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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 1, Pages 366–381 (2021)
DOI: 10.31801/cfsuasmas.766788
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: July 8, 2020; Accepted: January 6, 2021

ON THE LIFTS OF $F_a(5,1)-$STRUCTURE ON TANGENT AND COTANGENT BUNDLE

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Abstract. This paper consist of three main sections. In the first part, we obtain the complete lifts of the $F_a(5,1)-$structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)-$structure. Later we get the conditions of to be the almost holomorfic vector field with respect to the complete lifts of $F_a(5,1)-$structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)-$structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)-$structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)-$structure in cotangent bundle $T^*(M^n)$.

1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [17]. Also, the idea of $F-$structure manifold on a differentiable manifold developed by Yano [14], Ishıhara and Yano [7], Goldberg [6] and among others. Moreover, Yano and Patterson [15, 16] studied on the horizontal and complete lifts from a differentiable manifold M^n of class C^∞ to its cotangent bundles. Andreu has studied the structure defined by a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying $F^5 + F = 0$ [1]. Later Ram Nivas and C.S. Prasad [11] studied on more form $F_a(5,1)-$structure. This paper consist of three main sections. In the first part,

2020 *Mathematics Subject Classification.* Primary 15A72, 53A45; Secondary 47B47, 53C15.
Keywords and phrases. Integrability, Tachibana operators, lifts, Sasakian metric, tangent bundle, cotangent bundle.

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we obtain the complete lifts of the $F_a(5,1)-$structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)-$structure. Later we get the conditions of to be the almost holomorfic vector field with respect to the complete lifts of $F_a(5,1)-$structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)-$structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)-$structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)-$structure in cotangent bundle $T^*(M^n)$.

Let M^n be an $n-$dimensional differentiable manifold of class C^∞. Suppose there exist on M^n, a $(1,1)$ tensor field $F(\neq 0)$ satisfying [11]

$$F^5 - a^2F = 0, \tag{1}$$

where a is a complex number not equal to zero. If $a = i$ where $i = \sqrt{-1}$, our structure takes the form $F^5 + F = 0$ studied by Andreou [1].

Let us define on M^n, the operators l and m as follows :

$$l = (F^4/a^2) \text{ and } m = I - (F^4/a^2). \tag{2}$$

I being unit tensor field.

In view of equations (1) and (2), we have

$$l^2 = l,\ m^2 = m \text{ and } l + m = I. \tag{3}$$

For a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying (1) the operators l and m defined by (2), when applied to the tangent space of M^n at a point, are complementary projection operators.

Thus there exist complementary distributions L and M corresponding to the projection operators l and m respectively. If the rank of F is constant every where or equal to r, the dimensions of L and M are r and $n-r$ respectively [10]. Us call such a structure as $F_a(5,1)-$structure of rank r [11].

For a tensor field $F(\neq 0)$ of type $(1,1)$ admitting $F_a(5,1)-$structure and for the projection operators l and m given by (2) we have

$$Fl = lF = F,\ Fm = mF = 0. \tag{4}$$

and

$$F^2l = lF^2 = F^2,\ F^2m = mF^2 = 0. \tag{5}$$

In the manifold M^n endowed with $F_a(5,1)-$structure, the $(1,1)$ tensor field $\tilde{F}$ given by $\tilde{F} = l - m = (2F^4/a^2) - I$ gives an almost product structure [9].

1.1. **Complete Lift of $F_a(5,1)-$Structure on Tangent Bundle.** Let M^n be an $n-$dimensional differentiable manifold of class C^∞ and $T_P(M^n)$ the tangent space at a point p of M^n and

$$T(M^n) = \underset{p\in M^n}{U} T_P(M^n)$$

is the tangent bundle over the manifold M^n.

Let us denote by $T^r_s(M^n)$, the set of all tensor fields of class C^∞ and of type (r,s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift of F^C of an element of $T^1_1(M^n)$ with local components F^h_i has components of the form [16]

$$F^C = \begin{bmatrix} F^h_i & 0 \\ \delta^h_i & F^h_i \end{bmatrix}. \tag{6}$$

Now we obtain the following results on the complete lift of F satisfying $F^5 - a^2F = 0$.

Let $F, G \in T^1_1(M^n)$. Then we have [16]

$$(FG)^C = F^C G^C. \tag{7}$$

Replacing G by F in (7) we obtain

$$(FF)^C = F^C F^C \text{ or } (F^2)^C = (F^C)^2. \tag{8}$$

Now putting $G = F^4$ in (7) since G is $(1,1)$ tensor field therefore F^4 is also $(1,1)$ so we obtain $(FF^4)^C = F^C(F^4)^C$ which in view of (8) becomes

$$(F^5)^C = (F^C)^5. \tag{9}$$

Taking complete lift on both sides of equation $F^5 - a^2F = 0$ we get

$$(F^5)^C - (a^2F)^C = 0$$

which in consequence of equation (9) gives

$$(F^C)^5 - a^2F^C = 0. \tag{10}$$

Let F satisfying $(1,1)$ be an $F-$structure of rank r in M^n. Then the complete lifts $l^C = (F^4)^C$ of l and $m^C = I - (F^4)^C$ of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions L^C and M^C determined by l^C and m^C, respectively.

