F^5)^H - a^2 F^H = 0\) on Cotangent Bundle. In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of \(F_a(5, 1)\)-structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of \(F_a(5, 1)\)-structure.
in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of the structure.

Let $F, G$ be two tensor fields of type $(1, 1)$ on the manifold $M$. If $F^H$ denotes the horizontal lift of $F$, we have

$$F^H G^H + G^H F^H = (F G + G F)^H$$

Taking $F$ and $G$ identical, we get

$$(F^H)^2 = (F^2)^H$$

Multiplying both sides by $F^H$ and making use of the same (20), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H$$

and so on. Thus

$$(F^H)^5 = (F^5)^H$$

Since $F$ gives on $M$ the $F_{(5,1)}$ structure, we have

$$F^5 - a^2 F = 0.$$ (23)

Taking horizontal lift, we obtain

$$(F^5)^H - a^2 F^H = 0.$$ (24)

In view of (23), we can write

$$(F^H)^5 - a^2 F^H = 0.$$ (25)

**Theorem 18.** The Nijenhuis tensor $N_{(F^5),H,(F^5),H}(X^H, Y^H)$ of the horizontal lift $F^5$ vanishes if $F = a^2 I$ on $M$.

**Proof.** The Nijenhuis tensor $N_{(F^5),H}(X^H, Y^H)$ for the horizontal lift of $F^5$ is given by

$$N_{(F^5),H,(F^5),H}(X^H, Y^H) = \begin{align*}
(F^5)^H [X^H, (F^5)^H Y^H] - (F^5)^H [(F^5)^H X^H, Y^H] \\
- (F^5)^H [X^H, (F^5)^H Y^H] + (F^5)^H [(F^5)^H X^H, Y^H] \\
- a^4 [([F^H X^H, (F^H)^H Y^H] - (F^H)^H [(F^H)^H X^H, Y^H] \\
- (F^H)^H [X^H, (F^H)^H Y^H] + (F^H)^H [(F^H)^H X^H, Y^H]) \\
+ F^2 [X, Y]^H + \gamma \{R(FX, FY) - R((FX), Y) F \\
- R(X, FY) F^2 + R(X, Y) F^2 \}]
\end{align*}$$

Let us suppose that $F = a^2 I$ on $M$. Thus, the equation becomes

$$N_{(F^5),H,(F^5),H}(X^H, Y^H) = a^4 \{[[X, Y] - \{X, Y\} - [X, Y] + [X, Y]]^H \\
+ \gamma \{R(X, Y) - R(X, Y) - R(X, Y) + R(X, Y)\}.\$$
Therefore, it follows
\[ N_{(F^5)H,(F^5)H}(X^H, Y^H) = 0 \]
\[
\square
\]

**Theorem 19.** The Nijenhuis tensor \( N_{(F^5)H,(F^5)H}(X^H, \omega^V) \) of the horizontal lift \( F^5 \) vanishes if \( \nabla F = 0 \).

**Proof.**

\[
N_{(F^5)H,(F^5)H}(X^H, \omega^V) = \left( (F^5)^H X^H, (F^5)^H \omega^V \right) - (F^5)^H \left( (F^5)^H X^H, \omega^V \right)
\]

\[
- (F^5)^H \left( X^H, (F^5)^H \omega^V \right) + (F^5)^H \left( F^5)^H \left[ X^H, \omega^V \right] \right)
\]

\[
= a^4 \left\{ (\nabla_{FX} (\omega \circ F)) \omega^V - ((\nabla_{FX} \circ F)) \bigg( (\nabla_X \omega) \circ F \bigg) \right\}^V
\]

\[
- (\bigg( (\nabla_X \omega) \circ F \bigg) \bigg( (\nabla X \omega) \circ F \bigg) \bigg) V
\]

where \( F \in \mathcal{F}_1(M), X \in \mathcal{X}_0(M), \omega \in \mathcal{X}_0^0(M) \). The theorem is proved.

\[
\square
\]

**Theorem 20.** The Nijenhuis tensor \( N_{(F^5)H,(F^5)H}(\omega^V, \theta^V) \) of the horizontal lift \( F^5 \) vanishes.

**Proof.** Because of \( [\omega^V, \theta^V] = 0 \) and \( \omega \circ F \in \mathcal{X}_0(M) \) on \( T^*(M^n) \), the equation becomes

\[
N_{(F^5)H,(F^5)H}(\omega^V, \theta^V) = 0.
\]

\[
\square
\]

**Theorem 21.** Let \( (F^5)^H \) be a tensor field of type \((1, 1)\) on \( T^*(M^n) \). If the Tachibana operator \( \phi^H \) applied to vector and covector fields according to horizontal lifts of \( F^5 \) defined by \([\mathcal{L}]\) on \( T^*(M^n) \), then we get the following results.

