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On the Characterizations of Convolution Manifolds Obtained by Helix Hypersurfaces

Helis Hiperyüzeyleri Tarafından Elde Edilen Konvolüsyon Manifoldların Karakterizasyonları Üzerine

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Abstract
In this study, a submanifold obtained by tensor product of the immersions of two helix hypersurfaces obtained by planar curves is constructed. It is seen that, this submanifold is a convolution manifold with convolution metric and its minimality is examined. After, some characterizations are given by looking at the totally geodesic of same submanifold.

Keywords: Convolution Manifold, Helix Hypersurface, Planar Curve

Öz
Bu çalışmada, düzlemsel eğrilerden elde edilen iki helis hiperyüzey immersiyonlarının tensör çarpımları tarafından elde edilen bir altmanifold oluşturuldu. Bu altmanifoldun, konvolüsyon metrik ile birlikte bir konvolüsyon manifold olduğu görüldü ve bu manifoldun minimalliği incelendi. Daha sonra aynı altmanifoldun tamamen geodezikliğine bakılarak bazı karakterizasyonlar verildi.

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1. Introduction

In 1993, B.Y.Chen has started the study of the tensor product immersion of two immersions of a given Riemannian manifold. Considering Chen's definition, F. Decruyenaere and his friends (Decruyenaere vd., 1993) have studied the tensor product of two immersions of different manifolds in general; under certain conditions, this realizes an immersion of the product manifold. In (Mihai vd., 1994/1995), tensor product surfaces of Euclidean plane curves have investigated. Also, authors have studied the tensor product of surfaces of a Euclidean space curve and a Euclidean plane curve in (Arslan vd., 2001). And, in Phd thesis (Aksoy, 2008), Aksoy has insvestigated the tensor products of a surface in Euclidean space and a curve in Euclidean plane.

On the other hand, Chen has introduced the notion of convolution manifolds, which is related to isometric immersions to Euclidean spaces in (Chen, 2003). It is note that, let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let f be a positive differentiable function on N_1. The well-known notion of $warped\ product\ manifold$ $N_1 \times_f N_2$ is defined as the product manifold $N_1 \times N_2$ equipped with the Riemannian metric given by $g_1 + f^2 g_2$. It is well-known that, the notion of warped product plays some important roles in differential geometry as well as in physics (O'neill, 1983). The notion of convolution can be regarded as a natural extension of warped products. The notion of convolution products is defined as follows: Let N_1 and N_2 be two Riemannian manifolds equipped with metrics g_1 and g_2, respectively. Consider the symmetric tensor field $g_{f,h}$ of type (0,2) on the product manifold $N_1 \times N_2$ defined by

$$g_{f,h} = h^2 g_1 + f^2 g_2 + 2fhdf \otimes dh \quad (1)$$

for some positive differentiable functions f and h on N_1 and N_2, respectively. The symmetric tensor $g_{f,h}$ is denoted by ${}_h g_1 *_f g_2$, which is called the $convolution$ of g_1 and g_2 (via h and f). The product manifold $N_1 \times N_2$ equipped with ${}_h g_1 *_f g_2$ is called a $convolution\ manifold$, which is denoted by ${}_h N_1 \star_f N_2$. When the scale functions f and h are irrelevant, we simply denote ${}_h N_1 \star_f N_2$ and ${}_h g_1 *_f g_2$ by $N_1 \times N_2$ and $g_1 * g_2$, respectively. We also investigate relations between usual product manifolds and convolution manifolds by considering the tensor product of a regular surface of Euclidean 3-space and a planar curve.

Let N be a Riemannian manifold equipped with a Riemannian metric g. The gradient $grad\varphi$ of a function φ on N is defined by $< grad\varphi, X >= X\varphi$, for vector fields X tangent to N. If N is a submanifold of a Riemannian manifold $\widetilde{M}$, the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad (2)$$
$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad (3)$$

respectively, for vector fields X, Y tangent to N and normal to N. Here $\widetilde{\nabla}$ denotes the Riemannian connection on $\widetilde{M}$, σ the second fundamental form, D the normal connection and A the shape operator of N in $\widetilde{M}$. The second fundamental form and the shape operator are related by $< A_\xi X, Y >=< \sigma(X,Y), \xi >$, where $<,>$ denotes the inner product on M as well as on $\widetilde{M}$. A submanifold in a Riemannian manifold is called totally geodesic if its second fundamental form vanishes identically, or equivalently, its shape operator vanishes identically.

The Gauss equation of N in $\widetilde{M}$ is given by

$$\tilde{R}(X,Y;Z,W) = R(X,Y;Z,W) + < \sigma(X,Z), \sigma(Y,W) > - < \sigma(X,W), \sigma(Y,Z) >, \quad (4)$$

for X, Y, Z, W tangent to M, where R and $\tilde{R}$ denote the curvature tensors of N and $\widetilde{M}$, respectively. The covariant derivative $\bar{\nabla}\sigma$ of σ with respect to the connection on $TM \oplus T^{\perp}M$ is defined by

$$(\bar{\nabla}_X \sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \quad (5)$$

The Codazzi equation is

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X \sigma)(Y,Z) - (\bar{\nabla}_Y \sigma)(X,Z), \quad (6)$$

where $(\tilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X,Y)Z$.

