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The Double $(G'/G, 1/G)$-Expansion Method and Its Applications for Some Nonlinear Partial Differential Equations

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ABSTRACT: The double $(G'/G, 1/G)$-expansion method is used to find exact travelling wave solutions to the fractional differantial equations in the sense of Jumarie's modified Riemann- Liouville derivative. We exploit this method for the combined KdV- negative-order KdV equation (KdV-nKdV) and the Calogero-Bogoyavlinskii-Schiff equation (CBS) of fractional order. We see that these solutions are concise and easy to understand the physical phenomena of the nonlinear partial differential equations. These solutions can be shown in terms of trigonometric, hyperbolic and rational functions.

Keywords: The double expansion method, the Calogero–Bogoyavlinskii–Schiff equation, the combined KdV- negative-order KdV equation.

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INTRODUCTION

In the past decade, partial differential equations (PDEs) and fractional differential equations (FDEs) have been intensively studied popular subjects among the scientists. FDEs are a generalization of classical differential equations of integer order. The traveling wave solutions of nonlinear partial differential equations (NLPDEs) hold an important place in applied mathematics, physics and engineering. These equations are mathematical models of a physical phenomenon. Therefore, it is very important to obtain the traveling wave solutions of NLPDEs (İnan et al., 2017). These solutions give information about the behaviour of the physical event. The FDEs have interesting structures that deals with many phenomena in physics, chemistry and engineering, for example; in fluid flow, plasma waves, mechanics, solid state physics, oceanic phenomena, atmospheric and so on. Many researchers have been proposed various different methods to find solutions for nonlinear fractional differential equations, and they discovered many useful methods in order to find exact solutions of FDEs. Namely, the sine-cosine method (Taşcan and Bekir, 2009), the homogeneous balance method (En-Gui and Hong-Qing, 1998; Wang et al., 1996), the hyperbolic tangent expansion method (Yang et al., 2001), the tanh-function expansion method (Fan, 2000), the exponential function method (He and Wu, 2006), fractional sub-equation method (Guo et al., 2012; Lu, 2012), the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method (Li et al., 2010), the sub-ODE method (Zhang et al., 2006; Wang et al., 2007), the theta function method (Fan,2002), the differential transformation method (Odibat and Momani, 2008; Ekici and Ayaz, 2017), F-expansion method (Wang and Zhou, 2003; Wang and Li, 2005), the homotopy anlysis method (Arafa et al.,2011), and so on.

In this paper, we use the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method to obtain solutions to the fractional partial differential equations in the sense of modified Riemann-Liouville derivative by Jumarie (Jumarie, 2006). The importance of this double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method is that the exact travelling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables, (G'/G) and $(1/G)$, in which $G = G(\xi)$ satisfies a second order linear ordinary differential equation (LODE), namely $G''(\xi) + \lambda G(\xi) = \mu$, where λ and μ are constants. Hence the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method can be considered as a generalisation of the (G'/G) −expansion method. The α order of modified Riemann-Liouville derivative is defined as $(n \geq 1)$

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^t (t-\xi)^{-\alpha-1}[f(t) - f(0)]d\xi & , \quad \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\xi)^{-\alpha}[f(\xi) - f(0)]d\xi & , \quad 0 < \alpha < 1 \\ \left(f^{(n)}(t)\right)^{(\alpha-n)} & , \quad n \leq \alpha < n+1 \end{cases},$$

where $f: \mathbb{R} \to \mathbb{R}$, $t \to f(t)$ is a continuous function and $\Gamma(\alpha)$ is the gamma function defined as:

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)}.$$

Many interesting properties are known to the Jumarie's modified Riemann-Liouville derivative such as;

- $D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}$,
- $D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t)$,

• $D_t^{\alpha} f(g(t)) = f_g'[g(t)] D_t^{\alpha} g(t) = D_g^{\alpha} f[g(t)](g'(t))^{\alpha}$.

The derivative of α constant is zero, in the sense of modified Riemann-Liouville derivative of order α. This derivative can be applied to the other functions no matter it is differentiable or not.

We organize the rest of this paper as follows; in section 2, we describe the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method and give some properties of the method. In section 3, we deduce exact travelling solutions for the combined KdV-negative-order KdV equation and the Calogero-Bogoyavlinskii-Schiff equation. Finally, a summary and further suggestion of this work is also given in the conclusion.

