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A fixed point theorem for Hardy-Rogers type on generalized fractional differential equations

Jayashree Patil[a], Basel Hardan[b], Mohammed S. Abdo[c], Amol Bachhav[d], Archana Chaudhari[b]

[a] *Department of Mathematics Vasantrao Naik Mahavidyalaya, Cidco, Aurangabad, India.*
[b] *Research Scholar, Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431004, India.*
[c] *Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen.*
[d] *Navin Jindal School of Management, University of Texas at Dallas, Dallas,75080.*

Abstract

In this paper, a version modified of contraction Hardy-Rogers type in a metric space and is proved. Moreover, we apply this modified version to investigate the existence of unique solution of boundary value problems for the differential equations and generalized fractional differential equations through help of the properties of Green function. We also provide an example in support of acquired results. These results extend various comparable results from literature.

Keywords: Hardy-Rogers-type contractions, fixed points, metric-like spaces, generalized fractional differential equations.
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1. Introduction and Preliminarries

It is well known that the Banach contraction principle plays an important role in various fields of science especially in functional analysis and applied mathematical. Banach in [20] proved the existence and uniqueness for a point $u \in L$ such that $f : L \to L$ is a contraction map, i.e.

$$\delta(fu, fv) \leq \varsigma\delta(u, v). \tag{1}$$

Email addresses: jv.patil29@gmail.com (Jayashree Patil), bassil2003@gmail.com (Basel Hardan), msabdo1977@gmail.com (Mohammed S. Abdo), amol.bachhav@utdallas.edu (Amol Bachhav), archubkharat@gmail.com (Archana Chaudhari)

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where (L, δ) be a metric space, for each $u, v \in L$ and $\varsigma \in [0, 1)$. Kannan in [27] introduced the same result by the following

$$\delta(fu, fv) \leq \varsigma[\delta(fu, u) + \delta(fv, v)], \tag{2}$$

for all $u, v \in L$ and $\varsigma \in (0, \frac{1}{2})$.
Chatterjee in [22] modified the equation (2) as follows

$$\delta(fu, fv) \leq \varsigma[\delta(fu, v) + \delta(fu, v)], \tag{3}$$

for all $u, v \in L$ and $\varsigma \in (0, \frac{1}{2})$.
Through the literature, Fisher in [24] developed the equation (1) as follows

$$\delta(fu, fv) \leq \varsigma\delta(fu, v), \tag{4}$$

for all $u, v \in L$.
Then many attempts were made for expanded and developed equation (1), for e.g. Reich in [40] obtained the next result

$$\delta(fu, fv) \leq [\ell_1\delta(u, v) + \ell_2\delta(fu, u) + \ell_3\delta(fv, v)], \tag{5}$$

for all $u, v \in L$ such that $\ell_1 + \ell_2 + \ell_3 < 1$.
Recently, Shukla in [48] developed the equation (5) as follows

$$\delta(fu, fv) \leq [\ell_1\delta(u, fu) + \ell_2\delta(v, fv) + \ell_3\delta(v, fu)], \tag{6}$$

for all $u, v \in L$ such that $\ell_1 + \ell_2 + \ell_3 < 1$.
Also, in [25] Hardy and Rogers introduced a generalization of Reich's fixed point theorem, as in the following theorem:

Theorem 1.1. *Let (L, δ) be a metric space and f a self mapping of L. Suppose, $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 \in \mathbb{R}^+$ and we set, $\ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 = \varsigma$ such that*

$$\delta(fu, fv) \leq \ell_1\delta(u, fu) + \ell_2\delta(v, fv) + \ell_3\delta(u, fv) + \ell_4\delta(v, fu) + \ell_5\delta(u, v),$$

$\forall u, v \in L$, under the conditions; L is complete and $\varsigma < 1$, then f has a unique fixed point.
Or,

$$\delta(fu, fv) < \ell_1\delta(u, fu) + \ell_2\delta(v, fv) + \ell_3\delta(u, fv) + \ell_4\delta(v, fu) + \ell_5\delta(u, v), \tag{7}$$

$\forall u, v \in L$, under the conditions; L is compact, f is continuous, $\varsigma = 1$ and $u \neq v$, then T has a unique fixed point.

Riech's theory has been extensively studied by many researchers (see [1, 11, 18, 19, 32, 33, 35, 44, 52]). Moreover, many of the studies about the Hardy-Rogers theory have been introduced. Among these studies, Hardy and Rogers type common fixed point theorem for a family of self-maps in cone 2-metric spaces was obtained by Rangamma in [39]. In the same way, Chifu in [23] presented some fixed point results in b-metric spaces by using a contractive condition of Hardy-Rogers type with respect to the functional H. Arshad et al., in [17] established common fixed point theorems for mappings fulfilling locally contraction conditions under a closed ball in an ordered complete dislocated metric space. In [16, 41] the authors established common fixed point results for multi-valued mappings via generalized rational type contractions on complete b-metric spaces. A new direction to the literature of common fixed point theorems related to T-Hardy-Rogers contraction mappings, Banach pair of mappings, and cone metric space due to Rhymend in[42]. A modified class of Hardy-Rogers p-proximate cyclic contraction in uniform spaces was introduced by Olisama in [36]. Abbas in [1] proved some fixed point theorems for a T-Hardy-Rogers contraction in the setting of partially ordered partial metric spaces. Some fixed point theorems for a generalized almost Hardy-Rogers type F-contraction in metric like space were established by Saipara in [43].