1.2. **Horizontal Lift of $F_a(5,1)-$Structure on Tangent Bundle.** Let F^h_i be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}^A_B$ in $\pi^{-1}(U)$ are given by

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F^h_i & 0 \\ -\Gamma^h_t F^t_i + \Gamma^t_i F^h_t & F^h_i \end{pmatrix}.$$

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H = F^H G^H. \tag{11}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H. \tag{12}$$

Multiplying both sides by F^H and making use of the same (12), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H,\ (F^H)^5 = (F^5)^H. \tag{13}$$

Taking horizontal lift on both sides of equation $F^5 - a^2F = 0$ we get

$$(F^5)^H - (a^2F)^H = 0$$

view of (13), we can write

$$(F^H)^5 - a^2F^H = 0. \tag{14}$$

2. Main Results

2.1. The Nijenhuis Tensor $N_{(F^5)^C(F^5)^C}(X^C, Y^C)$ of the Complete Lift F^5 on Tangent Bundle $T(M^n)$.

Definition 1. *Let F be a tensor field of type $(1,1)$ admitting $F_a(5,1)-$structure in M^n. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^n is given by*

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{15}$$

for any $X, Y \in \Im_0^1(M^n)$ [2,12,13]. The condition of $N_F(X,Y) = N(X,Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k\partial_k = (F_i^s\partial_s^k F_j^k - F_j^l\partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \Im_1^1(M^n)$.

Definition 2. *Let X and Y be any vector fields on a Riemannian manifold (M^n, g), we have [17]*

$$\begin{aligned} \left[X^H, Y^H\right] &= [X,Y]^H - (R(X,Y)u)^V, \\ \left[X^H, Y^V\right] &= (\nabla_X Y)^V, \\ \left[X^V, Y^V\right] &= 0, \end{aligned} \tag{16}$$

where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{17}$$

In particular, we have the vertical spray u^V and the horizontal spray u^H on $T(M^n)$ defined by

$$u^V = u^i(\partial_i)^V = u^i\partial_{\bar{i}},\ u^H = u^i(\partial_i)^H = u^i\delta_i, \tag{18}$$

where $\delta_i = \partial_i - u^j\Gamma_{ji}^s\partial_{\bar{s}}$. u^V is also called the canonical or Liouville vector field on $T(M^n)$.

Theorem 3. *The Nijenhuis tensor* $N_{(F^5)^C(F^5)^C}\left(X^C, Y^C\right)$ *of the complete lift of* F^5 *vanishes if the Nijenhuis tensor of the* F *is zero.*

Proof. In consequence of Definition 1 the Nijenhuis tensor of $\left(F^5\right)^C$ is given by

$$\begin{aligned} N_{(F^5)^C(F^5)^C}\left(X^C, Y^C\right) &= [\left(F^5\right)^C X^C, \left(F^5\right)^C Y^C] - \left(F^5\right)^C [\left(F^5\right)^C X^C, Y^C] \\ &\quad - \left(F^5\right)^C [X^C, \left(F^5\right)^C Y^C] + \left(F^5\right)^C \left(F^5\right)^C \left[X^C, Y^C\right] \\ &= a^4\{[(FX)^C, (FY)^C] - (F)^C [(FX)^C, Y^C] \\ &\quad - (F)^C [X^C, (FY)^C] + (F)^C (F)^C \left[X^C, Y^C\right]\} \\ &= a^4\{[FX, FY] - F[FX, Y] \\ &\quad -F[X, FY] + F^2 [X, Y]\}^C \\ &= a^4 N(X, Y)^C \end{aligned}$$

□

Theorem 4. *The Nijenhuis tensor* $N_{(F^5)^C(F^5)^C}\left(X^C, Y^V\right)$ *of the complete lift of* F^5 *vanishes if the Nijenhius tensor* F *is zero.*

Proof.

$$\begin{aligned} N_{(F^5)^C(F^5)^C}\left(X^C, Y^V\right) &= [\left(F^5\right)^C X^C, \left(F^5\right)^C Y^V] - \left(F^5\right)^C [\left(F^5\right)^C X^C, Y^V] \\ &\quad - \left(F^5\right)^C [X^C, \left(F^5\right)^C Y^V] + \left(F^5\right)^C \left(F^5\right)^C \left[X^C, Y^V\right] \\ &= a^4\{[(FX)^C, (FY)^V] - (F)^C [(FX)^C, Y^V] \\ &\quad - (F)^C [X^C, (FY)^V] + \left(F^2\right)^C [X, Y]^V\} \\ &= a^4\{[FX, FY]^V - (F[FX, Y])^V \\ &\quad - (F[X, FY])^V - \left(F^2 [X, Y]\right)^V\} \\ &= a^4 N(X, Y)^V \end{aligned}$$