\[
i) \phi_{(F^5)H}(X \circ Y)^H = a^2 \left\{ -((L_X Y)^H \circ F) \right\} V
\]

\[
+ \left\{ (pR(Y, F X))^V \right\} V
\]

\[
ii) \phi_{(F^5)H}(X \circ \omega)^V = a^2 \left\{ (\nabla_{FX} \omega)^V - ((\nabla_X \omega) \circ F)^V \right\} V
\]

\[
iii) \phi_{(F^5)H}(\omega \circ X)^H = -a^2 (\omega \circ (\nabla_X F))^V,
\]

\[
iv) \phi_{(F^5)H}(\omega \circ \theta)^V = 0,
\]

where horizontal lifts \( X^H, Y^H \in \mathcal{X}_0^0(T^*(M^n)) \) of \( X, Y \in \mathcal{X}_0^0(M) \) and the vertical lift \( \omega^V, \theta^V \in \mathcal{X}_0^0(T^*(M^n)) \) of \( \omega, \theta \in \mathcal{X}_0^0(M) \) are given, respectively.

**Proof.**

\[
i)
\phi_{(F^5)H}(X \circ Y)^H = -((L_X Y)^H \circ F) \]

\[
\square
\]
\[ \phi_{(F^5)H} X^H \omega^V = -(L_{X^H} (F^5)H) X^H \]
\[ = -L_{X^H} (F^5)H X^H + (F^5)H L_{X^H} X^H \]
\[ = -a^2 L_{X^H} (F^5)H X^H - a^2 (F^5)H (\nabla X \omega) V \]
\[ = a^2 \{(\nabla F X \omega) V - ((\nabla X \omega) \circ F) V \}, \]

\[ \phi_{(F^5)H} (F^5)H V = \frac{1}{2} \frac{(F \omega (X) V)}{V} + (F^5)H L_{(F \omega (X) V)} \]
\[ = 0 \]

**Definition 22.** A Sasakian metric \( S \) is defined on \( T^*(M^n) \) by the three equations

\[ S g(\omega^V, \theta^V) = (g^{-1}(\omega, \theta)) V = g^{-1}(\omega, \theta) \circ \pi, \] (26)

\[ S g(\omega^V, Y^H) = 0, \] (27)

\[ S g(X^H, Y^H) = (g(X, Y)) V = g(X, Y) \circ \pi. \] (28)

For each \( x \in M^n \) the scalar product \( g^{-1} = (g^{ij}) \) is defined on the cotangent space \( \pi^{-1}(x) = T^*_x (M^n) \) by

\[ g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j, \] (29)

where \( X, Y \in \mathfrak{X}(M^n) \) and \( \omega, \theta \in \Omega^1(M^n) \). Since any tensor field of type (0, 2) on \( T^*(M^n) \) is completely determined by its action on vector fields of type \( X^H \) and \( \omega^V \) (see \[17\], p.280), it follows that \( S \) is completely determined by equations (26), (27) and (28).

**Theorem 23.** Let \( (T^*(M^n), S) \) be the cotangent bundle equipped with Sasakian metric \( S \) and a tensor field \( (F^5)H \) of type \( (1, 1) \) defined by (25). Sasakian metric \( S \) is pure with respect to \( (F^5)H \) if \( F = a^2 I \) (\( I \) = identity tensor field of type \( (1, 1) \)).
Proof. We put
\[ S(\tilde{X}, \tilde{Y}) = S(g((F^5)^H \tilde{X}, \tilde{Y}) - S g(\tilde{X}, (F^5)^H \tilde{Y}). \]

If \( S(\tilde{X}, \tilde{Y}) = 0 \), for all vector fields \( \tilde{X} \) and \( \tilde{Y} \) which are of the form \( \omega^V, \theta^V \) or \( X^H, Y^H \), then \( S = 0 \). By virtue of \( (F^5)^H - a^2 F^H = 0 \) and \([26], [27], [28]\), we get
\[ S(\omega^V, \theta^V) = S g((F^5)^H \omega^V, \theta^V) - S g(\omega^V, (F^5)^H \theta^V) = S g((a^2 F)^H \omega^V, \theta^V) - S g(\omega^V, (a^2 F)^H \theta^V) = a^2 S g((\omega \circ F)^V, \theta^V) - S g(\omega^V, (\theta \circ F)^V)). \]

\[ S(X^H, \theta^V) = S g((F^5)^H X^H, \theta^V) - S g(X^H, (F^5)^H \theta^V) = S g((a^2 F)^H X^H, \theta^V) - S g(X^H, (a^2 F)^H \theta^V) = a^2 S g((FX)^H, \theta^V) - S g(X^H, (\omega \circ F)^V)) = 0. \]

\[ S(X^H, Y^H) = S g((F^5)^H X^H, Y^H) - S g(X^H, (F^5)^H Y^H) = S g((a^2 F)^H X^H, Y^H) - S g(X^H, (a^2 F)^H Y^H) = a^2 S g((FX)^H, Y^H) - S g(X^H, (FY)^H)). \]

Thus, \( F = a^2 I \), then \( S g \) is pure with respect to \( (F^5)^H \). \( \square \)

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