On the other hand, fractional calculus has played a very important role in different areas, see [30] and references mentioned in it. Generalized fractional derivatives with respect to another function ψ have been considered in [30] as a generalization of Riemann–Liouville fractional operator. This fractional derivative is different from the other classical fractional derivative because the kernel appears in terms of another function ψ. Recently, Almeida in [13] presented a version generalized of Caputo with some enjoyable properties. The investigation of the existence and uniqueness of solutions to several fractional differential equations (FDEs) is the main topic of applied mathematics research. Many interesting results with regard to the existence and uniqueness of solutions by using some fixed point theorem were discussed in the following references [5, 6, 7, 8, 4, 14, 21, 26, 31, 38].

Fixed point techniques are constantly applied to prove the existence and uniqueness of differential equations (DEs) and FDEs, see [10, 12, 15, 28, 34, 45, 46, 49, 50]. To investigate the existence of unique solutions for different types of DEs and FDEs, we refer to [2, 3, 9, 29, 37, 47, 51].

To our knowledge, a modified contraction Hardy-Rogers type in metric space has not been extensively studied. Moreover, the fixed point technique based on generalized Hardy-Rogers type contraction mappings has never been applied on the boundary value problems (BVPs) for generalized FDEs involving ψ-Caputo fractional operators. For this reason, and motivated by the recent evolutions in ψ-fractional calculus, in this paper, we introduce a modification of Hardy-Rogers type contraction in metric space and we also apply this approach to investigate the existence of unique solution of boundary value problems for a classical DEs and generalized FDEs. The main result of this paper is to study the modified conditions of Hardy-Rogers fixed point theorem and proved it. Moreover, some applications to justify our results.

2. Main Results

In this part, we shall prove the modified Hardy-Rogers fixed point theorem as following:

Theorem 2.1. *Let L be a complete metric space and let $f : L \to \mathbb{R}$ be a continuous self-mapping on L, suppose f satisfying the condition (7) for all $u, v \in L, u \neq v$ and for some $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 \in [0, 1)$ such that $\sum_{i=1}^{5} \ell_i = \varsigma < 1$. Then f has a unique fixed point.*

Proof. Let u_0 be an arbitrary point in L and $\{u\}_{n=1}^{\infty}$ be the sequence of iterations for f at u_0 such that

$$f(u_{n-1}) = u_n. \tag{8}$$

Consider $u_{n-1} \neq u_n$ for all $n \in \mathbb{N}$. Thus, $\delta(u_{n-1}, u_n) = \delta(f(u_{n-2}), f(u_{n-1}))$, by (7) we get

$$\begin{aligned}\delta(u_{n-1}, u_n) <& \ell_1(u_{n-2}, f(u_{n-2})) + \ell_2\delta(u_{n-1}, f(u_{n-1}))+\\ &\ell_3\delta(u_{n-2}, f(u_{n-1})) + \ell_4\delta(u_{n-1}, f(u_{n-2})) + \ell_5\delta(u_{n-2}, u_{n-1}).\end{aligned}$$

By (8) we get

$$\begin{aligned}\delta(u_{n-1}, u_n) <& \ell_1\delta(u_{n-2}, u_{n-1}) + \ell_2\delta(u_{n-1}, u_n) + \ell_3\delta(u_{n-2}, u_n)+\\ &\ell_4\delta(u_{n-1}, u_{n-1}) + \ell_5\delta(u_{n-2}, u_{n-1}).\end{aligned}$$

From the triangle inequality for some $u_{n-2} \leq u_{n-1} \leq u_n$, we obtain

$$\begin{aligned}\delta(u_{n-1}, u_n) \leq& \ell_1\delta(u_{n-2}, u_{n-1}) + \ell_2\delta(u_{n-1}, u_n) + \ell_3\delta(u_{n-2}, u_{n-1})+\\ &\ell_3\delta(u_{n-1}, u_n) + \ell_5\delta(u_{n-2}, u_{n-1})\\ =& \left(\frac{\varsigma - \ell_2 - \ell_4}{1 - \ell_2 - \ell_3}\right)\delta(u_{n-2}, u_{n-1}).\end{aligned} \tag{9}$$

If we repeat equation (9), we arrive to,

$$\delta(u_{n-1}, u_n) \leq \left(\frac{\varsigma - \ell_2 - \ell_4}{1 - \ell_2 - \ell_3}\right)^n \delta(u_0, u_1). \tag{10}$$

For some $r \geq n-1$, we have

$$\delta(u_{n-1}, u_r) \leq \delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1}) + ... + \delta(u_{r-1}, u_r).$$

It follows from (10) that

$$\delta(u_{n-1}, u_r) \leq \{\kappa^n + \kappa^{n+1} + ... + \kappa^s\}\delta(u_0, u_1),$$

where $\kappa = \left(\frac{\varsigma-\ell_2-\ell_4}{1-\ell_2-\ell_3}\right)$. Therefore,

$$\kappa^n \to 0 \ as \ n \to \infty.$$

Hence

$$\delta(u_{n-1}, u_s) \to 0 \quad s \to \infty.$$

Every Cauchy sequence$\{u_{n-1}\}_{n-1}^{\infty}$ in L is convergence, since L is complete space, i.e. there exist $u_1 \in L$ such that $u_n \to u_1$, also we have a continuous mapping