MATERIALS AND METHODS

In this section, we give a description for the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method in order to find traveling wave solutions of nonlinear evolution equations (NLEEs). First of all, the second-order linear ordinary differential equation (LODE) is given as (Li et al., 2010)

$$G''(\xi) + \lambda G(\xi) = \mu \tag{1}$$

and we define the ϕ and ψ as in the following form

$$\phi = G'/G \ , \quad \psi = 1/G \tag{2}$$

and the derivatives can be written as

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi. \tag{3}$$

The solution of the equation 1 can be written in three different cases, depends on the parameter λ's sign. These cases are derived as follows;

Case 1: For $\lambda < 0$, we have the general solution for the equation 1

$$G(\xi) = n_1 \sinh(\sqrt{-\lambda}\xi) + n_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda} \ , \tag{4}$$

where are n_1 and n_2 are the constants of the integration. Thus, substituting equation 4 into equation 2 and using equation 3, we get

$$\psi^2 = \frac{-\lambda(\phi^2 - 2\mu\psi + \lambda)}{\lambda^2\rho + \mu^2} \ , \tag{5}$$

where $\rho = n_1^2 - n_2^2$.

Case 2: For $\lambda > 0$, we have the general solution for the equation 1

$$G(\xi) = n_1 \sin(\sqrt{\lambda}\xi) + n_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda} \ , \tag{6}$$

where n_1 and n_2 are the constants of the integration. Thus, substituting equation 6 into equation 2 and using equation 3, we get

$$\psi^2 = \frac{\lambda(\phi^2 - 2\mu\psi + \lambda)}{\lambda^2\rho - \mu^2} \ ,$$

where $\rho = n_1^2 + n_2^2$.

Case 3: For $\lambda = 0$, we have the general solution for the equation 1

$$G(\xi) = \frac{\mu}{2}\xi^2 + n_1\xi + n_2 \ , \tag{7}$$

where n_1 and n_2 are the constants of the integration. Again, substituting equation 7 into equation 2 and using equation 3, we get

$$\psi^2 = \frac{(\phi^2 - 2\mu\psi)}{n_1^2 - 2\mu n_2}.$$

Next, our main interest is to apply the double $(\frac{G'}{G}, \frac{1}{G})$-expansion method to the general nonlinear evolution equation. Suppose that $u = u(x, y, t)$ is an unknown function depends on the x, y and t variables, and we define the polynomial P in $u(x, y, t)$ and its various order partial derivatives with nonlinear terms as

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{xt}, u_{xy} \ldots) = 0. \tag{8}$$

In order to solve equation 8, we use the following steps :

Step 1: We assign a new transformation such that

$$u(x, y, t) = U(\xi) \ , \quad \xi = x + y - st + \xi_0 \ , \tag{9}$$

where ξ_0 is a constant and s is the velocity of traveling wave. The transformation in equation 9 transforms equation 8 into an ordinary differential equation (ODE) for $u = U(\xi)$, in the following form

$$Q\left(U, -sU', U', s^2U'', U'', \ldots\right) = 0 \, , \tag{10}$$

where $U(\xi)$ and its derivatives with respect to ξ are the elements of the Q polynomial.

Step 2: Next we assume that the solution of equation 10 can be expressed in terms of polynomials $\phi(\xi)$ and $\psi(\xi)$ as:

$$U(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi \ , \tag{11}$$

where $G = G(\xi)$ satisfies equation 1, and a_i $(i = 0,1\ldots N)$, b_i $(i = 1,2\ldots N)$, s, λ and μ are constants to be determined later. To find the value of N, we use the homogeneous balance method.

Step 3: Having find the value of N, we substitute equation 11 into equation 10 and using equation 3 with equation 5 (as in the case 1), the equation 10 will be changed into a polynomial of ψ^j and ϕ^k, where the degree is $j \leq 1$ and $0 \leq k \leq n$, n is any integer. Hence by equating the same powers of the polynomials in the resulting equation to zero, gives a system of algebraic equations in a_i $(i = 0,1\ldots N)$, b_i $(i = 1,2\ldots N)$, s, λ $(\lambda < 0)$, μ, n_1 and n_2.