□

Theorem 5. *The Nijenhuis tensor* $N_{(F^5)^C(F^5)^C}\left(X^V, Y^V\right)$ *of the complete lift of* F^5 *vanishes.*

Proof. Thus $\left[X^V, Y^V\right] = 0$ for all $X, Y \in \Im_0^1(M^n)$, easily we get

$$N_{(F^5)^C(F^5)^C}\left(X^V, Y^V\right) = 0.$$

□

2.2. The Purity Conditions of Sasakian Metric with Respect to $(F^5)^C$ on $T(M^n)$.

Definition 6. *The Sasaki metric ${}^Sg$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on M as follows:*

$$\begin{aligned} {}^Sg\left(X^H,Y^H\right) &= g\left(X,Y\right), \\ {}^Sg\left(X^H,Y^V\right) &= {}^Sg\left(X^V,Y^H\right)=0, \\ {}^Sg\left(X^V,Y^V\right) &= g\left(X,Y\right) \end{aligned} \tag{19}$$

for all $X,Y\in\Im_0^1\left(M^n\right)$.

Theorem 7. *The Sasaki metric ${}^Sg$ is pure with respect to $\left(F^5\right)^C$ if $\nabla F=0$ and $F=a^2I$, where I=identity tensor field of type $(1,1)$.*

Proof. $S(\widetilde{X},\widetilde{Y})={}^S g((F^5)^C\,\widetilde{X},\widetilde{Y})-{}^S g(\widetilde{X},(F^5)^C\,\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.

i)

$$\begin{aligned} S\left(X^V,Y^V\right) &= {}^Sg((F^5)^C\,X^V,Y^V)-{}^S g(X^V,(F^5)^C\,Y^V) \\ &= a^2\{{}^Sg((FX)^V\,,Y^V)-{}^S g(X^V,(FY)^V)\} \\ &= a^2\{(g\left(FX,Y\right))^V-(g\left(X,FY\right))^V\} \end{aligned}$$

ii)

$$\begin{aligned} S\left(X^V,Y^H\right) &= {}^Sg((F^5)^C\,X^V,Y^H)-{}^S g(X^V,(F^5)^C\,Y^H) \\ &= -a^2\ {}^Sg(X^V,(FY)^H+(\nabla_\gamma F)\,Y^H) \\ &= -a^2\ {}^Sg\left(X^V,(\nabla_\gamma F)\,Y^H\right) \\ &= -a^2\ {}^Sg(X^V,(((\nabla F)\,u)\,Y)^V) \\ &= -a^2(\ g\left(X,((\nabla F)\,u)\,Y\right)^V) \end{aligned}$$

iii)

$$\begin{aligned} S\left(X^H,Y^H\right) &= {}^Sg((F^5)^C\,X^H,Y^H)-{}^S g(X^H,(F^5)^C\,Y^H) \\ &= a^2\ {}^Sg((F)^C\,X^H,Y^H)-a^2\ {}^Sg(X^H,(F)^C\,Y^H) \\ &= a^2\ {}^Sg((FX)^H+(\nabla_\gamma F)\,X^H,Y^H) \\ &\quad -a^2\ {}^Sg(X^H,(FY)^H+(\nabla_\gamma F)\,Y^H) \\ &= a^2\{g\left((FX)\,,Y\right)^V-g\left(X,(FY)\right)^V\} \end{aligned}$$

□

Definition 8. *Let $\varphi \in \Im_1^1(M^n)$, and $\Im(M^n) = \sum_{r,s=0}^{\infty} \Im_s^r(M^n)$ be a tensor alebra over R. A map $\phi_\varphi \mid_{r+s\rangle 0}$: $\overset{*}{\Im}(M^n) \to \Im(M^n)$ is called as Tachibana operatör or $\phi_\varphi$operatör on M^n if*

a) ϕ_φ is linear with respect to constant coefficient,

b) $\phi_\varphi : \overset{*}{\Im}(M^n) \to \Im_{s+1}^r(M^n)$ for all r and s,

c) $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in \overset{*}{\Im}(M^n)$,

d) $\phi_{\varphi X} Y = -(L_Y \varphi)X$ for all $X, Y \in \Im_0^1(M^n)$, where L_Y is the Lie derivation with respect to Y (see [3, 5, 8]),

e)

$$\begin{aligned} (\phi_{\varphi X}\eta)Y &= (d(\imath_Y\eta))(\varphi X) - (d(\imath_Y(\eta o \varphi)))X + \eta((L_Y\varphi)X) \\ &= \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta) + \eta((L_Y\varphi)X) \end{aligned}$$

for all $\eta \in \Im_1^0(M^n)$ and $X, Y \in \Im_0^1(M^n)$, where $\imath_Y\eta = \eta(Y) = \eta \overset{C}{\otimes} Y, \overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r, s) on M^n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C [2, 4, 12](see [13] for applied to pure tensor field).

