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COTTON TENSOR ON SASAKIAN 3-MANIFOLDS ADMITTING ETA RICCI SOLITONS

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Abstract. The object of the present paper is to characterize Cotton tensor on a 3-dimensional Sasakian manifold admitting η-Ricci solitons. After introduction, we study 3-dimensional Sasakian manifolds and introduce a new notion, namely, Cotton pseudo-symmetric manifolds. Next we deal with the study of Cotton tensor on a Sasakian 3-manifold admitting η-Ricci solitons. Among others we prove that such a manifold is a manifold of constant scalar curvature and Einstein manifold with some appropriate conditions. Also, we classify the nature of the soliton metric. Finally, we give an important remark.

1. Introduction

In differential geometry, the Weyl conformal curvature tensor vanishes on a 3-dimensional pseudo-Riemannian manifold and hence one can consider an another type of conformal invariant, which is the Cotton tensor. Cotton tensor C is a tensor of type (1,2), defined by

$$C(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{(Xr)Y - (Yr)X\},$$

for any smooth vector fields X, Y. Therefore, in a 3-dimensional pseudo-Riemannian manifold Cotton tensor vanishes if the metric be conformally flat and the idea is given by Eisenhart. At the present time, the 3-dimensional spaces becoming onto the dignity of interest, as the Cotton tensor restricts the relation between the Ricci tensor and the energy-momentum tensor of matter in the Einstein equations and plays an important role in the Hamiltonian formalism of general relativity.
The notion of Ricci flow was introduced [17] by R. S. Hamilton in 1982 to find a

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canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g = -2S, \tag{1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$, where ψ_t are homeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [12]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (2) are interesting and useful in physics and are often referred as quasi-Einstein ([13], [14]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo homeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([15], [16], [18], [19], [21], [29]) and many others.

The notion of η-Ricci soliton, which is a generalization of Ricci soliton, was introduced by CHM and Kiaora [11]. This notion has also been studied in [12] for Hope hyperuricemia in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3}$$

where S is the Ricci tensor associated to g. In this connection we may mention the works of Ayar et al. [2], Blaga ([3], [4], [5]), Prakasha et al. [24], Kar et al. ([20], [23]) and Turan et al. [27]. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton(g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named proper η-Ricci soliton.

In this paper, after introduction, in section 2, we study 3-dimensional Sasakian manifold. Section 3 deals with Cotton tensor on a Sasakian 3-manifold admitting η-Ricci solitons. In section 4, we prove that a Cotton flat Sasakian 3-manifold admitting η-Ricci solitons is a manifold of constant scalar curvature 6 and an Einstein manifold. We classify Sasakian 3-manifolds admitting η-Ricci solitons satisfying $Q \cdot C = 0$ and show that such manifolds are the manifolds of constant scalar curvature in section 5. After these, in section 6 we characterize concircularly-Cotton semisymmetric Sasakian 3-manifolds admitting η-Ricci solitons and establish a result. Then in section 7, we introduce a new notion call Cotton pseudo-symmetric manifold and accordingly we study Sasakian 3-manifolds admitting η-Ricci solitons . We complete our paper with a valuable remark.

2. Three dimensional Sasakian manifolds

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying ([7], [8])

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0. \tag{4}$$

The first and one of the remaining three relations in (4) imply the other two relations in (4). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{5}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η), structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{6}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{7}$$

and

$$g(X, \xi) = \eta(X), \tag{8}$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).
An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{9}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.
If the characteristic vector field ξ is a Killing vector field , the contact metric manifold (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. The contact structure on M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, \frac{fd}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where f is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian metrics are K-contact and K-contact 3-metrics are Sasakian. For a Sasakian manifold, the following hold ([7], [8]):

$$\nabla_X \xi = -\phi X, \tag{10}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{11}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{12}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{13}$$

$$Q\xi = 2n\xi, \tag{14}$$

where ∇, R and Q denotes respectively, the Riemannian connection, curvature tensor and the $(1,1)$-tensor metrically equivalent to the Ricci tensor of g. The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{15}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.
It is known that the Ricci tensor of a three dimensional Sasakian manifold is given by [9]

$$S(X,Y) = \frac{1}{2}\{(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)\}, \tag{16}$$

where r is the scalar curvature which need not be constant, in general. So, g is Einstein (hence has constant curvature 1) if and only if $r = 6$.
As a consequence of (16), we have

$$S(X,\xi) = 2\eta(X). \tag{17}$$

Contact metric manifolds have also been studied by several authors such as ([9]- [14], [20]- [29]) and many others.