Step 4: In order to solve the algebraic equations that we deduced in Step 3, we make use the computer software programme to get easily access to the solution. If there is a possible solution, we obtain values for a_i, b_i, s, λ $(\lambda < 0)$, μ, n_1 and n_2. Having substitute these values into the equation 11, hence we obtain the traveling wave solution for the equation 10 in terms of hyperbolic functions for the case 1. Next, we transform the variables back which we used in equation 9 and we obtain the solution for the nonlinear partial differential equation in equation 8. In order to get solutions for the cases 2 and 3, similarly step 3 and step 4 can be applied to obtain the traveling wave solutions of equation 10 (i.e. equation 8), in terms of trigonometric functions and rational functions respectively.

RESULTS AND DISCUSSION

In this section we apply the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method that we mentioned in the previous section for the combined KdV- negative order KdV equation (KdV-nKdV) and the Calogero-Bogoyavlinskii-Schiff equation (CBS) of fractional order.

The combined KdV- negative-order KdV equation

The combined KdV-nKdV equation is (Wazwaz, 2018)

$$u_t + 6uu_x + u_{xxx} + u_{xxt} + 4uu_t + 2u_x\partial_x^{-1}(u_t) = 0. \tag{12}$$

We make the following transformation to the equation 12

$$u(x,t) = v_x(x,t),$$

hence we get the following equation

$$v_{xt} + 6v_xv_{xx} + v_{xxxx} + v_{xxxt} + 4v_xv_{xt} + 2v_{xx}v_t = 0. \tag{13}$$

Now we apply the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method to the equation 13. Suppose that

$$v(x,t) = V(\xi), \quad \xi = x - st + \xi_0 \ , \tag{14}$$

where s and ξ_0 are constants with $s \neq 0$. Substituting equation 14 into equation 13, reduces to the nonlinear ordinary differential equation

$$-sV'' + 6V'V'' + V^{(4)} - sV^{(4)} - 4sV'V'' - 2sV''V' = 0 \ , \tag{15}$$

where $V' = \frac{dV}{d\xi}$. Integrating equation 15 once and substituting $V' = U$ back, gives the following ODE equation

$$c + sU + (s-1)(U'' + 3U^2) = 0. \tag{16}$$

In order to find N, we use homogeneous balance method, such as balancing the terms U'' and U^2 in equation 16, gives $N = 2$, hence, from equation 11 we get

$$U(\xi) = a_0 + a_1\phi(x) + b_1\psi(x) + a_2\phi^2(x) + b_2\phi(x)\psi(x). \tag{17}$$

Next we substitute the equation 17 into equation 16 and equating the same powers of the polynomials $\phi(x)$ and $\psi(x)$ in the resulting equation to zero, gives a system of algebraic equations. Solving these equations with the help of the mathematical programme Maple, we find the following solutions for $c, b_j(j = 1,2), a_i(i = 0,1,2)$:

Case 1: For $\lambda < 0$, we get

$$a_0 = \frac{5\lambda(s-1)+s}{6(1-s)}, a_1 = 0,\ a_2 = -1, b_1 = \mu, b_2 = \sqrt{\frac{\lambda^2\rho-\mu^2}{\lambda}}\ , c = \frac{\lambda^2(s-1)^2 - s^2}{12(1-s)}\ .$$

Substituting these results into equation 17, we obtain the following solutions of equation 16

$$U(\xi) = \frac{5\lambda(s-1)+s}{6(1-s)} - \left(\frac{n_1\cosh(\sqrt{-\lambda}\xi)+n_2\sinh(\sqrt{-\lambda}\xi)}{n_1\sinh(\sqrt{-\lambda}\xi)+n_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}}\right)^2 + \frac{\mu}{n_1\sinh(\sqrt{-\lambda}\xi)+n_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}} + \sqrt{\mu^2-\lambda^2\rho}\,\frac{n_1\cosh\left(\sqrt{-\lambda}\xi\right)+n_2\sinh\left(\sqrt{-\lambda}\xi\right)}{\left(n_1\sinh(\sqrt{-\lambda}\xi)+n_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}\right)^2}\ , \qquad (18)$$

where $\xi = x - st + \xi_0$ and $\rho = n_1^2 - n_2^2$.