Remark 9. *If $r = s = 0$, then from c), d) and e) of Definition8 we have $\phi_{\varphi X}(\imath_Y\eta) = \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta)$ for $\imath_Y\eta \in \Im_0^0(M^n)$, which is not well-defined $\phi_\varphi-$operator. Different choices of Y and η leading to same function $f = \imath_Y\eta$ do get the same values. Consider $M^n = R^2$ with standard coordinates x, y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\imath_Y\eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Nov the right-hand side of $\phi_{\varphi X}(\imath_Y\eta) = \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ [12].*

Remark 10. *From d) of Definition8 we have*

$$\phi_{\varphi X} Y = [\varphi X, Y] - \varphi[X, Y].$$

By virtue of

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

for any $f, g \in \Im_0^0(M^n)$, we see that $\phi_{\varphi X} Y$ is linear in X, but not Y [12].

Theorem 11. *Let ϕ_φ be the Tachibana operator and the structure $\left(F^5\right)^C - a^2 F^C = 0$ defined by Definition 8 and (10), respectively. If $L_Y F = 0$, then all results with respect to $\left(F^5\right)^C$ is zero, where $X, Y \in \Im_0^1(M)$, the complete lifts $X^C, Y^C \in \Im_0^1(T(M))$ and the vertical lift $X^V, Y^V \in \Im_0^1(T(M))$.*

$$i)\ \phi_{(F^5)^C X^C} Y^C = -a^2 \left((L_Y F) X\right)^C$$

$$
\begin{aligned}
ii)\ \phi_{(F^5)^C X^C} Y^V &= -a^2 \left((L_Y F)\, X \right)^V \\
iii)\ \phi_{(F^5)^C X^V} Y^C &= -a^2 \left((L_Y F)\, X \right)^V \\
iv)\ \phi_{(F^5)^C X^V} Y^V &= 0
\end{aligned}
$$

Proof. *i)*

$$
\begin{aligned}
\phi_{(F^5)^C X^C} Y^C &= -(L_{Y^C} \left(F^5\right)^C) X^C \\
&= a^2 \{ -L_{Y^C} \left(FX\right)^C + \left(F\right)^C L_{Y^C} X^C \} \\
&= -a^2 \left((L_Y F)\, X \right)^C
\end{aligned}
$$

ii)

$$
\begin{aligned}
\phi_{(F^5)^C X^C} Y^V &= -(L_{Y^V} \left(F^5\right)^C) X^C \\
&= -L_{Y^V} \left(F^5\right)^C X^C + \left(F^5\right)^C L_{Y^V} X^C \\
&= a^2 \{ -L_{Y^V} \left(FX\right)^C + \left(F\right)^C L_{Y^V} X^C \} \\
&= -a^2 \left((L_Y F)\, X \right)^V
\end{aligned}
$$

iii)

$$
\begin{aligned}
\phi_{(F^5)^C X^V} Y^C &= -(L_{Y^C} \left(F^5\right)^C) X^V \\
&= -L_{Y^C} \left(F^5\right)^C X^V + \left(F^5\right)^C L_{Y^C} X^V \\
&= a^2 \{ -L_{Y^C} \left(FX\right)^V + \left(F\right)^C L_{Y^C} X^V \} \\
&= -a^2 \left((L_Y F)\, X \right)^V
\end{aligned}
$$

iv)

$$
\begin{aligned}
\phi_{(F^5)^C X^V} Y^V &= -(L_{Y^V} \left(F^5\right)^C) X^V \\
&= -L_{Y^V} \left(F^5\right)^C X^V + \left(F^5\right)^C L_{Y^V} X^V \\
&= 0
\end{aligned}
$$

□

Theorem 12. *If $L_Y F = 0$ for $Y \in M$, then its complete lift Y^C to the tangent bundle is an almost holomorfic vector field with respect to the structure $\left(F^5\right)^C - a^2 F^C = 0$.*

Proof. *i)*

$$
\begin{aligned}
(L_{Y^C} \left(F^5\right)^C) X^C &= L_{Y^C} \left(F^5\right)^C X^C - \left(F^5\right)^C L_{Y^C} X^C \\
&= a^2 \{ L_{Y^C} \left(FX\right)^C - \left(F\right)^C L_{Y^C} X^C \} \\
&= a^2 \left((L_Y F)\, X \right)^C
\end{aligned}
$$

ii)

$$\begin{aligned}(L_{Y^C}\left(F^5\right)^C)X^V &= L_{Y^C}\left(F^5\right)^C X^V-\left(F^5\right)^C L_{Y^C}X^V\\ &= a^2\{L_{Y^C}\left(FX\right)^V-\left(F\right)^C L_{Y^C}X^V\}\\ &= a^2\left(\left(L_Y F\right)X\right)^V\end{aligned}$$

□

2.3. **The Structure** $\left(F^5\right)^H-a^2F^H=0$ **on Tangent Bundle** $T\left(M^n\right)$.