Let $E^m \otimes E^n$ denote the tensor product of two Euclidean spaces E^m and E^n. Then $E^m \otimes E^n$ is isometric to E^{mn}. The Euclidean inner product $<,>$ on $E^m \otimes E^n$ is given by

$$< \alpha \otimes \beta, \gamma \otimes \delta > = < \alpha, \gamma >< \beta, \delta >, \tag{7}$$

where $< \alpha, \gamma >$ denotes the Euclidean inner product of $\alpha, \gamma \in E^m$ and $< \beta, \delta >$ the Euclidean inner product of $\beta, \delta \in E^n$ (for more details about convolution manifolds, we refer to (Chen, 2002, 2003)).

Let M, N be two differentiable manifolds and $f: M \to E^m$, $h: N \to E^n$ be two immersions. The direct sum map $f \oplus h: M \times N \to E^{m+n}$ and tensor product map $f \otimes h: M \times N \to E^{mn}$ are defined by

$$(f \oplus h)(p,q) = \big(f(p), h(q)\big), \tag{8}$$

$$(f \otimes h)(p,q) = f(p) \otimes h(q), \tag{9}$$

respectively. Necessary and sufficient conditions for $f \otimes h$ to be an immersion have obtained in (Decruyenaere vd., 1993).

Proposition 1.1. Let $x: (N_1, g_1) \to E_*^n \subset E^n$ and $y: (N_2, g_2) \to E_*^m \subset E^m$ be isometric immersions of Riemannian manifolds (N_1, g_1) and (N_2, g_2) into E_*^n and E_*^m, respectively. Then, the map

$$\psi: N_1 \times N_2 \to E^n \otimes E^m = E^{nm}; (u,v) \to x(u) \otimes y(v),\ u \in N_1,\ v \in N_2 \tag{10}$$

gives rise to a convolution manifold $N_1 \star N_2$ equipped with

$$_{\rho_2}g_1 *_{\rho_1} g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2\rho_1\rho_2 d\rho_1 \otimes d\rho_2, \tag{11}$$

where $\rho_1 = \sqrt{\Sigma_{j=1}^n x_j^2}$ and $\rho_2 = \sqrt{\Sigma_{\alpha=1}^m y_\alpha^2}$ denote the distance functions of x and y and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are Euclidean coordinate systems of E^n and E^m, respectively (Chen, 2003).

Proof. For vector fields X, Y tangent to N_1 and Z, W tangent to N_2, we have

$$d\psi(X) = X\psi = X \otimes y, d\psi(Z) = Z\psi = x \otimes Z. \tag{12}$$

Also, it follows from the definition of gradient of $\rho_1 = |x|$ that

$$< X, x > = \frac{1}{2} X < x, x > = \rho_1(X\rho_1) = \rho_1 d\rho_1(X). \tag{13}$$

Similarly, we have

$$\rho_2 d\rho_2(Z) = < Z, y >. \tag{14}$$

From (7),(12),(13) and (14), we obtain Proposition 1.1.

Example 1.1. If $y: (N_2, g_2) \to E_*^m \subset E^m$ *is* an isometric immersion such that $y(N_2)$ is contained in the unit hypersphere S^{m-1} of E^m centered at the origin. Then, the convolution $g_1 * g_2$ *of* g_1 and g_2 on the convolution manifold $N_1 \star N_2$ defined by (11) is nothing but the warped product metric*:* $g_1 + |x|^2 g_2$ (Chen, 2003).

2. Some Characterizations for Convolution Manifolds Obtained By Helix Hypersurfaces

Let $\phi: U \subset R^2 \to E_*^3 \subset E^3$, $\phi(u,v) = \alpha(u) + v(\sin\theta N^\alpha + \cos\theta B^\alpha)$ and $\psi: W \subset R^2 \to E_*^3 \subset E^3$, $\psi(w,z) = \beta(w) + z(\sin\theta N^\beta + \cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I \subset R \to E_*^3 \subset E^3$, $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u))$ and $\beta: J \subset R \to E_*^3 \subset E^3$, $\beta(w) =$

$(\beta_1(w), \beta_2(w), \beta_3(w))$, respectively. Here, N^α and B^α denote elements of the Frenet frame $\{T^\alpha = V_1{}^\alpha, N^\alpha = V_2{}^\alpha, B^\alpha = V_3{}^\alpha\}$ of the curve α and N^β and B^β denote elements of the Frenet frame $\{T^\beta = V_1{}^\beta, N^\beta = V_2{}^\beta, B^\beta = V_3{}^\beta\}$ of the curve β. Then, their tensor product is given by

$$f(u,v,w,z) = (\phi \otimes \psi): U \times W \subseteq \mathrm{R}^4 \to E^3 \otimes E^3 \cong E^9,$$

$$= (\phi_1(u,v)\psi_1(w,z), \phi_1(u,v)\psi_2(w,z), \phi_1(u,v)\psi_3(w,z), \phi_2(u,v)\psi_1(w,z), \phi_2(u,v)\psi_2(w,z),$$

$$\phi_2(u,v)\psi_3(w,z), \phi_3(u,v)\psi_1(w,z), \phi_3(u,v)\psi_2(w,z), \phi_3(u,v)\psi_3(w,z)).$$

A basis of tangent space of manifold $U \times W$ is $\{X = \frac{\partial f}{\partial u}, Y = \frac{\partial f}{\partial v}, W = \frac{\partial f}{\partial w}, Z = \frac{\partial f}{\partial z}\}$, such that