$$f(\lim_{n\to\infty} u_n) = f(u_1), \lim_{n\to\infty} u_n = u_1.$$

Hence, u_1 is a fixed point of $f \ in \ L$.
Now to prove that u_1 is a unique fixed point of $f \ in \ L$, there exist another fixed point $u_2 \in L$ such that $u_1 \neq u_2$, $f(u_1) = u_1$ and $f(u_2) = u_2$. By (7) we have

$$\begin{aligned}\delta(u_1, u_2) <& \ell_1\delta(u_1, f(u_1)) + \ell_2\delta(u_2, f(u_2)) + \ell_3\delta(u_1, f(u_2)) + \ell_4\delta(u_2, f(u_1)) + \\ & \ell_5\delta(u_1, z_2), \\ <& (\varsigma - \ell_1 - \ell_2)\delta(u_1, u_2)\end{aligned}$$

which implies $u_1 = u_2$. So u_1 is a unique fixed point of $f \ in \ L$. □

Theorem 2.2. *Let L be a complete metric space and let f, g are two continuous self-mapping on L satisfy*

$$\delta(f(u), g(v)) < \ell_1\delta(u, f(u)) + \ell_2\delta(v, g(v)) + \ell_3\delta(u, g(v)) + \ell_4\delta(v, f(u)) + \ell_5\delta(u, v)$$

for all $u, v \in L, u \neq v$ and for some $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 \in [0, 1)$ such that $\sum_{i=1}^{5} \ell_i = \varsigma < 1$. Then f and g having a unique fixed point.

Proof. For $u_0, v_0 \in L$ we take $f(u_{n-1}) = u_n, g(v_{n-1}) = v_n$, it follows that

$$\begin{aligned}&\delta(u_k, v_k) = \delta(f(u_{k-1}, g(v_{k-1})) \\ &< \ell_1\delta(u_{k-1}, f(u_{k-1})) + \ell_2\delta(v_{k-1}, g(v_{k-1}) + \ell_3\delta(u_{k-1}, g(v_{k-1}) \\ &+ \ell_4\delta(v_{k-1}, f(u_{k-1} + \ell_5\delta(u_{k-1}, v_{k-1})),\end{aligned}$$

$for \ k \in \mathbb{N}$. Also, we have

$$\begin{aligned}&\sum_{k=1}^{n}\delta(u_k, v_k) = \sum_{k=1}^{n}\delta(f_1(u_{k-1}, f_2(v_{k-1})) \\ &< \sum_{k=1}^{n}[\ell_1\delta(u_{k-1}, u_k) + \ell_2\delta(v_{k-1}, v_k) + \ell_3\delta(u_{k-1}, u_k) + \ell_4\delta(v_{k-1}, u_k) + \\ &\ell_5 d(u_{k-1}, v_{k-1})], \\ &\leq [\ell_1\delta(u_0, u_n) + \ell_2\delta(v_0, v_n) + \ell_3\sum_{k=1}^{n}\delta(u_{k-1}, v_k) + \sum_{k=1}^{n}\ell_4\delta(v_{k-1}, u_k) + \\ &\sum_{k=1}^{n}\ell_5\delta(u_{k-1}, v_{k-1})],\end{aligned}$$

and

$$\sum_{k=1}^{n}\delta(u_{k+1},v_k) \leq \ell_1\delta(u_1,u_n)+\ell_2\delta(v_0,v_n)+\ell_3\sum_{k=1}^{n}\delta(u_k,v_k) + \sum_{k=1}^{n}\ell_4\delta(v_{k-1},u_k)+\sum_{k=1}^{n}\ell_5\delta(u_{k-1},v_{k-1}).$$

Therefore,

$$\sum_{k=1}^{n}\delta(u_k,u_{k+1}) \leq (\ell_1+\ell_5)\delta(u_0,u_n)+(\ell_2+\ell_3)\delta(u_1,u_{n+1}).$$

Hence

$$\sum_{k=1}^{n}\delta(u_k,u_{k+1}) \leq \sum_{k=1}^{n}\delta(u_k,v_k) \leq \sum_{k=1}^{n}\delta(u_{k+1},v_k) < \infty.$$

This means $\sum_{k=1}^{n} d(u_k,u_{k+1}) \to 0 \quad as \quad k\to\infty$, so $\{u_k\}$ is a Cauchy sequence in L.
By the same way, we can show that $\{v_k\}$ is a Cauchy sequence in L. Since L is complete metric space, there exist a common fixed point in L. To get it, we suppose

$$u_1=\lim_{n\to\infty}u_n, \qquad u_2=\lim_{n\to\infty}v_n, \qquad \forall u_1,u_2\in L,$$

Therefore,

$$\delta(u_n,u_1)\to 0, \qquad n\to\infty,$$
$$\delta(v_n,v_1)\to 0, \qquad n\to\infty.$$

Since f *and* g are continuous mappings, we obtain

$$d\delta(f(u_n),f(u_1))\to 0, \qquad n\to\infty,$$
$$\delta(g(v_n),g(u_2))\to 0, \qquad n\to\infty.$$