Theorem 13. *The Nijenhuis tensor* $N_{(F^5)^H(F^5)^H}\left(X^H,Y^H\right)$ *of the horizontal lift of* F^5 *vanishes if the Nijenhuis tensor of the* F *is zero and* $\{-(\hat{R}\left(FX,FY\right)u)+(F(\hat{R}\left(FX,Y\right)u))+(F\left(R\left(X,FY\right)u\right))-((F)^2\left(\hat{R}\left(X,Y\right)u\right))\}^V=0$.

Proof.

$$\begin{aligned}N_{(F^5)^H(F^5)^H}\left(X^H,Y^H\right) &= [\left(F^5\right)^H X^H,\left(F^5\right)^H Y^H]-\left(F^5\right)^H[\left(F^5\right)^H X^H,Y^H]\\ &\quad -\left(F^5\right)^H[X^H,\left(F^5\right)^H Y^H]+\left(F^5\right)^H\left(F^5\right)^H\left[X^H,Y^H\right]\\ &= a^4\{([FX,FY]-(F)\left[FX,Y\right]\\ &\quad -(F)\left[X,FY\right]-(F)\left(F\right)\left[X,Y\right])^H\\ &\quad -(\hat{R}\left(FX,FY\right)u)^V+(F(\hat{R}\left(FX,Y\right)u))^V\\ &\quad +(F(\hat{R}\left(X,FY\right)u))^V-((F)^2\left(\hat{R}\left(X,Y\right)\right)u)^V\}\\ &= a^4\{(N_{FF}\left(X,Y\right))^H-(\hat{R}\left(FX,FY\right)u)^V\\ &\quad +(F(\hat{R}\left(FX,Y\right)u))^V+(F(\hat{R}\left(X,FY\right)u))^V\\ &\quad -((F)^2\left(\hat{R}\left(X,Y\right)u\right))^V\}.\end{aligned}$$

□

If $N_{FF}\left(X,Y\right)=0$ and $\{-\hat{R}\left(FX,FY\right)u+(F(\hat{R}\left(FX,Y\right)u))+(F(\hat{R}\left(X,FY\right)u))-((F)^2\left(\hat{R}\left(X,Y\right)u\right))\}^V=0$, then we get $N_{(F^5)^H(F^5)^H}\left(X^H,Y^H\right)=0$. The theorem is proved.

Where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_X Y=\nabla_Y X+[X,Y]$ (see [17] p.88-89).

Theorem 14. *The Nijenhuis tensor* $N_{(F^5)^H(F^5)^H}\left(X^H,Y^V\right)$ *of the horizontal lift of* F^5 *vanishes if the Nijenhuis tensor of the* F *is zero and* $\nabla F=0$.

Proof.

$$\begin{aligned}N_{(F^5)^H(F^5)^H}\left(X^H,Y^V\right) &= [\left(F^5\right)^H X^H,\left(F^5\right)^H Y^V]-\left(F^5\right)^H[\left(F^5\right)^H X^H,Y^V]\\ &\quad -\left(F^5\right)^H[X^H,\left(F^5\right)^H Y^V]+\left(F^5\right)^H\left(F^5\right)^H\left[X^H,Y^V\right]\\ &= a^4\{[FX,FY]^V-\left(F\left[FX,Y\right]\right)^V-\left(F\left[X,FY\right]\right)^V\\ &\quad +((F)^2\left[X,Y\right])^V+\left(\nabla_{FY}FX\right)^V-\left(F\left(\nabla_Y FX\right)\right)^V\end{aligned}$$

$$\begin{aligned} & -\left(F\left(\nabla_{FY}X\right)\right)^V + \left((F)^2\nabla_Y X\right)^V\} \\ = \; & a^4\{\left(N_{FF}\left(X,Y\right)\right)^V + \left(\nabla_{FY}F\right)X - \left(F\left(\left(\nabla_Y F\right)X\right)\right)^V\} \end{aligned}$$

□

Theorem 15. *The Nijenhuis tensor* $N_{(F^5)^H(F^5)^H}\left(X^V,Y^V\right)$ *of the horizontal lift of* F^5 *vanishes.*

Proof. Because of $\left[X^V,Y^V\right]=0$ for $X,Y\in M$, easily we get

$$N_{(F^5)^H(F^5)^H}\left(X^V,Y^V\right)=0.$$

□

Theorem 16. *The Sasakian metric* ${}^Sg$ *is pure with respect to* $\left(F^5\right)^H$ *if* $F=a^2I$, *where* I *=identity tensor field of type* $(1,1)$.