$X = ((1 - vk_1^\alpha \sin\theta)T_1^\alpha[\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)], (1 - vk_1^\alpha \sin\theta)T_1^\alpha[\beta_2(w) + z(\sin\theta + \cos\theta B_2^\beta)],$
$(1 - vk_1^\alpha \sin\theta)T_1^\alpha[\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)], (1 - vk_1^\alpha \sin\theta)T_2^\alpha[\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)],$
$(1 - vk_1^\alpha \sin\theta)T_2^\alpha[\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)], (1 - vk_1^\alpha \sin\theta)T_2^\alpha[\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)],$
$(1 - vk_1^\alpha \sin\theta)T_3^\alpha[\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)], (1 - vk_1^\alpha \sin\theta)T_3^\alpha[\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)],$
$(1 - vk_1^\alpha \sin\theta)T_3^\alpha[\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)]),$

$Y = ((\sin\theta N_1^\alpha + \cos\theta B_1^\alpha)[\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)], (\sin\theta N_1^\alpha + \cos\theta B_1^\alpha)[\beta_2(w) + z(\sin\theta N_2^\beta$
$+\cos\theta B_2^\beta)], (\sin\theta N_1^\alpha + \cos\theta B_1^\alpha)[\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)], (\sin\theta N_2^\alpha + \cos\theta B_2^\alpha)[\beta_1(w)$
$+z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)], (\sin\theta N_2^\alpha + \cos\theta B_2^\alpha)[\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)], (\sin\theta N_2^\alpha$
$+\cos\theta B_2^\alpha)[\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)], (\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)[\beta_1(w) + z(\sin\theta N_1^\beta$
$+\cos\theta B_1^\beta)], (\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)[\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)], (\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)[\beta_3(w)$
$+z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)]),$

$W = (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)\left(1 - zk_1^\beta \sin\theta\right)T_1^\beta, (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)\left(1 -\right.$
$\left.zk_1^\beta \sin\theta\right)T_2^\beta, (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)\left(1 - zk_1^\beta \sin\theta\right)T_3^\beta, (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)\left(1 -\right.$
$\left.zk_1^\beta \sin\theta\right)T_1^\beta, (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)\left(1 - zk_1^\beta \sin\theta\right)T_2^\beta, (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)\left(1 -\right.$
$\left.zk_1^\beta \sin\theta\right)T_3^\beta, (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)\left(1 - zk_1^\beta \sin\theta\right)T_1^\beta, (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)\left(1 -\right.$
$\left.zk_1^\beta \sin\theta\right)T_2^\beta, (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)\left(1 - zk_1^\beta \sin\theta\right)T_3^\beta),$

$Z = (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\sin\theta N_1^\alpha + \cos\theta B_1^\alpha), (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\sin\theta N_2^\alpha$
$+\cos\theta B_2^\alpha), (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\sin\theta N_3^\alpha + \cos\theta B_3^\alpha), (\alpha_2(u) + v\sin\theta N_2^\alpha +$
$v\cos\theta B_2^\alpha)(\sin\theta N_1^\alpha + \cos\theta B_1^\alpha), (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\sin\theta N_2^\alpha + \cos\theta B_2^\alpha), (\alpha_2(u) +$
$v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\sin\theta N_3^\alpha + \cos\theta B_3^\alpha), (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\sin\theta N_1^\alpha + \cos\theta B_1^\alpha), (\alpha_3(u) +$
$v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\sin\theta N_2^\alpha + \cos\theta B_2^\alpha), (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)).$

The coefficients of the Riemannian metric g induced on Imf by the Euclidean metric of E^9 can be given as in the following lemma:

Lemma 2.1. Let $\phi: U \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\phi(u,v) = \alpha(u) + v(\sin\theta N^\alpha + \cos\theta B^\alpha)$ and $\psi: W \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\psi(w,z) = \beta(w) + z(\sin\theta N^\beta + \cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I \subset \mathrm{R} \to E_*^3 \subset E^3$, $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u))$ and $\beta: J \subset \mathrm{R} \to E_*^3 \subset E^3$, $\beta(w) = (\beta_1(w), \beta_2(w), \beta_3(w))$, respectively. Here, N^α and B^α denote elements of the Frenet frame $\{T^\alpha = V_1{}^\alpha, N^\alpha = V_2{}^\alpha, B^\alpha = V_3{}^\alpha\}$ of the curve α and N^β and B^β denote elements of the Frenet frame $\{T^\beta = V_1{}^\beta, N^\beta = V_2{}^\beta, B^\beta = V_3{}^\beta\}$ of the curve β. For their tensor product which is given by $f(u,v,w,z) = (\phi \otimes \psi): U \times W \subseteq \mathrm{R}^4 \to E^3 \otimes E^3 \cong E^9$, the coefficients of the Riemannian metric g induced on Imf by the Euclidean metric of E^9 are

$g_{11}=\rho_2^2 g_1(\phi_u,\phi_u),\ \ g_{22}=\rho_2^2 g_1(\phi_v,\phi_v),\ g_{33}=\rho_1^2 g_2(\psi_w,\psi_w),\ g_{44}=\rho_1^2 g_2(\psi_z,\psi_z),$

$g_{12}=\rho_2^2 g_1(\phi_u,\phi_v)=g_{21},\ \ g_{13}=\rho_1\rho_2 d\rho_1(\phi_u)\rho_2(\psi_w)=g_{31},\ g_{14}=\rho_1\rho_2 d\rho_1(\phi_u)\rho_2(\psi_z)=g_{41},$

$g_{23}=\rho_1\rho_2 d\rho_1(\phi_v)\rho_2(\psi_w)=g_{32},\ g_{24}=\rho_1\rho_2 d\rho_1(\phi_v)\rho_2(\psi_z)=g_{42},\ g_{34}=\rho_1^2 g_2(\psi_w,\psi_z)=g_{43}.$

Here, $\rho_1=\sqrt{\Sigma_{i=1}^3\phi_i^2}$ and $\rho_2=\sqrt{\Sigma_{j=1}^2\psi_j^2}$ denote the distance functions of ϕ and ψ; $\phi=(\phi_1,\phi_2,\phi_3)$ and $\psi=(\psi_1,\psi_2,\psi_3)$ are Euclidean coordinate systems of E^3, respectively.