That is,

$$\begin{aligned}\delta(u_1,f(u_1)) &= \delta(f^{-1}(f(u_1)),f(u_1))\\ &< \ell_1\delta(f^{-1}(f(u_1)),f(u_1))+\ell_2\delta(u_1,f(u_1))\\ &+\ell_3\delta(f(u_1),f(u_1))+\ell_4\delta(u_1,f^{-1}(f(u_1)))+\ell_5\delta(f(u_1),u_1)\\ &< (\varsigma-\ell_3-\ell_4)\delta(u_1,f(u_1)),\end{aligned}$$

which implies $f(u_1)=u_1$. Similarly, we get $g(u_2)=u_2$.
Now, we shall prove that u_1 is common fixed point of f and g in L as follows

$$\begin{aligned}\delta(u_1,u_2) &< \ell_1\delta(u_1,f_1(u_1))+\ell_2\delta(u_2,f_2(u_2))+\ell_3\delta(u_1,f_2(u_2))\\ &+\ell_4\delta(u_2,f_1(u_1))+\ell_5\delta(u_1,u_2)\\ &< (\varsigma-\ell_1-\ell_1)\delta(u_1,u_2).\end{aligned}$$

To prove the uniqueness of u_1, we must suppose another point $u_3\in L$ such that

$$f(u_3)=u_3, \quad and \quad g(u_3)=u_3.$$

Therefore

$$\begin{aligned}&\delta(u_1,u_3)=(f_1(u_1),f_3(u_3))\\ &<\ell_1\delta(u_1,f_1(u_1))+\ell_2\delta(u_3,f_2(u_3))+\ell_3\delta(u_1,f_2(u_3))\\ &+\ell_4\delta(u_3,f_1(u_1))+\ell_5\delta(z_1,z_3)\\ &=(\varsigma-\ell_1-\ell_2)\delta(u_1,u_3).\end{aligned}$$

Hence $u_1=u_3$. Thus, u_1 is the unique fixed point of f and g in L. □

In next theorem, we will generalize Theorems 2.1 and 2.2.

Theorem 2.3. *Let f_ϑ be a family continuous self-mapping in complete metric space L, suppose that*

$$\begin{aligned}\delta(f_\vartheta(u), f_\beta(v)) \leq & \ell_1\delta(u, f_\vartheta(u)) + \ell_2\delta(v.f_\beta(v)) + \ell_3\delta(u, f_\beta(v)) + \ell_4\delta(v, f_\vartheta(u)) \\ & + \ell_5\delta(u, v),\end{aligned}$$

for every $u, v \in L, u \neq v$ and $\sum_{i=1}^{5} \ell_i = \varsigma < 1$. Then $f_\vartheta(u)$ has a unique fixed point $u_1 \in L$.

Proof. By repeat the same way in Theorem 2.2 with replacing f and g by f_ϑ and f_β respectively, we get

$$f_\vartheta(u_1) = f_\beta(u_1) = u_1.$$

□

We can reformulate the theorem as follows:

Theorem 2.4. *Let f^k be a self-mappings on a complete metric space L such that $f^k(u^k) = u^k$, for all u, $u^k \in L$ $\forall k$ respectively, such that*

$$\begin{aligned}\delta(f^k(u), f^k(v)) < & \ell_1\delta(u, f^k(u)) + \ell_2\delta(v, f^k(v)) + \ell_3\delta(u, f^k(v)) + \ell_4\delta(v, f^k(u)) \\ & + \ell_5\delta(u, v).\end{aligned}$$

$\forall u, v \in L, u \neq v$ and $\sum_{i=1}^{5} \ell_i = \varsigma < 1$.

Proof. We need to prove that $f^k(u_1) = u_1$. Therefore, by same technique used to prove the Theorem 2.2, Theorem 2.4 can be proven. □

Example 2.1. *Assume that $L = [0, 1]$ is a complete metric space. Suppose that $f(u) = u/3$, at $u \in [0, \frac{1}{3})$ and $f(v) = v/4$ at $v \in (\frac{1}{3}, 1]$. Clearly, f is discontinues, so (1) is not hold. Take $\varsigma = 1/3$. Hence, all conditions of Theorem 2.1 is satisfied and a unique fixed point is $u = 0 \in L$.*

3. Applications

3.1. An application without necessary continuity condition

In the next theorem, we can apply our results to study the existence and uniqueness of common fixed points for mappings without continuity condition.

Theorem 3.1. *Let f_{k_1}, f_{k_2} be two self-mappings on complete metric space L satisfies*

$$\begin{aligned}\delta(f_{k_1}(u), f_{k_2}(v)) < & \ell_1\delta(u, f_{k_1}(u)) + \ell_2\delta(v, f_{k_2}(v)) + \ell_3\delta(u, f_{k_2}(v)) + \\ & \ell_4\delta(v, f_{k_1}(u)) + \ell_5\delta(u, v),\end{aligned}$$

$\forall u, v \in L, u \neq v$ and $\sum_{i=1}^{5} \ell_i = \varsigma < 1$. Suppose that $f_{k_1} f_{k_2} = f_{k_2} f_{k_1}$ is continuous then f_{k_1} and f_{k_2} having a unique common fixed point in L.