If we take $s = 2$, $\mu = 0$, $n_1 = \cosh(\xi_0)$, $n_2 = \sinh(\xi_0)$ in equation 18 we get the hyperbolic solution for the equation 12

$$U(\xi) = -\frac{5}{6}\lambda - \frac{1}{3} + \lambda\coth^2\left(\sqrt{\lambda}\xi+\xi_0\right) + i\sqrt{5}\lambda\coth\left(\sqrt{\lambda}\xi+\xi_0\right)csch\left(\sqrt{\lambda}\xi+\xi_0\right).$$

Case 2: For $\lambda > 0$, we get

$$a_0 = \frac{5\lambda(s-1)+s}{6(1-s)}, \quad a_1 = 0, \quad a_2 = -1, \quad b_1 = \mu, b_2 = \sqrt{\frac{\lambda^2\rho-\mu^2}{\lambda}}\ , \quad c = \frac{\lambda^2(s-1)^2 - s^2}{12(1-s)}$$

substituting these results into equation 17, we obtain the following solutions for the equation 16

$$U(\xi) = \frac{5\lambda(s-1)+s}{6(1-s)} - \lambda\left(\frac{n_2\sin\left(\sqrt{\lambda}\xi\right)+n_1\cos\left(\sqrt{\lambda}\xi\right)}{n_1\sin\left(\sqrt{\lambda}\xi\right)+n_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)^2 + \frac{\mu}{n_2\cos(\sqrt{\lambda}\xi)+n_1\sin(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}$$

$$+\sqrt{\lambda^2\rho-\mu^2}\ \frac{n_1\cos(\sqrt{\lambda}\xi)+n_2\sin(\sqrt{\lambda}\xi)}{\left(n_1\sin(\sqrt{\lambda}\xi)+n_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}\right)^2}\ , \qquad (19)$$

where $\xi = x - st + \xi_0$ and $\rho = n_1^2 + n_2^2$. If we take $s = 2, \mu = 0, n_1 = \cos(\xi_0)$, $n_2 = \sin(\xi_0)$ in equation 19, we get the trigonometric solution for the equation 12

$$U(\xi) = -\frac{5}{6}\lambda - \frac{1}{3} - \lambda\cot^2\left(\sqrt{\lambda}\xi+\xi_0\right) + \sqrt{5}\lambda\cot\left(\sqrt{\lambda}\xi+\xi_0\right)\csc\left(\sqrt{\lambda}\xi+\xi_0\right)\ ,$$

where $\xi = x - st + \xi_0$.

Case 3: For $\lambda = 0$, we get

$$a_0 = \frac{-s+\sqrt{s^2+12c-12cs}}{6(s-1)}, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = 0, \quad b_2 = 0,$$

substituting these results into equation 17, we obtain the following solutions for the equation 16

$$U(\xi)=\frac{-s+\sqrt{s^2+12c-12cs}}{6(s-1)}\ .$$

The Calogero–Bogoyavlinskii–Schiff equation

In this section, the double $(\mathbf{G}'/\mathbf{G},\mathbf{1}/\mathbf{G})$-expansion method is used to find new traveling wave solutions to the nonlinear Calogero–Bogoyavlinskii–Schiff equation of fractional order (Mohyud-Din and Saba, 2017). The CBS equation with the Jumarie's modified Riemann-Liouville fractional derivative (Jumarie, 2006) can be written in the following form

$$D_t^{2\alpha}u_x+u_{xxxy}+4u_xu_{xy}+2u_{xx}u_y=0,\qquad 0<\alpha\leq 1\ , \tag{20}$$

where α is defined in Section 1. Next we transform the equation 20 into an ordinary differential equation by using the following transformation

$$u(x,y,t)=U(\xi),\quad \xi=x+y+s\frac{t^\alpha}{\Gamma(\alpha+1)}+\xi_0\ ,$$

where s and ξ_0 are constants with $s\neq 0$. These transformations reduce the equation 20 to the following nonlinear ODE

$$sU''+U^{(4)}+6U'U''=0\,,$$

integrating this equation gives

$$c+sU'+U'''+3(U')^2=0\ , \tag{21}$$

where $U'=\frac{dU}{d\xi}$. Balancing the terms U''' and $(U')^2$, gives $N=1$, hence from equation 11 we write the following solution

$$U(\xi)=a_0+a_1\phi(x)+b_1\psi(x). \tag{22}$$

Next we substitute the equation 22 into the equation 21 and equating the same powers of the polynomials $\phi(x)$ and $\psi(x)$ in the resulting equation to zero, gives a system of algebraic equations. Solving these equations with the help of the mathematical software Maple, we find the following solutions for s,a_1,b_1:

