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A Finite Difference Method to Solve a Special Type of Second Order Differential Equations

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Abstract: In this study, we give a finite difference scheme to solve a special type of second order differential equations. Our numerical method based on finite difference relation which is obtained the Lagrange polynomial interpolations. By applying this method the equation is made discrete using appropriate finite difference approaches instead of derivatives. The approximate solutions are obtained by using Maple 13. Absolute errors are calculated. The results are analyzed with tables. The graphics of errors for different mesh size are given.

Keywords: Boundary Value Problem, Finite Difference Method, Lagrange Polynomial Interpolation.

1 Introduction

The finite difference method is one of the simplest and of the oldest methods. The principle of finite difference methods is a numerical scheme which used to solve ordinary differential equations. The basic idea of this method is the differential by replacing the derivatives in the equation using differential quotients. The domain of the independent variable of the differential equation is partitioned and approximations of the solution are computed at the space or time points. Basicly, there are two main derivations to approximate the derivatives [1] (For details p.335). Recently, many authors have obtained or generated finite difference relations approximation to derivative [1]-[10]. In this paper, we consider the following second order boundary value problem

$$y^{''}+y^{'}+y=f(x),\ \ 0\leq x\leq 1 \tag{1}$$

with boundary conditions

$$y(0)=a,\ \ \ y(1)=b. \tag{2}$$

We use interpolation to approximate derivatives. Then we have numerical solutions, exact solutions and obtaine absolute errors on any grid point.

2 Method of Solution

In this section, we try to find formulas for approximating the derivatives using polynomials interpolation. For this purpose, we use the second order interpolation polynomials. Let take the Lagrange interpolating polynomial form

$$y(x)\approx p_2(x)=y(x_0)(L_0^{(2)})(x)+y(x_1)(L_1^{(2)})(x)+y(x_2)(L_2^{(2)})(x). \tag{3}$$

Then, the approximate derivatives are

$$y^{'}(x_0)\approx p_2^{'}(x_0)=y(x_0)(L_0^{(2)})^{'}(x_0)+y(x_1)(L_1^{(2)})^{'}(x_0)+y(x_2)(L_2^{(2)})^{'}(x_0), \tag{4}$$

$$y^{''}(x_0)\approx p_2^{''}(x_0)=y(x_0)(L_0^{(2)})^{''}(x_0)+y(x_1)(L_1^{(2)})^{''}(x_0)+y(x_2)(L_2^{(2)})^{''}(x_0) \tag{5}$$

where $(L_j^{(2)})(x)$ is a polynomial of degree 2 which is called Lagrange interpolation polynomials of degree 2 and $x_1=x_0+h$ and $x_2=x_0+2h$.

We evaluate the values of $(L_0^{(2)})^{'}(x_0)$ and $(L_0^{(2)})^{''}(x_0)$ in Eqs.4 and 5. If we compute these values, we obtain,

$$(L_0^{(2)})^{'}(x_0) = -3/2h, \ \ (L_1^{(2)})^{'}(x_0) = 2/h, \ \ (L_2^{(2)})^{'}(x_0) = -1/2h, \tag{6}$$

$$(L_0^{(2)})^{''}(x_0) = 1/h^2, \ \ (L_1^{(2)})^{''}(x_0) = -2/h^2, \ \ (L_2^{(2)})^{''}(x_0) = 1/h^2. \tag{7}$$

Thus, we have the approximate of the first and second derivatives

$$y^{'}(x) = \frac{1}{2h}(-y(x+2h) + 4y(x+h) - 3y(x)), \tag{8}$$

$$y^{''}(x) = \frac{1}{h^2}(y(x+2h) - 2y(x+h) + y(x)). \tag{9}$$

If we display $y(x_k) = y_k, \ \ y(x_k + h) = y_{k+1}, \ \ y(x_k + 2h) = y_{k+2}$, the above relations can be written as

$$y^{'}(x) = \frac{1}{2h}(-y_{k+2} + 4y_{k+1} - 3y_k), \tag{10}$$

$$y^{''}(x) = \frac{1}{h^2}(y_{k+2} - 2y_{k+1} + y_k) \tag{11}$$

where $1 \leq k \leq n-1$. Now, Eq.10 and Eq.11 are put in Eq.1, we get the following difference equation

$$\frac{1}{h^2}(y_{k+2} - 2y_{k+1} + y_k) + \frac{1}{2h}(-y_{k+2} + 4y_{k+1} - 3y_k) = f(x_k) \tag{12}$$

which can be simplified

$$(\frac{1}{h^2} - \frac{1}{2h})y_{k+2} + (-\frac{2}{h^2} + \frac{4}{2h})y_{k+1} + (\frac{1}{h^2} - \frac{3}{2h} + 1)y_k = f(x_k) \tag{13}$$

and so,

$$(1 - \frac{h}{2})y_{k+2} + (-2 + 2h)y_{k+1} + (1 - \frac{3h}{2} + h^2)y_k = h^2 f(x_k). \tag{14}$$