Proof. Take $u_n = f_{k_1}(u_{n-1})$, $u_{n+1} = f_{k_2}(u_n)$ *and* $f_{k_1}(u_{n-1}) \neq f_{k_2}(u_{n-1})$, $u_n \neq u_{n-1}, \forall n \in \mathbb{N}$.
Therefore,

$$\begin{aligned}\delta(u_{2n+1}, u_{2n}) &= \delta(f_{k_1}(u_{2n}), f_{k_2}(u_{2n-1})) \\ &< \ell_1\delta(u_{2n}, x_{2n+1}) + \ell_2\delta(u_{2n-1}, u_{2n}) + \ell_3\delta(u_{2n}, u_{2n}) \\ &+ \ell_4\delta(u_{2n-1}, u_{2n+1}) + \ell_5\delta(u_{2n}, u_{2n-1}).\end{aligned}$$

So, we have

$$\delta(u_{2n+1}, u_{2n}) \leq \Big(\frac{\varsigma - \ell_1 - \ell_3}{1 - \ell_2 - \ell_4}\Big)\delta(u_{2n}, u_{2n-1}). \tag{11}$$

From (11) we obtain

$$\delta(u_{2n+1}, u_{2n}) \leq \Big(\frac{\varsigma - \ell_1 - \ell_3}{1 - \ell_2 - \ell_4}\Big)^{2n}\delta(u_1, u_0).$$

Therefore,

$$f_{k_1}f_{k_2}(u_1) = f_{k_2}f_{k_1}(u_1) = f_{k_1}f_{k_2}(\lim_{k\to\infty} u_{n_k}) = \lim_{k\to\infty} u_{n_{k+1}} = u_1.$$

Suppose that u_1 is a fixed point of $f_{k_1}f_{k_2}$ *in* L such that $f_{k_1}f_{k_2}(u_1) = u_1$. Now, we must show that u_1 is a fixed point of f_{k_1} *and* f_{k_2} *in* L, i.e.

$$f_{k_1}(u_1) = u_1 \quad and \quad f_{k_2}(u_1) = u_1.$$

For that, let

$$f_{k_1}(u_1) \neq u_1 \quad and \quad f_{k_2}(u_1) \neq u_1.$$

Then by using(7), we have

$$\begin{aligned}\delta(z_1, f_1(z_1)) &= \delta(f_2f_1(z_1), f_1(z_1)) \\ &< \ell_1\delta(f_1(z_1), f_2f_1(z_1) + \ell_2\delta(z_1, f_1(z_1)) + \ell_3\delta(f_1(z_1, f_1(z_1) \\ &+ \ell_4\delta(z_1, f_1(f_1(z_1)) + \ell_5\delta(f_1(z_1), z_1)\end{aligned}$$

Hence, u_1 is a fixed point of f_{k_1} *in* L. Similarly, we get $f_{k_2}(u_1) = u_1$. This indicates that f_{k_1} and f_{k_2} having a common fixed point in L. That was proof of existence result.
Again we apply (7) for proving the uniqueness result. Suppose $u_2 \in L$ $(u_2 \neq u_1)$ are another fixed points of f_{k_1} and f_{k_2} such that

$$\begin{aligned}\delta(u_1, u_2) &= \delta(f_1(u_1), f_2(u_2)) \\ &< \ell_1\delta(u_1, f_1(u_1)) + \ell_2\delta(u_2, f_2(u_2)) + \ell_3\delta(u_1, f_2(u_2) \\ &+ \ell_4\delta(u_2, f_1(u_1) + \ell_5\delta(u_1, u_2) \\ &= (\varsigma - \ell_1 - \ell_2)\delta(u_1, u_2).\end{aligned}$$

This means that $u_1 = u_2$. So, we have proven the uniqueness result. The proof is completed. □

3.2. An application on DEs

:

Consider the following nonlinear DE

$$\begin{cases} u''(t) = -g(t, u(t)), & t \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \tag{12}$$

where the function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous.
Now, by using Theorem 2.1, we discuss the existence and uniqueness of the problem (12).

Problem (12) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t, \tau)g(\tau, u(\tau))d\tau, \ \forall t \in [0, 1], \tag{13}$$

where $G(t, \tau)$ is the Green's function defined by

$$G(t, \tau)\begin{cases} t - \tau t, & 0 \leq t \leq \tau \leq 1, \\ \tau - \tau t, & 0 \leq \tau \leq t \leq 1. \end{cases}$$

Therefore, if $u \in C^2([0,1])$, then a solution of (12) it will be u if and only if it is a solution of (13). Denote the space of all continuous functions by $L = C([0,1])$. Let δ satisfying

$$\delta(u,v) = \|u\|_\infty + \|v\|_\infty + \|u-v\|_\infty, \text{ for all } u,v \in L$$

such that $\|u\|_\infty = max_{t\in[0,1]}|u(t)| \; for \; all \; u \in L.$

Theorem 3.2. *Assume that*

i) $|g(\tau,a) - g(\tau,b)| \leq 8f_1(\tau)|a-b|, \forall a,b \in \mathbb{R}, \tau \in [0,1]$ *such that* $f_1 : [0,1] \to [0,\infty]$ *a continuous functions.*

ii) $|g(\tau,a)| \leq 8f_2(\tau)|a|, \forall a \in \mathbb{R}, \tau \in [0,1]$ *such that* $f_2 : [0,1] \to [0,\infty]$ *a continuous functions.*

iii) $max_{\tau\in[0,1]}f_1(\tau) = \eta k_1 < \frac{1}{81}, \quad 0 \leq \eta < \frac{1}{9}.$

iv) $max_{\tau\in[0,1]}f_2(\tau) = \eta k_2 < \frac{1}{81}, \quad 0 \leq \eta < \frac{1}{9}.$

Then, $u \in L = C([0,1],\mathbb{R})$ *is a unique solution to problem (12).*

Proof. Consider the operator $f : L \to L$ defined by

$$fu(t) = \int_0^1 G(t,\tau)g(\tau,u(\tau))d\tau, \; \forall u \in L, \; t \in [0,1].$$