Case 1: For $\lambda<0$, we get

$$a_1=1,\quad b_1=\sqrt{-\frac{\lambda^2\rho+\mu^2}{\lambda}},\qquad s=\lambda,$$

substituting these results into the equation 22, we obtain the following solution for the equation 20

$$U(\xi)=a_0+\left(\sqrt{-\lambda}\right)\frac{n_1\cosh\left(\sqrt{-\lambda}\xi\right)+n_2\sinh\left(\sqrt{-\lambda}\xi\right)}{n_1\sinh\left(\sqrt{-\lambda}\xi\right)+n_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}+\frac{\sqrt{-\frac{\lambda^2\rho+\mu^2}{\lambda}}}{n_1\sinh\left(\sqrt{-\lambda}\xi\right)+n_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}, \tag{23}$$

where $\xi=x+y+\lambda\frac{t^\alpha}{\Gamma(\alpha+1)}+\xi_0$ and $\rho=n_1^2-n_2^2$.

If we take $s=\lambda=-2,\mu=0,\ a_0=1,n_1=\cosh(\xi_0)$ and $n_2=\sinh(\xi_0)$ in equation 23, we get the hyperbolic solution for the equation 20

$$U(\xi) = 1 + \sqrt{2}\coth\left(\sqrt{2}\xi + \xi_0\right) + \sqrt{2}\mathrm{csch}\left(\sqrt{2}\xi + \xi_0\right) \ ,$$

where $\xi = x + y - 2t + \xi_0$.

Case 2: For $\lambda > 0$, we get

$$a_1 = 1, \quad b_1 = \sqrt{\frac{\lambda^2\rho - \mu^2}{\lambda}}, \qquad s = \lambda$$

substituting these results into the equation 22, we obtain the following solution of the equation 20

$$U(\xi) = a_0 + \left(\sqrt{\lambda}\right)\frac{n_1\cos(\sqrt{\lambda}\xi) - n_2\sin(\sqrt{\lambda}\xi)}{n_1\sin(\sqrt{\lambda}\xi) + n_2\cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}} + \frac{\sqrt{\frac{\lambda^2\rho-\mu^2}{\lambda}}}{n_1\sin(\sqrt{\lambda}\xi) + n_2\cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}}, \tag{24}$$

where $\xi == x + y + \lambda\frac{t^\alpha}{\Gamma(\alpha+1)} + \xi_0$ and $\rho = n_1^2 + n_2^2$.

If we take $s = \lambda = 2, \mu = 0,\ a_0 = 1$, and $n_1 = \cos(\xi_0), n_2 = \sin(\xi_0)$ in equation 24 we get the trigonometric solution for the equation 20

$$U(\xi) = 1 + \sqrt{2}\cot\left(\sqrt{2}\xi + \xi_0\right) + \sqrt{2}\csc\left(\sqrt{2}\xi + \xi_0\right) \ ,$$

where $\xi = x + y + 2t + \xi_0$.

Case 3: For $\lambda = 0$, we get

$$a_1 = 2, \qquad b_1 = 0, \qquad c = 0,$$

substituting these results into the equation 22, we obtain the following solution for the equation 20

$$U(\xi) = a_0 + 2\frac{\mu\xi + n_1}{\frac{\mu}{2}\xi^2 + n_1\xi + n_2}.$$

CONCLUSION

In this paper, we proposed various types of travelling wave solutions for the KdV-nKdV equation and for the Calogero–Bogoyavlinskii–Schiff equation that are successfully found by using the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method. These solutions are rational, hyperbolic and trigonometric solutions.

The main idea of this method is to reduce the partial differential equation to an ODE by using the travelling wave transformation (equation 9), after integrating the ODE in equation 10, once or more, then express the ODE in a compact form. This ODE can be written by a m-th degree polynomial in terms of $(\mathbf{G}'/\mathbf{G})$ and $(\mathbf{1}/\mathbf{G})$, where $G = G(\xi)$ is the general solution of the second order LODE in equation 1. In order to find the positive integer m, we use the homogeneous balance method, that is balancing between the highest order derivative term and nonlinear term. The coefficients of the polynomials can be obtained by solving a set of algebraic equations. Because of the tedious calculation, these equations can be solved easily by using computer software programs such as Maple and Mathematica. Generally, it is possible to find a solution of the algebraic equations.