Proof. $S(\widetilde{X},\widetilde{Y}) ={}^S g(\left(F^5\right)^H\widetilde{X},\widetilde{Y}) -{}^S g(\widetilde{X},\left(F^5\right)^H\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.

i)

$$\begin{aligned} S\left(X^V,Y^V\right) &= {}^Sg(\left(F^5\right)^H X^V,Y^V) -{}^S g(X^V,\left(F^5\right)^H Y^V) \\ &= a^2\{{}^Sg(\left(FX\right)^V,Y^V) -{}^S g(X^V,\left(FY\right)^V)\} \\ &= a^2\{\left(g\left(FX,Y\right)\right)^V - \left(g\left(X,FY\right)\right)^V\} \end{aligned}$$

ii)

$$\begin{aligned} S\left(X^V,Y^H\right) &= {}^Sg(\left(F^5\right)^H X^V,Y^H) -{}^S g(X^V,\left(F^5\right)^H Y^H) \\ &= -a^2\ {}^Sg(X^V,\left(FY\right)^H) \\ &= 0 \end{aligned}$$

iii)

$$\begin{aligned} S\left(X^H,Y^H\right) &= {}^Sg(\left(F^5\right)^H X^H,Y^H) -{}^S g(X^H,\left(F^5\right)^H Y^H) \\ &= a^2\ \{({}^Sg\left(FX\right)^H,Y^H) -{}^S g(X^H,\left(FY\right)^H)\} \\ &= a^2\ \{\left(g\left(FX\right),Y\right)^V - (g(X,\left(FY\right)^H))^V\} \end{aligned}$$

□

Theorem 17. *Let* ϕ_φ *be the Tachibana operator and the structure* $\left(F^5\right)^H - a^2F^H = 0$ *defined by Definition 8 and (14), respectively. if* $L_Y F=0$ *and* $F=a^2I$, *then all results with respect to* $\left(F^5\right)^H$ *is zero, where* $X,Y\in\Im_0^1\left(M\right)$, *the horizontal lifts* $X^H,Y^H\in\Im_0^1\left(T\left(M^n\right)\right)$ *and the vertical lift* $X^V,Y^V\in\Im_0^1\left(T\left(M^n\right)\right)$

i) $\phi_{(F^5)^H X^H}Y^H = -a^2\{-\left(\left(L_Y F\right)X\right)^H + \left(\hat{R}\left(Y,FX\right)u\right)^V - \left(F\left(\hat{R}\left(Y,X\right)u\right)\right)^V\},$

ii) $\phi_{(F^5)^H X^H}Y^V = a^2\{-\left(\left(L_Y F\right)X\right)^V + \left(\left(\nabla_Y F\right)X\right)^V\},$

$$iii)\ \phi_{(F^5)^H X^V} Y^H = a^2\{-((L_Y F)X)^V - (\nabla_{FX} Y)^V + (F(\nabla_X Y))^V\},$$

$$iv)\ \phi_{(F^5)^H X^V} Y^V = 0,$$

Proof. $i)$

$$\begin{aligned}
\phi_{(F^5)^H X^H} Y^H &= -(L_{Y^H}(F^5)^H) X^H \\
&= -L_{Y^C}(F^5)^H X^H + (F^5)^H L_{Y^H} X^H \\
&= -a^2[Y,FX]^H + a^2\gamma\hat{R}[Y,FX] \\
&\quad + a^2(F[Y,X])^H - a^2(F)^H(\hat{R}(Y,X)u)^V \\
&= -a^2\{-((L_Y F)X)^H + (\hat{R}(Y,FX)u)^V \\
&\quad -(F(\hat{R}(Y,X)u))^V\}
\end{aligned}$$

$ii)$

$$\begin{aligned}
\phi_{(F^5)^H X^H} Y^V &= -(L_{Y^V}(F^5)^H) X^H \\
&= -L_{Y^V}(F^5 X)^H + (F^5)^H L_{Y^V} X^H \\
&= -a^2[Y,FX]^V + a^2(\nabla_Y FX)^V \\
&\quad + a^2(F[Y,X])^V - a^2(F(\nabla_Y X))^V \\
&= a^2\{-((L_Y F)X)^V + ((\nabla_Y F)X)^V\}
\end{aligned}$$

$iii)$

$$\begin{aligned}
\phi_{(F^5)^H X^V} Y^H &= -(L_{Y^H}(F^5)^H) X^V \\
&= -L_{Y^H}(F^5 X)^V + (F^5)^H L_{Y^H} X^V \\
&= a^2[Y,FX]^V - a^2(\nabla_{FX} Y)^V \\
&\quad + a^2(F[Y,X])^H + a^2(F(\nabla_X Y))^V \\
&= a^2\{-((L_Y F)X)^V - (\nabla_{FX} Y)^V + (F(\nabla_X Y))^V\}
\end{aligned}$$

$iv)$

$$\begin{aligned}
\phi_{(F^5)^H X^V} Y^V &= -(L_{Y^V}(F^5)^H) X^V \\
&= -a^2 L_{Y^V}(FX)^V + a^2(F)^H L_{Y^V} X^V \\
&= 0
\end{aligned}$$

□

2.4. **The Structure** $(F^5)^H - a^2 F^H = 0$ **on Cotangent Bundle.** In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F_a(5,1)-$structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F_a(5,1)-$structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of the structure.