Let $u,v \in L$, we have

$$\begin{aligned} |fu(t) - fv(t)| &= \left| \int_0^1 G(t,\tau)g(\tau,u(\tau))d\tau - \int_0^1 G(t,r)g(\tau,v(\tau))d\tau \right| \\ &\leq \int_0^1 G(t,\tau)|g(\tau,u(\tau) - g(\tau,v(\tau))|d\tau \\ &\leq 8\int_0^1 G(t,\tau)f_1(\tau)|u(\tau) - v(\tau)|d\tau \\ &\leq 8\eta k_1\|u-v\|_\infty \int_0^1 G(t,\tau)d\tau \\ &\leq \eta k_1\|u-v\|_\infty \end{aligned}$$

where we used fact that $\int_0^1 G(t,\tau)d\tau = \frac{t}{2} - \frac{t^2}{2}$ for all $t \in [0,1]$ and so $\sup_{t\in[0,1]} \int_0^1 G(t,\tau)d\tau = \frac{1}{8}$. Therefore,

$$\|fu - fv\|_\infty \leq \eta k_1 \|u-v\|_\infty . \tag{14}$$

On the other hand, we have

$$\begin{aligned} |fu(t)| &\leq \left| \int_0^1 G(t,\tau) \left|g(\tau,u(\tau))\right| d\tau \right| \\ &\leq 8\int_0^1 G(t,\tau)f_2(\tau)|u(\tau)|d\tau \\ &\leq \eta k_2\|u\|_\infty. \end{aligned}$$

Then,

$$\|fu\|_\infty \leq \eta k_2\|u\|_\infty. \tag{15}$$

Similarly, we get

$$\|fv\|_\infty \leq \eta k_2\|v\|_\infty. \tag{16}$$

By (14),(15) and (16), we obtain

$$\begin{aligned}\delta(fu,fv) &= \|fu\|_\infty + \|fv\|_\infty + \|fu-fv\|_\infty \\ &\le \eta k_2\|u\|_\infty + \eta k_2\|v\|_\infty + \eta k_1\|u-v\|_\infty \\ &< \eta(2k_2+k_1^2)(\|u\|_\infty + \|v\|_\infty + \|u-v\|_\infty) \\ &< \eta(2k_2+k_1^2)\delta(u,v),\end{aligned}$$

where $(2k_2+k_1^2)<1$. Suppose $\ell_1,\ell_2,\ell_3,\ell_4,\ell_5>0$, such that

$$\ell_1<\frac{1}{9},\ell_2<\frac{1}{9},\ell_3<\frac{1}{9},\ell_4<\frac{1}{9},\ell_5<\frac{1}{9} \quad \textit{and} \quad \eta\in[0,1).$$

Then, the following is satisfied

$$\delta(fu,fv)<\ell_1\delta(u,fu)+\ell_2\delta(v,fv)+\ell_3\delta(u,fv)+\ell_4\delta(v,fu)+\ell_5\delta(u,v).$$

Hence, by Theorem 2.1, the problem (12) has a unique solution $u\in L$. □

3.3. An application on FDEs

:

More definitions and properties of the generalized fractional calculus can be found in [13, 30].

Lemma 3.1. *Let $1<\theta<2$, $h:[0,1]\to\mathbb{R}^+$ are continuous function and $\psi:[0,1]\to\mathbb{R}^+$ an increasing function with $\psi'(t)\neq 0$ for $t\in[0,1]$. Then the function $u(t)\in C[0,1]$ is a solution of the following problem*

$$\begin{cases} {}^{C}D_{0^+}^{\theta,\psi}u(t)+h(t)=0, & t\in[0,1] \\ u(0)=u(1)=0. \end{cases} \tag{17}$$

if and only if $u\in C[0,1]$ is a solution of the following fractional integral equation

$$u(t)=\frac{1}{\Gamma(\theta)}\int_0^1 \psi'(\tau)G(t,\tau)h(\tau)d\tau.$$

where

$$G(t,\tau)=\begin{cases} \frac{[\psi(t)-\psi(0)]\left[[\psi(1)-\psi(\tau)]^{\theta-1}-[\psi(t)-\psi(\tau)]^{\theta-1}\right]}{[\psi(1)-\psi(0)]}, & 0\le\tau\le t\le 1, \\ \frac{[\psi(t)-\psi(0)][\psi(1)-\psi(\tau)]^{\theta-1}}{[\psi(1)-\psi(0)]}, & 0\le t\le\tau\le 1. \end{cases} \tag{18}$$

Here $G(t,\tau)$ is called Green function of BVP (17).

Proof. Applying $I_{0^+}^{\theta,\psi}$ on both sides of the first equation of (17),

$$I_{0^+}^{\theta,\psi}\,{}^{C}D_{0^+}^{\theta,\psi}u(t)+I_{0^+}^{\theta,\psi}h(t)=0.$$

Using Theorem 4 (see [13]) we get

$$u(t)=c_0+c_1[\psi(t)-\psi(0)]-\frac{1}{\Gamma(\theta)}\int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\theta-1}h(\tau)d\tau.$$

The condition $u(0)=0$ means $c_0=0$, and we have

$$u(1)=c_1[\psi(1)-\psi(0)]-\frac{1}{\Gamma(\theta)}\int_0^1 \psi'(\tau)[\psi(1)-\psi(\tau)]^{\theta-1}h(\tau)d\tau.$$