Proof: Let $\phi: U\subset \mathrm{R}^2\to E_*^3\subset E^3$, $\phi(u,v)=\alpha(u)+v(\sin\theta N^\alpha+\cos\theta B^\alpha)$ and $\psi: W\subset \mathrm{R}^2\to E_*^3\subset E^3$, $\psi(w,z)=\beta(w)+z(\sin\theta N^\beta+\cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I\subset \mathrm{R}\to E_*^3\subset E^3$, $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u))$ and $\beta: J\subset \mathrm{R}\to E_*^3\subset E^3$, $\beta(w)=(\beta_1(w),\beta_2(w),\beta_3(w))$, respectively. Then their tensor product $(\phi\otimes\psi): U\times W\subseteq \mathrm{R}^4\to E^3\otimes E^3\cong E^9$ is obtained from (10). Also, using the Riemannian metric g, its coefficients are obtained as following:

$g_{11}=<X,X>=\|\phi_u\|^2\|\psi\|^2, g_{22}=<Y,Y>=\|\phi_v\|^2\|\psi\|^2, g_{33}=<W,W>=\|\psi_w\|^2\|\phi\|^2,$

$g_{44}=<Z,Z>=\|\psi_z\|^2\|\phi\|^2, g_{12}=<X,Y>=<\phi_u,\phi_v>\|\psi\|^2, g_{13}=<X,W>=<\phi_u,\phi><\psi,\psi_w>,$

$g_{14}=<X,Z>=<\phi_u,\phi><\psi,\psi_z>, g_{23}=<Y,W>=<\phi_v,\phi><\psi,\psi_w>,$

$g_{24}=<Y,Z>=<\phi_v,\phi><\psi,\psi_z>, g_{34}=<W,Z>=<\psi_w,\psi_z\|\phi\|^2.$

Then, we can give our main theorem:

Theorem 2.1. Let $\phi: U\subset R^2\to E_*^3\subset E^3$ and $\psi: W\subset R\to E_*^3\subset E^3$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I\subset R\to E_*^3\subset E^3$, $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u))$ and $\beta: J\subset R\to E_*^3\subset E^3$, $\beta(w)=(\beta_1(w),\beta_2(w),\beta_3(w))$. Then, the map

$f: U\times W\to E^3\otimes E^3=E^9; (u,v,w,z)\to\phi(u,v)\otimes\psi(w,z),\quad u,v\in U,\ \ w,z\in W,$

gives rise to a convolution manifold $U\star W$ equipped with ${}_{\rho_2}g_1 *_{\rho_1} g_2=\rho_2^2 g_1+\rho_1^2 g_2+2\rho_1\rho_2 d\rho_1\otimes d\rho_2$.

Proof: The proof is obvious from (1) and Lemma 2.1.

The normal space of $U\times W$ is spanned by $\{n_1,n_2,n_3,n_4\}$. So, we can give the following lemma:

Lemma 2.2. Let $\phi: U\subset \mathrm{R}^2\to E_*^3\subset E^3$, $\phi(u,v)=\alpha(u)+v(\sin\theta N^\alpha+\cos\theta B^\alpha)$ and $\psi: W\subset \mathrm{R}^2\to E_*^3\subset E^3$, $\psi(w,z)=\beta(w)+z(\sin\theta N^\beta+\cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I\subset \mathrm{R}\to E_*^3\subset E^3$, $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u))$ and $\beta: J\subset \mathrm{R}\to E_*^3\subset E^3$, $\beta(w)=(\beta_1(w),\beta_2(w),\beta_3(w))$, respectively. Here, N^α and B^α denote elements of the Frenet frame $\{T^\alpha={V_1}^\alpha, N^\alpha={V_2}^\alpha, B^\alpha={V_3}^\alpha\}$ of the curve α and N^β and B^β denote elements of the Frenet frame $\{T^\beta={V_1}^\beta, N^\beta={V_2}^\beta, B^\beta={V_3}^\beta\}$ of the curve β. Then, the normal space of $U\times W$ is spanned by

$n_1=(-(\alpha_3(u)+v\sin\theta N_3^\alpha+v\cos\theta B_3^\alpha)\left(\beta_2(w)+z\left(\sin\theta N_2^\beta+\cos\theta B_2^\beta\right)\right),(\alpha_3(u)+v\sin\theta N_3^\alpha+v\cos\theta B_3^\alpha)\left(\beta_1(w)+z\left(\sin\theta N_1^\beta+\cos\theta B_1^\beta\right)\right),0,0,0,0,(\alpha_1(u)+v\sin\theta N_1^\alpha+v\cos\theta B_1^\alpha)\left(\beta_2(w)+z\left(\sin\theta N_2^\beta+\cos\theta B_2^\beta\right)\right),-(\alpha_1(u)+v\sin\theta N_1^\alpha+v\cos\theta B_1^\alpha)\left(\beta_1(w)+z\left(\sin\theta N_1^\beta+\cos\theta B_1^\beta\right)\right),0),$