This is a tridiagonal system of linear equations. If we write in matrix-vector form $\mathbf{AU} = \mathbf{B}$, we have

$$\mathbf{A} = \begin{pmatrix} -2+2h & 1-h/2 & 0 & \cdots & \cdots & 0 \\ 1-3h/2+h^2 & -2+2h & 1-h/2 & 0 & \cdots & 0 \\ 0 & 1-3h/2+h^2 & -2+2h & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1-3h/2+h^2 & -2+2h \end{pmatrix}_{(n-1)x(n-1)}$$

$$\mathbf{U} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}_{(n-1)x1}$$

$$\mathbf{B} = \begin{pmatrix} h^2 f(x_1) - (1 - 3h/2 + h^2) \\ h^2 f(x_2) \\ \vdots \\ \vdots \\ h^2 f(x_{n-2}) \\ h^2 f(x_{n-1}) - (1 - h/2)y_n \end{pmatrix}_{(n-1)x1}$$

where $y_k \approx y(x_k)$ and $x_k = kh$. If this system is solve by Maple 13, we get the approximate solutions.

Above the numerical method can be extended the following variable coefficient differential equation,

$$p(x)y^{''} + q(x)y^{'} + r(x)y = f(x), \ \ 0 \leq x \leq 1 \tag{15}$$

with boundary conditions

$$y(0) = a, \ \ y(1) = b. \tag{16}$$

If Eq.10 and Eq.11 are put in Eq.15, we get the following difference equation

$$\frac{p(x_k)}{h^2}(y_{k+2} - 2y_{k+1} + y_k) + \frac{q(x_k)}{2h}(-y_{k+2} + 4y_{k+1} - 3y_k) + r(x_k)y = f(x_k) \tag{17}$$

which can be simplified

$$y_{k+2}(\frac{p(x_k)}{h^2}-\frac{q(x_k)}{2h})+y_{k+1}(\frac{-2p(x_k)}{h^2}+\frac{2q(x_k)}{h})+y_k(\frac{p(x_k)}{h^2}-\frac{3q(x_k)}{2h}+r(x_k)). \quad (18)$$

This is a tridiagonal system of linear equations. If we write in matrix-vector form $\mathbf{AU}=\mathbf{B}$, we have

$$\mathbf{A}=\begin{pmatrix} A(x_0) & C(x_0) & 0 & \cdots & \cdots & 0 \\ B(x_1) & A(x_1) & C(x_1) & 0 & \cdots & 0 \\ 0 & B(x_2) & A(x_2) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & B(x_{n-2}) & A(x_{n-2}) \end{pmatrix}_{(n-1)x(n-1)}$$

where, $A(x_i)=(\frac{-2p(x_i)}{h^2}+\frac{2q(x_i)}{h})$, $B(x_i)=(\frac{p(x_i)}{h^2}-\frac{3q(x_i)}{2h}+r(x_i))$, $C(x_i)=(\frac{p(x_i)}{h^2}-\frac{q(x_i)}{2h})$

$$\mathbf{U}=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}_{(n-1)x1}$$

$$\mathbf{B}=\begin{pmatrix} f(x_0)-y_0((\frac{p(x_0)}{h^2}-\frac{3q(x_0)}{2h}+r(x_0))) \\ f(x_k) \\ \vdots \\ \vdots \\ f(x_{n-3}) \\ f(x_{n-2})-y_n((\frac{p(x_{n-2})}{h^2}-\frac{q(x_{n-2})}{2h})) \end{pmatrix}_{(n-1)x1}$$

3 Illustrative Examples

Example1.

Let we consider the following second order boundary value problem,

$$y^{''}-y^{'}+y=e^x \quad (19)$$

with the boundary condition

$$y(0)=1, \; y(1)=e$$

with exact solution $y_e=e^x$.

If we put , Eqs.(10) and (11) in Eq.(19) and simplify it we obtain the following equation,

$$y_{k+2}(1+\frac{h}{2})+y_{k+1}(-2-2h)+y_k(1+\frac{3h}{2}+h^2)=h^2e^x$$

.