Since $u(0) = u(1) = 0$, then

$$c_1 = \frac{[\psi(1)-\psi(0)]^{-1}}{\Gamma(\theta)} \int_0^1 \psi'(\tau)[\psi(1)-\psi(\tau)]^{\theta-1} h(\tau) d\tau$$

The equation ${}^C D_{0+}^{\theta,\psi} u(t) + h(t) = 0$ is reduces to the equivalent integral equation

$$\begin{aligned} u(t) &= \frac{[\psi(t)-\psi(0)]}{[\psi(1)-\psi(0)]} \frac{1}{\Gamma(\theta)} \int_0^1 \psi'(\tau)[\psi(1)-\psi(s)]^{\theta-1} h(\tau) d\tau \\ &\quad - \frac{1}{\Gamma(\theta)} \int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\theta-1} h(\tau) d\tau, \end{aligned}$$

which implies

$$\begin{aligned} u(t) &= \frac{[\psi(t)-\psi(0)]}{[\psi(1)-\psi(0)]} \frac{1}{\Gamma(\theta)} \int_0^t \psi'(\tau)[\psi(1)-\psi(s)]^{\theta-1} h(\tau) d\tau \\ &\quad + \frac{[\psi(t)-\psi(0)]}{[\psi(1)-\psi(0)]} \frac{1}{\Gamma(\theta)} \int_t^1 \psi'(\tau)[\psi(1)-\psi(\tau)]^{\theta-1} h(\tau) d\tau \\ &\quad - \frac{1}{\Gamma(\theta)} \int_0^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\theta-1} h(\tau) d\tau \\ &= \int_0^t \frac{\psi'(\tau)[\psi(t)-\psi(0)]\left[[\psi(1)-\psi(\tau)]^{\theta-1} - [\psi(t)-\psi(\tau)]^{\theta-1}\right]}{[\psi(1)-\psi(0)]\Gamma(\theta)} h(\tau) d\tau \\ &\quad + \int_t^1 \frac{\psi'(\tau)[\psi(t)-\psi(0)][\psi(1)-\psi(\tau)]^{\theta-1}}{[\psi(1)-\psi(0)]\Gamma(\theta)} h(\tau) d\tau \\ &= \frac{1}{\Gamma(\theta)} \int_0^1 \psi'(\tau) G(t,\tau) h(\tau) d\tau. \end{aligned}$$

□

Now, we consider the following nonlinear FDE

$$\begin{cases} \mathcal{D}_{0+}^{\theta;\psi} u(t) + g(t,u(t)) = 0, & t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{19}$$

where $\mathcal{D}_{0+}^{\theta;\psi}$ is generalized fractional derivative in the sense of Caputo and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous. Now, by using Theorem 2.1 we will discuss the existence and uniqueness of solutions for (19). Problem (19) is equivalent to the following fractional integral equation

$$u(t) = \frac{1}{\Gamma(\theta)} \int_0^1 \psi'(\tau) G(t,\tau) g(t,\tau) d\tau, \quad t \in [0,1], \tag{20}$$

where $G(t,\tau)$ defined by (18).

Therefore, if $u \in C^2([0,1])$, then a solution of (19) it will be u if and only if it is a solution of (20). Denote the space of all continuous functions by $L = C([0,1])$. Let δ a metric like defined on L as

$$\delta(u,v) = \|u\|_\infty + \|v\|_\infty + \|u-v\|_\infty, \ for \ all \ u, v \in L$$

such that $\|u\|_\infty = max_{t\in[0,1]} |u(t)| \ for \ all \ u \in L.$

Theorem 3.3. *Assume that*

i) $\psi \in C^1[0,1]$ *and there exists* $\xi > 0$ *such that* $\sup_{\tau\in[0,1]} |\psi'(\tau)| \le \xi$.

ii) $|g(\tau, a) - g(\tau, b)| \le \frac{\Gamma(\theta+1)}{[\psi(1)-\psi(0)]^\theta} f_1(\tau)|a - b|, \forall a, b \in \mathbb{R}, \tau \in [0, 1]$ *such that* $f_1 : [0, 1] \to [0, \infty)$ *is a continuous functions.*

iii) $|g(\tau, a)| \le \frac{\Gamma(\theta+1)}{[\psi(1)-\psi(0)]^\theta} f_2(\tau)|a|, \forall a \in \mathbb{R}, \tau \in [0, 1]$ *such that* $f_2 : [0, 1] \to [0, \infty]$ *a continuous functions.*

iv) $max_{\tau\in[0,1]} f_1(\tau) = \eta k_1 < \frac{1}{81}, \quad 0 \le \eta < \frac{1}{9}.$

v) $max_{\tau\in[0,1]} f_2(\tau) = \eta k_2 < \frac{1}{81}, \quad 0 \le \eta < \frac{1}{9}.$

Then, the problem (19) has a unique solution $u \in L$.