$n_2=(0,0,0,-(\alpha_3(u)+v\sin\theta N_3^\alpha+v\cos\theta B_3^\alpha)(\beta_2(w)+z(\sin\theta N_2^\beta+\cos\theta B_2^\beta)),(\alpha_3(u)+v\sin\theta N_3^\alpha+v\cos\theta B_3^\alpha)(\beta_1(w)+z\left(\sin\theta N_1^\beta+\cos\theta B_1^\beta\right),0,(\alpha_2(u)+v\sin\theta N_2^\alpha+v\cos\theta B_2^\alpha)\left(\beta_2(w)+z\left(\sin\theta N_2^\beta+\cos\theta B_2^\beta\right)\right),-(\alpha_2(u)+v\sin\theta N_2^\alpha+v\cos\theta B_2^\alpha)\left(\beta_1(w)+z\left(\sin\theta N_1^\beta+\cos\theta B_1^\beta\right)\right),0),$

$n_3 = (-(\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)),\ (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)), 0, (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)), -(\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)),0,0,0,0),$

$n_4 = (0,0,0,0, -(\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)), (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)),0, (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)), -(\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)).$

Proof: Let $\phi: U \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\phi(u,v) = \alpha(u) + v(\sin\theta N^\alpha + \cos\theta B^\alpha)$ and $\psi: W \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\psi(w,z) = \beta(w) + z(\sin\theta N^\beta + \cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I \subset \mathrm{R} \to E_*^3 \subset E^3$, $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u))$ and $\beta: J \subset \mathrm{R} \to E_*^3 \subset E^3$, $\beta(w) = (\beta_1(w), \beta_2(w), \beta_3(w))$, respectively. Then using the Riemannian metric g and the base $\{X, Y, W, Z\}$ of tangent space of manifold $U \times W$, the normal space of $U \times W$ easily can be obtained.

After stating this lemma, we can state following theorem:

Theorem 2.2. Let $\phi: U \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\phi(u,v) = \alpha(u) + v(\sin\theta N^\alpha + \cos\theta B^\alpha)$ and $\psi: W \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\psi(w,z) = \beta(w) + z(\sin\theta N^\beta + \cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I \subset \mathrm{R} \to E_*^3 \subset E^3$, $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u))$ and $\beta: J \subset \mathrm{R} \to E_*^3 \subset E^3$, $\beta(w) = (\beta_1(w), \beta_2(w), \beta_3(w))$, respectively. Here, N^α and B^α denote elements of the Frenet frame $\{T^\alpha = {V_1}^\alpha, N^\alpha = {V_2}^\alpha, B^\alpha = {V_3}^\alpha\}$ of the curve α and N^β and B^β denote elements of the Frenet frame $\{T^\beta = {V_1}^\beta, N^\beta = {V_2}^\beta, B^\beta = {V_3}^\beta\}$ of the curve β. Then, $U \star W$ convolution manifold is a minimal submanifold.

Proof: Let $\{X, Y, W, Z\}$ be the base of tangent space of manifold $U \times W$. Then we can give the derivatives $\{X_u = \frac{\partial^2 f}{\partial u^2}, Y_v = \frac{\partial^2 f}{\partial v^2}, W_w = \frac{\partial^2 f}{\partial w^2}, Z_z = \frac{\partial^2 f}{\partial z^2}\}$ as

$$X_u = ([-vk_1^{\alpha\prime}\sin\theta T_1^\alpha + (1 - vk_1^\alpha\sin\theta)T_1^{\alpha\prime}][\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_1^\alpha + (1 - vk_1^\alpha\sin\theta)T_1^{\alpha\prime}][\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_1^\alpha + (1 - vk_1^\alpha\sin\theta)T_1^{\alpha\prime}][\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_2^\alpha + (1 - vk_1^\alpha\sin\theta)T_2^{\alpha\prime}][\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_2^\alpha + (1 - vk_1^\alpha\sin\theta)T_2^{\alpha\prime}][\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_2^\alpha + (1 - vk_1^\alpha\sin\theta)T_2^{\alpha\prime}][\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_3^\alpha + (1 - vk_1^\alpha\sin\theta)T_3^{\alpha\prime}][\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta)],$$
$$[-vk_1^{\alpha\prime}\sin\theta T_3^\alpha + (1 - vk_1^\alpha\sin\theta)T_3^{\alpha\prime}][\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta)],$$
$$\left[-vk_1^{\alpha\prime}\sin\theta T_3^\alpha + (1 - vk_1^\alpha\sin\theta)T_3^{\alpha\prime}\right]\left[\beta_3(w) + z\left(\sin\theta N_3^\beta + \cos\theta B_3^\beta\right)\right]),$$

$$Y_v = 0,$$

$$W_w = ([\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha][-zk_1^{\beta\prime}\sin\theta T_1^\beta + (1 - zk_1^\beta\sin\theta)T_1^{\beta\prime}],$$
$$[\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha][-zk_1^{\beta\prime}\sin\theta T_2^\beta + (1 - zk_1^\beta\sin\theta)T_2^{\beta\prime}],$$
$$[\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha][-zk_1^{\beta\prime}\sin\theta T_3^\beta + (1 - zk_1^\beta\sin\theta)T_3^{\beta\prime}],$$
$$[\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha][-zk_1^{\beta\prime}\sin\theta T_1^\beta + (1 - zk_1^\beta\sin\theta)T_1^{\beta\prime}],$$
$$[\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha][-zk_1^{\beta\prime}\sin\theta T_2^\beta + (1 - zk_1^\beta\sin\theta)T_2^{\beta\prime}],$$
$$[\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha][-zk_1^{\beta\prime}\sin\theta T_3^\beta + (1 - zk_1^\beta\sin\theta)T_3^{\beta\prime}],$$