The difference between the $(\mathbf{G}'/\mathbf{G})$-expansion method and the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method is; if we take $\mu = 0$ in equation 1 and $b_i = 0$ in equation 11, the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method reduces to the $(\mathbf{G}'/\mathbf{G})$-expansion method. So the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method is an extension of the $(\mathbf{G}'/\mathbf{G})$ expansion method. The proposed method in this paper is more effective and general than the $(\mathbf{G}'/\mathbf{G})$-expansion method, since it gives exact solutions in general form. In summary, the advantage of the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method over the $(\mathbf{G}'/\mathbf{G})$-expansion method is that the solutions from the first method recover the solutions from the second method.

Finally, the double $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method is useful, practical, and concise method to find travelling wave solutions to the nonlinear partial differential equations, hence this method can be applied for many other nonlinear partial differential equations.

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REFERENCES

Arafa AAM, Rida SZ, Mohamed H, 2011. Homotopy analysis method for solving biological population model. Communications in Theoretical Physics, 56(5), 797.

Ekici M, Ayaz F, 2017. Solution of model equation of completely passive natural convection by improved differential transform method. Research on Engineering Structures and Materials, 3(1), 1-10.

En-Gui F, Hong-Qing Z, 1998. The homogeneous balance method for solving nonlinear soliton equations. Acta Phys Sinica, 47(3), 363.

Fan E, 2000. Extended tanh-function method and its applications to nonlinear equations. Physics Letters A, 277(4), 212-218.

Fan E, 2002. Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method. Journal of Physics A: Mathematical and General, 35(32), 6853.

Guo S, Mei L, Li Y, Sun Y, 2012. The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics. Physics Letters A, 376(4), 407-411.

He JH, Wu XH, 2006. Exp-function method for nonlinear wave equations. Chaos, Solitons & Fractals, 30(3), 700-708.

Inan IE, Duran S, Uğurlu Y, 2017. TAN (F(ξ2))-expansion method for traveling wave solutions of AKNS and Burgers-like equations. Optik, 138, 15-20.

Jumarie G, 2006. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Computers & Mathematics with Applications, 51(9-10), 1367-1376.

Li LX, Li EQ, Wang ML, 2010. The $(\mathbf{G}'/\mathbf{G}, \mathbf{1}/\mathbf{G})$-expansion method and its application to travelling wave solutions of the Zakharov equations. Applied Mathematics-A Journal of Chinese Universities, 25(4), 454-462.

Lu B. 2012. Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations. Physics Letters A, 376(28-29), 2045-2048.

Mohyud-Din ST, Saba F, 2017. Extended $(\mathbf{G}'/\mathbf{G})$-expansion method for Calogero–Bogoyavlinskii–Schiff equation of fractional order. Journal of Taibah University for Science, 11(6), 1099-1109.

Odibat Z, Momani S. 2008. A generalized differential transform method for linear partial differential equations of fractional order. Applied Mathematics Letters, 21(2), 194-199.

Taşcan F, Bekir A, 2009. Analytic solutions of the (2+ 1)-dimensional nonlinear evolution equations using the sine–cosine method. Applied Mathematics and Computation, 215(8), 3134-3139.

Wang M, Zhou Y, Li Z, 1996. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. Physics Letters A, 216(1-5), 67-75.

Wang, M, Zhou Y, 2003. The periodic wave solutions for the Klein–Gordon–Schrödinger equations. Physics Letters A, 318(1-2), 84-92.

Wang M, Li X, 2005. Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations. Physics Letters A, 343(1-3), 48-54.

Wang M, Li X, Zhang J, 2007. Various exact solutions of nonlinear Schrödinger equation with two nonlinear terms. Chaos, Solitons & Fractals, 31(3), 594-601.

Wazwaz AM, 2018. A new integrable equation that combines the KdV equation with the negative-order KdV equation. Mathematical Methods in the Applied Sciences, 41(1), 80-87.

Yang L, Liu J, Yang K, 2001. Exact solutions of nonlinear PDE, nonlinear transformations and reduction of nonlinear PDE to a quadrature. Physics Letters A, 278(5), 267-270.

Zhang JL, Wang ML, Li XZ, 2006. The subsidiary ordinary differential equations and the exact solutions of the higher order dispersive nonlinear Schrödinger equation. Physics Letters A, 357(3), 188-195.