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have [17]

$$F^H G^H + G^H F^H = (FG + GF)^H$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H \tag{20}$$

Multiplying both sides by F^H and making use of the same (20), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H \tag{21}$$

and so on. Thus

$$(F^H)^5 = (F^5)^H \tag{22}$$

Since F gives on M the $F_a(5,1)-$structure, we have

$$F^5 - a^2 F = 0. \tag{23}$$

Taking horizontal lift, we obtain

$$(F^5)^H - a^2 F^H = 0. \tag{24}$$

In view of (22), we can write

$$(F^H)^5 - a^2 F^H = 0. \tag{25}$$

Theorem 18. *The Nijenhuis tensor $N_{(F^5)^H(F^5)^H}(X^H, Y^H)$ of the horizontal lift F^5 vanishes if $F = a^2 I$ on M.*

Proof. The Nijenhuis tensor $N(X^H, Y^H)$ for the horizontal lift of F^5 is given by

$$\begin{aligned} N_{(F^5)^H,(F^5)^H}(X^H, Y^H) &= [(F^5)^H X^H, (F^5)^H Y^H] - (F^5)^H[(F^5)^H X^H, Y^H] \\ &\quad -(F^5)^H[X^H, (F^5)^H Y^H] + (F^5)^H (F^5)^H [X^H, Y^H] \\ &= a^4\{[(F)^H X^H, (F)^H Y^H] - (F)^H[(F)^H X^H, Y^H] \\ &\quad -(F)^H[X^H, (F)^H Y^H] + (F)^H (F)^H [X^H, Y^H]\} \\ &= a^4\{\{[FX, FY] - F[(FX), Y] - F[X, FY] \\ &\quad +F^2[X,Y]\}^H + \gamma\{R(FX, FY) - R((FX), Y)F \\ &\quad -R(X, FY)F^2 + R(X,Y)F^2\}\} \end{aligned}$$

Let us suppose that $F = a^2 I$ on M. Thus, the equation becomes

$$\begin{aligned} N_{(F^5)^H,(F^5)^H}(X^H, Y^H) &= a^4\{\{[X,Y] - [X,Y] - [X,Y] + [X,Y]\}^H \\ &\quad +\gamma\{R(X,Y) - R(X,Y) - R(X,Y) + R(X,Y)\}. \end{aligned}$$

Therefore, it follows

$$N_{(F^5)^H,(F^5)^H}(X^H, Y^H) = 0$$

□

Theorem 19. *The Nijenhuis tensor* $N_{(F^5)^H(F^5)^H}(X^H, \omega^V)$ *of the horizontal lift* F^5 *vanishes if* $\nabla F = 0$.

Proof.

$$\begin{aligned} N_{(F^5)^H,(F^5)^H}(X^H, \omega^V) &= [(F^5)^H X^H, (F^5)^H \omega^V] - (F^5)^H[(F^5)^H X^H, \omega^V] \\ &\quad -(F^5)^H[X^H, (F^5)^H \omega^V] + (F^5)^H(F^5)^H[X^H, \omega^V] \\ &= a^4\{(\nabla_{FX}(\omega \circ F))^V - ((\nabla_{FX}) \circ F)^V \\ &\quad -((\nabla_X(\omega \circ F)) \circ F)^V + ((\nabla_X \omega) \circ F^2)^V\} \\ &= a^4\{(\omega \circ (\nabla_{FX} F) - (\omega \circ (\nabla_X F)F\}^V \end{aligned}$$

where $F \in \Im_1^1(M)$, $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$. The theorem is proved. □

Theorem 20. *The Nijenhuis tensor* $N_{(F^5)^H,(F^5)^H}(\omega^V, \theta^V)$ *of the horizontal lift* F^5 *vanishes.*

Proof. Because of $[\omega^V, \theta^V] = 0$ and $\omega \circ F \in \Im_1^0(M^n)$ on $T^*(M^n)$, the equation becomes

$$N_{(F^5)^H,(F^5)^H}(\omega^V, \theta^V) = 0.$$

□

Theorem 21. *Let* $(F^5)^H$ *be a tensor field of type* $(1,1)$ *on* $T^*(M^n)$. *If the Tachibana operator* ϕ_φ *applied to vector and covector fields according to horizontal lifts of* F^5 *defined by (25) on* $T^*(M^n)$, *then we get the following results.*

$$\begin{aligned} i)\ \phi_{(F^5)^H X^H} Y^H &= a^2\{-((L_Y F)X)^H - (pR(Y, FX))^V \\ &\quad +((pR(Y, X))F)^V\}, \end{aligned}$$

$$ii)\ \phi_{(F^5)^H X^H} \omega^V = a^2\{(\nabla_{FX}\omega)^V - ((\nabla_X \omega) \circ F)^V\},$$

$$iii)\ \phi_{(F^5)^H \omega^V} X^H = -a^2(\omega \circ (\nabla_X F))^V,$$

$$iv)\ \phi_{(F^5)^H \omega^V} \theta^V = 0,$$

where horizontal lifts $X^H, Y^H \in \Im_0^1(T^*(M^n))$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_1^0(M^n)$ are given, respectively.