Definition 1. *In a n-dimensional Riemannian manifold the concircular curvature tensor of type (1,3) is defined by*

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{18}$$

Then in a 3-dimensional Riemannian manifold the concircular curvature tensor is given by

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{19}$$

Definition 2. *A Riemannian manifold is said to be concircularly flat if the concircular curvature tensor $\mathcal{Z}$ vanishes.*

Let us consider a Riemannian manifold (M,g) and let the Levi-Civita connection ∇ of (M,g). A Riemannian manifold is called locally symmetric [10] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). A Riemannian or a semi-Riemannian manifold $(M,g), n \geq 3$, is called semisymmetric if

$$R.R = 0 \tag{20}$$

holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds were first studied by Cartan, Lichnerowich, Couty and Sinjukov. A fundamental study on Riemannian semisymmetric manifolds was made by Szabó [26], Boeckx et al [6], Kowalski [22] and Prakasha et al. [25]. A semi-Riemannian manifold $(M, g), n \geq 3$, is said to be Ricci-semisymmetric if on M we have

$$R.S = 0, \tag{21}$$

where S is the Ricci tensor. Alegre et al. [1] have studied semi-Riemannian generalized Sasakian space-forms.

The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci semisymmetric manifolds were investigated by several authors.
For a $(0, k+2)$-tensor field $Q(g, T)$ associated with any $(0, k)$-tensor field T on a Riemannian manifold (M, g) is defined as follows [28]:

$$\begin{aligned}(Q(g,T))(X_1, ..., X_k; X, Y) &= ((X \wedge_g Y).T)(X_1, ..., X_k)\\ &= -T((X \wedge_g Y))X_1, X_2, ..., X_k)\\ &\quad -... - T(X_1, ...X_{k-1}, (X \wedge_g Y)X_k),\end{aligned} \tag{22}$$

where $X \wedge Y$ is the endomorphism given by

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{23}$$

We define the subsets U_R, U_S of a Riemannian Manifold M by $U_R = \{x \in M : R - \frac{r}{n(n-1)}G \neq 0 \quad at \quad x\}$ and $U_S = \{x \in M : S - \frac{r}{n}g \neq 0 \quad at \quad x\}$ respectively, where $G(X, Y)Z = g(Y, Z)X - g(X, Z)Y$. Evidently we have $U_S \subset U_R$. A Riemannian manifold is said to be pseudo-symmetric [28] if at every point of M the tensor $R.R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$R.R = f_R Q(g, R)$$

on U_R, where f_R is some function on U_R. Clearly, every semi-symmetric manifold is pseudo-symmetric but the converse is not true [28].
A Riemannian manifold M is said to Ricci pseudo-symmetric if $R.S$ and $Q(g, S)$ on M are linearly dependent. This is equivalent to

$$R.S = f_S Q(g, S)$$

holds on U_S, where f_S is a function defined on U_S.

In the present work we introduce a new notion, namely Cotton pseudo-symmetric manifold for the first time as follows:

Definition 3. *A Riemannian manifold M is said to Cotton pseudo-symmetric if $R.C$ and $Q(g, C)$ on M are linearly dependent. This is equivalent to*

$$R.C = f_S Q(g, C)$$

holds on U_S, where f_S is a function defined on U_S.

Lemma 4. *(Proposition 2.1 of [23]) The Ricci tensor of a three dimensional Sasakian manifold admitting η-Ricci soliton is of the form:*

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{24}$$

As a consequence of the above Lemma we have

$$QX = -\lambda X - \mu\eta(X)\xi. \tag{25}$$

Lemma 5. *(Proposition 2.2 of [23]) For an η-Ricci soliton on a three dimensional Sasakian manifold we have*

$$\lambda + \mu = -2. \tag{26}$$

In view of (25) and (26) we have

$$Q\xi = 2\xi. \tag{27}$$

On contraction, (24) gives

$$r = -3\lambda - \mu. \tag{28}$$

We use the above Lemmas in the next sections to develop our results.