$[\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha][-zk_1^{\beta'}\sin\theta T_1^\beta + (1 - zk_1^\beta\sin\theta)T_1^{\beta'}]$,
$[\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha][-zk_1^{\beta'}\sin\theta T_2^\beta + (1 - zk_1^\beta\sin\theta)T_2^{\beta'}]$,
$[\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha]\left[-zk_1^{\beta'}\sin\theta T_3^\beta + \left(1 - zk_1^\beta\sin\theta\right)T_3^{\beta'}\right]$),

$Z_z = 0.$

Thus, we obtain $< X_u, n_i >= 0, < Y_v, n_i >= 0, < W_w, n_i >= 0, < Z_z, n_i >= 0,\ i = 1,2,3,4.$

Hence, from the Gauss formula (2), it is obtained $h(X,X) = 0, h(Y,Y) = 0, h(W,W) = 0, h(Z,Z) = 0.$ From (Chen, 1973), one can recall that, a submanifold of a Riemannian manifold is said to be minimal, if mean curvature vector H vanishes identically. Since we have $< h(X,X) + h(Y,Y) + h(W,W) + h(Z,Z), n_i >= 0$, the proof completes.

Now, we can give the following theorem:

Theorem 2.3. Let $\phi: U \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\phi(u,v) = \alpha(u) + v(\sin\theta N^\alpha + \cos\theta B^\alpha)$ and $\psi: W \subset \mathrm{R}^2 \to E_*^3 \subset E^3$, $\psi(w,z) = \beta(w) + z(\sin\theta N^\beta + \cos\theta B^\beta)$ be two immersions of helix hypersurfaces obtained by planar curves $\alpha: I \subset \mathrm{R} \to E_*^3 \subset E^3$, $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u))$ and $\beta: J \subset \mathrm{R} \to E_*^3 \subset E^3$, $\beta(w) = (\beta_1(w), \beta_2(w), \beta_3(w))$, respectively. Here, N^α and B^α denote elements of the Frenet frame $\{T^\alpha = {V_1}^\alpha, N^\alpha = {V_2}^\alpha, B^\alpha = {V_3}^\alpha\}$ of the curve α and N^β and B^β denote elements of the Frenet frame $\{T^\beta = {V_1}^\beta, N^\beta = {V_2}^\beta, B^\beta = {V_3}^\beta\}$ of the curve β. Then the convolution manifold $U \star W$ is totally geodesic submanifold if and only if the functions $(\frac{\phi_3}{\phi_1}), (\frac{\phi_3}{\phi_2}), (\frac{\phi_2}{\phi_1})$ and $(\frac{\psi_1}{\psi_2}), (\frac{\psi_2}{\psi_3})$ are constant.

Proof: Let $\{X, Y, W, Z\}$ be the base of tangent space of manifold $U \times W$. Now, we obtain the derivatives with respect to parameters *u*, *v*, *w*, *z* of the base $\{X, Y, \mathrm{W}, Z\}$ on manifold $U \star W$ as following:

$X_v = (-k_1^{\alpha'}\sin\theta T_1^\alpha(\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta), -k_1^{\alpha'}\sin\theta T_1^\alpha(\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta),$
$-k_1^{\alpha'}\sin\theta T_1^\alpha(\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta), -k_1^{\alpha'}\sin\theta T_2^\alpha(\beta_1(w) + z(\sin\theta N_1^\beta +$
$\cos\theta B_1^\beta), -k_1^{\alpha'}\sin\theta T_2^\alpha(\beta_2(w) + z(\sin\theta N_2^\beta + \cos\theta B_2^\beta), -k_1^{\alpha'}\sin\theta T_2^\alpha(\beta_3(w) + z(\sin\theta N_3^\beta +$
$\cos\theta B_3^\beta), -k_1^{\alpha'}\sin\theta T_3^\alpha(\beta_1(w) + z(\sin\theta N_1^\beta + \cos\theta B_1^\beta), -k_1^{\alpha'}\sin\theta T_3^\alpha(\beta_2(w) + z(\sin\theta N_2^\beta +$
$\cos\theta B_2^\beta), -k_1^{\alpha'}\sin\theta T_3^\alpha(\beta_3(w) + z(\sin\theta N_3^\beta + \cos\theta B_3^\beta)),$

$X_w = ((1 - vk_1^{\alpha'}\sin\theta)T_1^\alpha(\beta_1'(w) + z(\sin\theta N_1^{\beta'} + \cos\theta B_1^{\beta'}), (1 - vk_1^{\alpha'}\sin\theta)T_1^\alpha(\beta_2'(w) + z(\sin\theta N_2^{\beta'}$
$+\cos\theta B_2^{\beta'}), (1 - vk_1^{\alpha'}\sin\theta)T_1^\alpha(\beta_3'(w) + z(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'}), ((1 - vk_1^{\alpha'}\sin\theta)T_2^\alpha(\beta_1'(w)$
$+z(\sin\theta N_1^{\beta'} + \cos\theta B_1^{\beta'}), (1 - vk_1^{\alpha'}\sin\theta)T_2^\alpha(\beta_2'(w) + z(\sin\theta N_2^{\beta'} + \cos\theta B_2^{\beta'}), (1$
$-vk_1^{\alpha'}\sin\theta)T_2^\alpha(\beta_3'(w) + z(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'}), (1 - vk_1^{\alpha'}\sin\theta)T_3^\alpha(\beta_1'(w) + z(\sin\theta N_1^{\beta'}$
$+\cos\theta B_1^{\beta'}), (1 - vk_1^{\alpha'}\sin\theta)T_3^\alpha(\beta_2'(w) + z(\sin\theta N_2^{\beta'} + \cos\theta B_2^{\beta'}), (1 - vk_1^{\alpha'}\sin\theta)T_3^\alpha(\beta_3'(w)$
$+z(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'})),$