For values of k we have system of linear equations. If we solve this system we obtain numerical solutions y_k on grid points.

k	Exact Solution	Numerical Solution	Error
1	1.105170918	1.105397191	.226273e-3
2	1.221402758	1.221833837	.431079e-3
3	1.349858808	1.350464503	.605695e-3
4	1.491824698	1.492564994	.740296e-3
5	1.648721271	1.649545153	.823882e-3
6	1.822118800	1.822963006	.844206e-3
7	2.013752707	2.014540404	.787697e-3
8	2.225540928	2.226180312	.639384e-3
9	2.459603111	2.459985933	.382822e-3

Table 1 Exact Solutions, Numerical Solutions and Errors for $N=10$

k	N=20	N=30	N=40
1	.30677e-4	.93302e-5	.39910e-5
2	.60205e-4	.18452e-4	.79190e-5
3	.88318e-4	.27323e-4	.11771e-4
4	.11473e-3	.35909e-4	.15542e-4
5	.13914e-3	.44173e-4	.19221e-4
6	.16124e-3	.52076e-4	.22800e-4
7	.18068e-3	.59575e-4	.26269e-4
8	.19710e-3	.66629e-4	.29621e-4
9	.21013e-3	.73192e-4	.32843e-4

Table 2 Errors for Different N Values

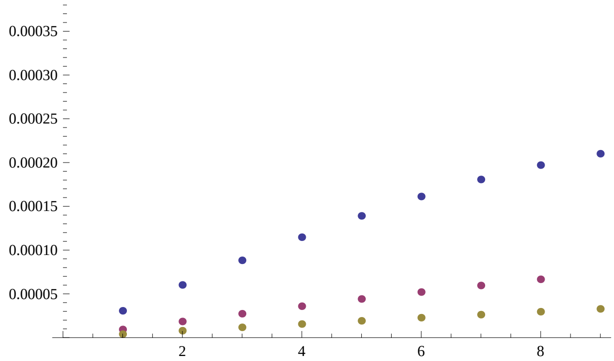

Fig. 1: Errors for different N values

Example2.
Let we consider the following second order variable coefficient boundary value problem,

$$y^{''} - 2x^3 y^{'} + 8x^2 y = 0 \tag{20}$$

with the boundary condition

$$y(0) = 1, \;\; y(1) = \frac{1}{3}$$

with exact solution $y_e = 1 - \frac{2}{3}x^4$.
If we put , Eqs.(10) and (11) in Eq.(20) and simplify it we obtain the following equation,

$$y_{k+2}(\frac{p(x_k)}{h^2} - \frac{q(x_k)}{2h}) + y_{k+1}(\frac{-2p(x_k)}{h^2} + \frac{2q(x_k)}{h}) + y_k(\frac{p(x_k)}{h^2} - \frac{3q(x_k)}{2h} + r(x_k)) = h^2 e^x$$

.
For values of k we have system of linear equations. If we solve this system we obtain numerical solutions y_k on grid points.

k	Exact Solution	Numerical Solution	Error
1	0.999999893	1.000751494	.751600e-3
2	0.999998293	1.001502987	.150469e-2
3	0.999991360	1.002251918	.226055e-2
4	0.999972693	1.002985460	.301276e-2
5	0.999933333	1.003672800	.373946e-2
6	0.999861760	1.004257392	.439563e-2
7	0.999743893	1.004649214	.490553e-2
8	0.999563093	1.004717032	.515393e-2
9	0.999300160	1.004280762	.498060e-2

Table 3 Exact Solutions, Numerical Solutions and Errors for $N = 50$

k	Exact Solution	Numerical Solution	Error
1	0.999999739	0.974111727	.258880e-1
2	0.999995833	0.948223455	.517723e-1
3	0.999978906	0.922329155	.776497e-1
4	0.999933333	0.896399782	.103533550
5	0.999837239	0.870368289	.129468949
6	0.999662500	0.844116607	.155545892
7	0.999374739	0.817464650	.181910089
8	0.998933333	0.790161498	.208771834
9	0.998291406	0.761879033	.236412372

Table 4 Exact Solutions, Numerical Solutions and Errors for $N = 40$

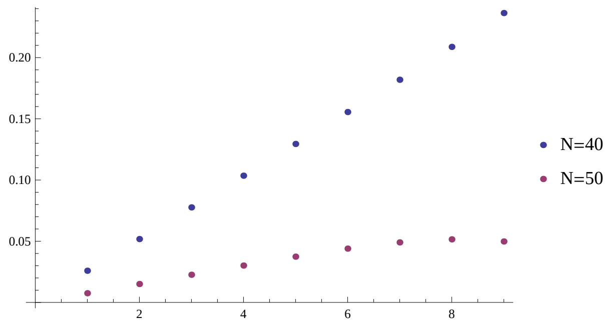

Fig. 2: Errors for different N values

4 Conclusion

A finite difference scheme is considered to solve a special type of second order differential equations. Lagrange interpolation polynomials have been successfully applied to obtain difference scheme. From the numerical results, it can be concluded that the given method is accurate and effective.

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