3. Cotton tensor on Sasakian 3-manifolds admitting η-Ricci solitons

In this section, we consider a skewsymmetric tensor of type (1,2) on Sasakian 3-manifold, called Cotton tensor C, defined by

$$C(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{(Xr)Y - (Yr)X\}, \tag{29}$$

for all smooth vector fields X, Y.
Making use of (7), (10), (12) and (25) in (29) we get

$$C(X,Y) = \mu[\eta(Y)\phi X - \eta(X)\phi Y + 2g(\phi X, Y)\xi] - \frac{1}{4}[(Xr)Y - (Yr)X]. \tag{30}$$

The Cotton tensor can also be exhibited as a tensor of type (0,3) as follows:

$$C(X,Y,Z) = g(C(X,Y),Z). \tag{31}$$

By the virtue of (30) and (31), it follows that

$$\begin{aligned} C(X,Y,Z) = {} & \mu[2g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)] \\ & - \frac{1}{4}[(Xr)g(Y,Z) - (Yr)g(X,Z)]. \end{aligned} \tag{32}$$

As a consequence of (30) and (32), we derived the following results:

$$C(X,\xi) = \mu\phi X - \frac{1}{4}(Xr)\xi, \tag{33}$$

$$\eta(C(X,Y)) = 2\mu g(\phi X,Y) - \frac{1}{4}[(Xr)\eta(Y) - (Yr)\eta(X)], \tag{34}$$

$$\eta(C(X,\xi)) = -\frac{1}{4}(Xr), \tag{35}$$

$$\begin{aligned} C(\phi X, Y) = \mu[3\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \eta(Y)X] \\ - \frac{1}{4}[((\phi X)r)Y - (Yr)\phi X], \end{aligned} \tag{36}$$

$$\eta(C(\phi X, Y)) = -2\mu g(X,Y) + 2\mu\eta(X)\eta(Y) - \frac{1}{4}((\phi X)r)\eta(Y), \tag{37}$$

$$\eta(C(\phi X, \phi Y)) = -2\mu g(X, \phi Y), \tag{38}$$

$$\eta(C(\phi X, \xi)) = -\frac{1}{4}(\phi X)r, \tag{39}$$

$$\begin{aligned} C(\phi X, \phi Y, \phi Z) = -\frac{1}{4}((\phi X)r)[g(Y,Z) - \eta(Y)\eta(Z)] \\ + \frac{1}{4}((\phi Y)r)[g(X,Z) - \eta(X)\eta(Z)]. \end{aligned} \tag{40}$$

4. Cotton flat Sasakian 3-manifolds admitting η-Ricci solitons

In this section we characterize Cotton flat Sasakian 3-manifolds admitting η-Ricci solitons. Then we have

$$C(X,Y,Z) = 0. \tag{41}$$

By the virtue of (32) and (41) we get

$$\begin{aligned} \mu[2g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)] \\ -\frac{1}{4}[(Xr)g(Y,Z) - (Yr)g(X,Z)] = 0. \end{aligned} \tag{42}$$

Replacing Z by ξ in the above equation we find

$$2\mu g(\phi X, Y) = \frac{1}{4}[(Xr)\eta(Y) - (Yr)\eta(X)]. \tag{43}$$

Putting $Y = \xi$ in (43) gives

$$Xr = 0 \tag{44}$$

and hence r becomes constant.

Since r is constant, from (43) it follows that

$$2\mu g(\phi X, Y) = 0. \tag{45}$$

Substituting Y by ϕY in (45) and then in the light of (6), after contraction, we obtain

$$\mu = 0. \tag{46}$$

Thus η-Ricci soliton is not proper and so we have the following:

Theorem 6. *A Cotton flat Sasakian 3-manifold does not admit proper η-Ricci soliton.*

Making use of (46) in (26) entails that

$$\lambda = -2. \tag{47}$$

Thus the η-Ricci soliton is shrinking and hence we can state the following:

Theorem 7. *An η-Ricci soliton (g, ξ, λ, μ) on a Cotton flat Sasakian 3-manifold is shrinking.*

Making use of (46) and (47) in (28) yields

$$r = 6. \tag{48}$$

Therefore we are in a position to state the following:

Theorem 8. *A Cotton flat Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ) is of constant scalar curvature* 6*.*

In view of the Theorem 8, from (16) we get

$$S(X, Y) = 2g(X, Y), \tag{49}$$

that is, the manifold becomes Einstein manifold. Thus we can conclude the following:

Corollary 9. *A Cotton flat Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ) is an Einstein manifold.*

5. Sasakian 3-manifolds admitting η-Ricci solitons satisfying $Q \cdot C = 0$

In the present section, we classify Sasakian 3-manifolds admitting η-Ricci solitons satisfying $Q \cdot C = 0$. Then we have

$$(Q \cdot C)(X, Y) = 0, \tag{50}$$

for any smooth vector fields X, Y.
From (50) we get

$$QC(X, Y) - C(QX, Y) - C(X, QY) = 0. \tag{51}$$

With the help of (25), (26), (30), (33) and (34) in the preceding equation yields

$$-2\mu\eta(Y)\phi X + 2\mu\eta(X)\phi Y - 4(\mu + 1)\mu g(\phi X, Y)\xi - \frac{\lambda}{4}[(Xr)Y - (Yr)X] = 0. \tag{52}$$