$X_z = ((1 - vk_1^{\alpha'}\sin\theta)T_1^\alpha(\sin\theta N_1^\beta + \cos\theta B_1^\beta), (1 - vk_1^{\alpha'}\sin\theta)T_1^\alpha(\sin\theta N_2^\beta + \cos\theta B_2^\beta),$
$(1 - vk_1^{\alpha'}\sin\theta)T_1^\alpha(\sin\theta N_3^\beta + \cos\theta B_3^\beta), (1 - vk_1^{\alpha'}\sin\theta)T_2^\alpha(\sin\theta N_1^\beta + \cos\theta B_1^\beta),$
$(1 - vk_1^{\alpha'}\sin\theta)T_2^\alpha(\sin\theta N_2^\beta + \cos\theta B_2^\beta), (1 - vk_1^{\alpha'}\sin\theta)T_2^\alpha(\sin\theta N_3^\beta + \cos\theta B_3^\beta),$
$(1 - vk_1^{\alpha'}\sin\theta)T_3^\alpha(\sin\theta N_1^\beta + \cos\theta B_1^\beta), (1 - vk_1^{\alpha'}\sin\theta)T_3^\alpha(\sin\theta N_2^\beta + \cos\theta B_2^\beta),$
$(1 - vk_1^{\alpha'}\sin\theta)T_3^\alpha(\sin\theta N_3^\beta + \cos\theta B_3^\beta)),$

$Y_w = ((\sin\theta N_1^\beta + \cos\theta B_1^\beta)(\beta_1'(w) + z(\sin\theta N_1^{\beta'} + \cos\theta B_1^{\beta'}), (\sin\theta N_1^\beta + \cos\theta B_1^\beta)(\beta_2'(w) + z(\sin\theta N_2^{\beta'}$
$+\cos\theta B_2^{\beta'}), (\sin\theta N_1^\beta + \cos\theta B_1^\beta)(\beta_3'(w) + z(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'}), (\sin\theta N_2^\beta + \cos\theta B_2^\beta)(\beta_1'(w)$
$+z(\sin\theta N_1^{\beta'} + \cos\theta B_1^{\beta'}), (\sin\theta N_2^\beta + \cos\theta B_2^\beta)(\beta_2'(w) + z(\sin\theta N_2^{\beta'} + \cos\theta B_2^{\beta'}), (\sin\theta N_2^\beta$
$+\cos\theta B_2^\beta)(\beta_3'(w) + z(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'}), (\sin\theta N_3^\beta + \cos\theta B_3^\beta)(\beta_1'(w) + z(\sin\theta N_1^{\beta'}$
$+\cos\theta B_1^{\beta'}), (\sin\theta N_3^\beta + \cos\theta B_3^\beta)(\beta_2'(w) + z(\sin\theta N_2^{\beta'} + \cos\theta B_2^{\beta'}), (\sin\theta N_3^\beta + \cos\theta B_3^\beta)(\beta_3'(w)$
$+z(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'})),$

$Y_z = ((\sin\theta N_1^\alpha + \cos\theta B_1^\alpha)(\sin\theta N_1^\beta + \cos\theta B_1^\beta), (\sin\theta N_1^\alpha + \cos\theta B_1^\alpha)(\sin\theta N_2^\beta + \cos\theta B_2^\beta),$
$(\sin\theta N_1^\alpha + \cos\theta B_1^\alpha)(\sin\theta N_3^\beta + \cos\theta B_3^\beta), (\sin\theta N_2^\alpha + \cos\theta B_2^\alpha)(\sin\theta N_1^\beta + \cos\theta B_1^\beta),$
$(\sin\theta N_2^\alpha + \cos\theta B_2^\alpha)(\sin\theta N_2^\beta + \cos\theta B_2^\beta), (\sin\theta N_2^\alpha + \cos\theta B_2^\alpha)(\sin\theta N_3^\beta + \cos\theta B_3^\beta),$
$(\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)(\sin\theta N_1^\beta + \cos\theta B_1^\beta), (\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)(\sin\theta N_2^\beta + \cos\theta B_2^\beta),$
$(\sin\theta N_3^\alpha + \cos\theta B_3^\alpha)(\sin\theta N_3^\beta + \cos\theta B_3^\beta)),$

$W_z = ((\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\sin\theta N_1^{\beta'} + \cos\theta B_1^{\beta'}), (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\sin\theta N_2^{\beta'}$
$+\cos\theta B_2^{\beta'}), (\alpha_1(u) + v\sin\theta N_1^\alpha + v\cos\theta B_1^\alpha)(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'}), (\alpha_2(u) + v\sin\theta N_2^\alpha$
$+v\cos\theta B_2^\alpha)(\sin\theta N_1^{\beta'} + \cos\theta B_1^{\beta'}), (\alpha_2(u) + v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\sin\theta N_2^{\beta'} + \cos\theta B_2^{\beta'}), (\alpha_2(u)$
$+v\sin\theta N_2^\alpha + v\cos\theta B_2^\alpha)(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'}), (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\sin\theta N_1^{\beta'}$
$+\cos\theta B_1^{\beta'}), (\alpha_3(u) + v\sin\theta N_3^\alpha + v\cos\theta B_3^\alpha)(\sin\theta N_2^{\beta'} + \cos\theta B_2^{\beta'}), (\alpha_3(u) + v\sin\theta N_3^\alpha$
$+v\cos\theta B_3^\alpha)(\sin\theta N_3^{\beta'} + \cos\theta B_3^{\beta'})).$