Proof. i)

$$\phi_{(F^5)^H X^H} Y^H = -(L_{Y^H}(F^5)^H)X^H$$

$$\begin{aligned} &= -L_{Y^H}(F^5)^H X^H + (F^5)^H L_{Y^H} X^H \\ &= a^2\{-((L_Y F)X)^H - (pR(Y,FX))^V \\ &\quad +((pR(Y,X))F)^V\} \end{aligned}$$

ii)

$$\begin{aligned} \phi_{(F^5)^H X^H}\omega^V &= -(L_{\omega^V}(F^5)^H)X^H \\ &= -L_{\omega^V}(F^5)^H X^H + (F^5)^H L_{\omega^V} X^H \\ &= -a^2 L_{\omega^V}(FX)^H - a^2(F)^H(\nabla_X\omega)^V \\ &= a^2\{(\nabla_{FX}\omega)^V - ((\nabla_X\omega)\circ F)^V\}, \end{aligned}$$

iii)

$$\begin{aligned} \phi_{(F^5)^H\omega^V} X^H &= -(L_{X^H}(F^5)^H)\omega^V \\ &= -a^2(\nabla_X(\omega\circ F))^V + a^2((\nabla_X\omega)\circ F)^V \\ &= -a^2(\omega\circ(\nabla_X F))^V \end{aligned}$$

iv)

$$\begin{aligned} \phi_{(F^5)^H\omega^V}\theta^V &= -(L_{\theta^V}(F^5)^H)\omega^V \\ &= -L_{\theta^V}(F^5)^H\omega^V + (F^5)^H L_{\theta^V}\omega^V \\ &= 0 \end{aligned}$$

□

Definition 22. *A Sasakian metric ${}^S g$ is defined on $T^*(M^n)$ by the three equations*

$${}^S g(\omega^V,\theta^V) = (g^{-1}(\omega,\theta))^V = g^{-1}(\omega,\theta)o\pi, \tag{26}$$

$${}^S g(\omega^V, Y^H) = 0, \tag{27}$$

$${}^S g(X^H, Y^H) = (g(X,Y))^V = g(X,Y)\circ\pi. \tag{28}$$

For each $x\in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x(M^n)$ by

$$g^{-1}(\omega,\theta) = g^{ij}\omega_i\theta_j, \tag{29}$$

where $X, Y \in \Im^1_0(M^n)$ and $\omega,\theta\in\Im^0_1(M^n)$. Since any tensor field of type $(0,2)$ on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [17], p.280), it follows that ${}^S g$ is completely determined by equations (26), (27) and (28).

Theorem 23. *Let $(T^*(M^n), {}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^5)^H$ of type $(1,1)$ defined by (25). Sasakian metric ${}^S g$ is pure with respect to $(F^5)^H$ if $F = a^2 I$ (I = identity tensor field of type $(1,1)$).*

Proof. We put

$$S(\tilde{X},\tilde{Y}) = ^S g((F^5)^H\tilde{X},\tilde{Y}) - ^S g(\tilde{X},(F^5)^H\tilde{Y}).$$

If $S(\tilde{X},\tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^V,θ^V or X^H,Y^H, then $S=0$. By virtue of $(F^5)^H - a^2F^H = 0$ and (26), (27), (28), we get

i)

$$\begin{aligned} S(\omega^V,\theta^V) &= {}^S g((F^5)^H\omega^V,\theta^V) -^S g(\omega^V,(F^5)^H\theta^V) \\ &= {}^S g((a^2F)^H\omega^V,\theta^V) -^S g(\omega^V,(a^2F)^H\theta^V) \\ &= a^2({}^S g((\omega\circ F)^V,\theta^V) -^S g(\omega^V,(\theta\circ F)^V)). \end{aligned}$$

ii)

$$\begin{aligned} S(X^H,\theta^V) &= {}^S g((F^5)^H X^H,\theta^V) -^S g(X^H,(F^5)^H\theta^V) \\ &= {}^S g((a^2F)^H X^H,\theta^V) -^S g(X^H,(a^2F)^H\theta^V) \\ &= a^2({}^S g((FX)^H,\theta^V) -^S g(X^H,(\omega\circ F)^V)) \\ &= 0. \end{aligned}$$

iii)

$$\begin{aligned} S(X^H,Y^H) &= {}^S g((F^5)^H X^H,Y^H) -^S g(X^H,(F^5)^H Y^H) \\ &= {}^S g((a^2F)^H X^H,Y^H) -^S g(X^H,(a^2F)^H Y^H) \\ &= a^2({}^S g((FX)^H,Y^H) -^S g(X^H,(FY)^H)). \end{aligned}$$

Thus, $F = a^2I$, then ${}^S g$ is pure with respect to $(F^5)^H$. □

Declaration of Competing Interest There is no conflict of interests regarding the publication of this paper.

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