Proof. Consider the mapping $F^* : L \to L$ defined by

$$F^* u(t) = \frac{1}{\Gamma(\theta)} \int_0^1 \psi'(\tau) G(t, \tau) g(t, \tau) d\tau, \quad u \in L, \ t \in [0, 1].$$

Let $u, v \in L$, we have

$$\begin{aligned} |F^* u(t) - F^* v(t)| &\le \frac{1}{\Gamma(\theta)} \int_0^1 \psi'(\tau) G(t, \tau) \left| g(\tau, u(\tau) - g(\tau, v(\tau)) \right| d\tau \\ &\le \frac{\theta}{[\psi(1) - \psi(0)]^\theta} \int_0^1 \sup_{\tau\in[0,1]} \left|\psi'(\tau)\right| G(t, \tau)_{\tau\in[0,1]} \\ &\max f_1(\tau) |u(\tau) - v(\tau)| d\tau, \\ &\le \frac{\theta\xi}{[\psi(1) - \psi(0)]^\theta} \eta k_1 \|u - v\|_\infty \int_0^1 G(t, \tau) d\tau. \end{aligned} \tag{21}$$

Since $\psi \in C^1[0, 1]$, for $0 \le \tau \le t \le 1$, we have

$$\begin{aligned} \int_0^1 G(t, \tau) d\tau &= \int_0^1 \frac{[\psi(t) - \psi(0)] \left[[\psi(1) - \psi(\tau)]^{\theta-1} - [\psi(t) - \psi(\tau)]^{\theta-1} \right]}{[\psi(1) - \psi(0)]} d\tau \\ &= \frac{[\psi(t) - \psi(0)]}{[\psi(1) - \psi(0)]} \int_0^1 \psi'(\tau) [\psi(1) - \psi(\tau)]^{\theta-1} \left[\psi'(\tau)\right]^{-1} d\tau \\ &\quad - \frac{[\psi(t) - \psi(0)]}{[\psi(1) - \psi(0)]} \int_0^1 [\psi(t) - \psi(\tau)]^{\varsigma-1} d\tau \\ &\le \frac{[\psi(t) - \psi(0)]}{[\psi(1) - \psi(0)]} \int_0^1 \psi'(\tau) [\psi(1) - \psi(\tau)]^{\theta-1} \left[\psi'(\tau)\right]^{-1} d\tau \\ &\le \frac{1}{\theta\xi} [\psi(t) - \psi(0)][\psi(1) - \psi(0)]^{\theta-1} \\ &\le \frac{1}{\theta\xi} [\psi(1) - \psi(0)]^\theta. \end{aligned} \tag{22}$$

Similarly, for $0 \le t \le \tau \le 1$, we have

$$\begin{aligned} \int_0^1 G(t, \tau) d\tau &= \int_0^1 \frac{[\psi(t) - \psi(0)][\psi(1) - \psi(\tau)]^{\theta-1}}{[\psi(1) - \psi(0)]} d\tau \\ &= \frac{[\psi(t) - \psi(0)]}{[\psi(1) - \psi(0)]} \int_0^1 \psi'(\tau) [\psi(1) - \psi(\tau)]^{\theta-1} \left[\psi'(\tau)\right]^{-1} d\tau \\ &\le \frac{1}{\theta\xi} [\psi(t) - \psi(0)][\psi(1) - \psi(0)]^{\theta-1} \\ &\le \frac{1}{\theta\xi} [\psi(1) - \psi(0)]^\theta. \end{aligned} \tag{23}$$

From (22) and (23), we get

$$\int_0^1 G(t,\tau)d\tau \leq \frac{1}{\theta\xi}[\psi(1)-\psi(0)]^{\theta}, \text{ for all } t\in[0,1] \text{ and } 1<\theta<2.$$

Hence, the inequality (21) becomes

$$\|F^*x - F^*y\|_\infty \leq \eta\tau_1\|x-y\|_\infty. \tag{24}$$

Also, we have from (22), (23) and the condition (iii) that

$$\begin{aligned}|F^*u(t)| &\leq \frac{1}{\Gamma(\theta)}\left|\int_0^1 G(t,\tau)\,|g(\tau,u(\tau))|\,d\tau\right|\\ &\leq \frac{\theta}{[\psi(1)-\psi(0)]^{\theta}}\int_0^1 G(t,\tau)f_2(\tau)|u(\tau)|d\tau\\ &\leq \eta k_2\|u\|_\infty.\end{aligned}$$

Thus,

$$\|F^*u\|_\infty \leq \eta k_2\|u\|_\infty. \tag{25}$$

Similarly, we get

$$\|F^*v\|_\infty \leq \eta k_2\|v\|_\infty. \tag{26}$$

By (24),(25) and (26), we get

$$\begin{aligned}\delta(F^*u,F^*v) &= \|F^*u\|_\infty + \|F^*v\|_\infty + \|F^*u - F^*v\|_\infty\\ &\leq \eta k_2\|u\|_\infty + \eta k_2\|v\|_\infty + \eta k_1\|u-v\|_\infty\\ &\leq \eta(2k_2+k_1)(\|u\|_\infty + \|v\|_\infty + \|u-v\|_\infty)\\ &< \eta(2k_2+k_1^2)\delta(u,v)\end{aligned}$$

where $(2k_2+k_1^2)<1$. Suppose $\ell_1,\ell_2,\ell_3,\ell_4,\ell_5>0$, such that

$$\ell_1<\frac{1}{9},\ell_2<\frac{1}{9},\ell_3<\frac{1}{9},\ell_4<\frac{1}{9},\ell_5<\frac{1}{9} \quad \textit{and} \quad \eta\in[0,1).$$

Then, the following relation is satisfied

$$\delta(F^*u,F^*v) < \ell_1\delta(u,F^*u) + \ell_2\delta(v,F^*v) + \ell_3\delta(u,F^*v) + \ell_4\delta(y,F^*u) + \ell_5\delta(u,v).$$

Hence, by Theorem 2.1, the problem (19) has a unique solution $u\in L$. □

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