Similarly, as a result of long operations, one can calculate the derivatives Y_u, W_u, W_v, Z_u, Z_v, Z_w. Thus, we can give the following results:

We know that, $h(X,X) = 0, h(Y,Y) = 0, h(W,W) = 0, h(Z,Z) = 0$. Also, we have

$$< X_v, n_i >= 0 \quad and \quad < Y_u, n_i >= 0, \tag{15}$$

$$< X_w, n_1 >= (\frac{\phi_3(u,v)}{\phi_1(u,v)})_u \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_w \psi_2^2(w,z) =< W_u, n_1 >,$$

$$< X_w, n_2 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_u \phi_2^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_w \psi_2^2(w,z) =< W_u, n_2 >, \tag{16}$$

$$< X_w, n_3 >= (\frac{\phi_2(u,v)}{\phi_1(u,v)})_u \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_w \psi_2^2(w,z) =< W_u, n_3 >,$$

$$< X_w, n_4 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_u \phi_2^2(u,v)(\frac{\psi_2(w,z)}{\psi_3(w,z)})_w \psi_3^2(w,z) =< W_u, n_4 >,$$

$$< X_z, n_1 >= (\frac{\phi_3(u,v)}{\phi_1(u,v)})_u \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_z \psi_2^2(w,z) =< Z_u, n_1 >,$$

$$< X_z, n_2 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_u \phi_2^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_z \psi_2^2(w,z) =< Z_u, n_2 >, \tag{17}$$

$$< X_z, n_3 >= (\frac{\phi_2(u,v)}{\phi_1(u,v)})_u \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_z \psi_2^2(w,z) =< Z_u, n_3 >,$$

$$< X_z, n_4 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_u \phi_2^2(u,v)(\frac{\psi_2(w,z)}{\psi_3(w,z)})_z \psi_3^2(w,z) =< Z_u, n_4 >,$$

$$< Y_w, n_1 >= (\frac{\phi_3(u,v)}{\phi_1(u,v)})_v \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_w \psi_2^2(w,z) =< W_v, n_1 >,$$

$$< Y_w, n_2 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_v \phi_2^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_w \psi_2^2(w,z) =< W_v, n_2 >, \quad (18)$$

$$< Y_w, n_3 >= (\frac{\phi_2(u,v)}{\phi_1(u,v)})_v \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_w \psi_2^2(w,z) =< W_v, n_3 >,$$

$$< Y_w, n_4 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_v \phi_2^2(u,v)(\frac{\psi_2(w,z)}{\psi_3(w,z)})_w \psi_3^2(w,z) =< W_v, n_4 >,$$

$$< Y_z, n_1 >= (\frac{\phi_3(u,v)}{\phi_1(u,v)})_v \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_z \psi_2^2(w,z) =< Z_v, n_1 >,$$

$$< Y_z, n_2 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_v \phi_2^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_z \psi_2^2(w,z) =< Z_v, n_2 >, \quad (19)$$

$$< Y_z, n_3 >= (\frac{\phi_2(u,v)}{\phi_1(u,v)})_v \phi_1^2(u,v)(\frac{\psi_1(w,z)}{\psi_2(w,z)})_z \psi_2^2(w,z) =< Z_v, n_3 >,$$

$$< Y_z, n_4 >= (\frac{\phi_3(u,v)}{\phi_2(u,v)})_v \phi_2^2(u,v)(\frac{\psi_2(w,z)}{\psi_3(w,z)})_z \psi_3^2(w,z) =< Z_v, n_4 >,$$

$$< W_z, n_i >= 0, \quad (20)$$

$< Z_w, n_i >= 0$, , for all i=1,2,3,4. (21)

Considering that h is symmetric, we obtain $h(X,W) \neq 0$, $h(X,Z) \neq 0$, $h(Y,W) \neq 0$, $h(Y,Z) \neq 0$. Thus, take into account this statement and the equations (15-21), the proof is complete.

3. Conclusion

In 1802, M.A. Lancret has stated a classical result which is “A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant” and in 1845, B. De Saint Venant has proved this result. Also, if both of κ and τ are non-zero constant it is a general helix and we call it a circular helix. Its known that, straight line and circle are degenerate-helix examples (κ=0, if the curve is straight line and τ=0, if the curve is a circle) (Kula vd., 2010). After these studies, many authors have studied helix curves, helix surfaces and helix submanifolds (for instance, one can see (Barrera Cadena vd, 2015; Di Scala and Ruiz-Hernández, 2009, 2010, 2016; Fetcu, 2015; Küçükarslan and Yıldırım, 2018; Zıplar, 2012 and etc.). Some other motivations for the study of helix submanifolds comes from the physics of interfaces of liquid crystals and that they appear contained in the shadow boundary of a submanifold (Di Scala and Ruiz-Hernández, 2010).

In this study, considering two helix hypersurfaces which are obtained by planar curves, we obtain a convolution manifold with the aid of the immersions of these hypersurfaces and we give some important characterizations about this manifold. Consequently, the researchers who are dealing with some special areas of physics and medicine can use these characterizations about the geometry of the tensor product of helix hypersurfaces and this approach can bring a new perspective to researchers of these fields.

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