Taking inner product of the above with an arbitrary smooth vector field Z and then contracting X and Z and using $\phi\xi = \operatorname{Tr}\phi = 0$, we obtain

$$\lambda(Yr) = 0 \tag{53}$$

from which it follows that either $\lambda = 0$ or r is constant. Hence we have the following:

Theorem 10. *Let M^3 be a Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ) satisfying $Q \cdot C = 0$. Then either g is steady or M^3 is a manifold of constant scalar curvature.*

6. Concircularly Cotton semisymmetric Sasakian 3-manifolds admitting η-Ricci solitons

This section deals with the study of Concircularly Cotton semisymmetric Sasakian 3-manifolds admitting η-Ricci solitons. Then we have the following:

$$(\mathcal{Z}(X,Y)\cdot C)(U,V)=0, \tag{54}$$

which implies that

$$\mathcal{Z}(X,Y)C(U,V)-C(\mathcal{Z}(X,Y)U,V)+C(\mathcal{Z}(X,Y)V,U)=0. \tag{55}$$

Using (15), (24) and (25) in (19) we get

$$\begin{aligned}\mathcal{Z}(X,Y)Z &= 2(\lambda+\frac{r}{3})[g(X,Z)Y-g(Y,Z)X] \\ &\quad +\mu[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X \\ &\quad -g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi].\end{aligned} \tag{56}$$

As a consequence of (56) we derived the following:

$$\mathcal{Z}(X,\xi)Z=(2\lambda+\frac{2r}{3}+\mu)[g(X,Z)\xi-\eta(Z)X] \tag{57}$$

and

$$\mathcal{Z}(X,\xi)\xi=(2\lambda+\frac{2r}{3}+\mu)[\eta(X)\xi-X]. \tag{58}$$

With the help of (56), from (55) it follows that

$$\begin{aligned}&2(\lambda+\frac{r}{3})[g(X,C(U,V))Y-g(Y,C(U,V))X] \\ &+\mu[\eta(X)\eta(C(U,V))Y-\eta(Y)\eta(C(U,V))X \\ &-g(Y,C(U,V))\eta(X)\xi+g(X,C(U,V))\eta(Y)\xi] \\ &-C(\mathcal{Z}(X,Y)U,V)+C(\mathcal{Z}(X,Y)V,U)=0.\end{aligned} \tag{59}$$

Putting $Y=V=\xi$ in the above equation we have

$$\begin{aligned}&(2\lambda+\frac{2r}{3}+\mu)[g(X,C(U,\xi))-\eta(C(U,\xi))X] \\ &+C(\mathcal{Z}(X,\xi)\xi,U)-C(\mathcal{Z}(X,\xi)U,\xi)=0.\end{aligned} \tag{60}$$

On the application of (57) and (58), the above equation reduces to the following equation

$$\begin{aligned}&(2\lambda+\frac{2r}{3}+\mu)[g(C(U,\xi),X)\xi-\eta(C(U,\xi))X-\eta(X)C(U,\xi) \\ &-C(X,U)+\eta(U)C(X,\xi)]=0.\end{aligned} \tag{61}$$

Using (30), (33) and (35) in the last equation gives

$$(2\lambda+\frac{2r}{3}+\mu)[3\mu g(X,\phi U)\xi+\frac{1}{4}(Xr)U-\frac{1}{4}(Xr)\eta(U)\xi]=0. \tag{62}$$

Substituting $U = \phi U$ in (62) and the using (4) yields

$$(2\lambda + \frac{2r}{3} + \mu)[-3\mu g(X,U)\xi + 3\mu\eta(X)\eta(U)\xi + \frac{1}{4}(Xr)\phi U] = 0. \tag{63}$$

Taking inner product of (63) with ξ and then contracting X, U we obtain

$$(2\lambda + \frac{2r}{3} + \mu)\mu = 0. \tag{64}$$

By the virtue of (26) and (64) we get

$$(\lambda + \frac{2r}{3} - 2)(\lambda + 2) = 0, \tag{65}$$

which implies that $r = \frac{3}{2}(2-\lambda)$ or $\lambda = -2$. Hence we can state our next theorem as follows:

Theorem 11. *Let M^3 be a Concircularly Cotton semisymmetric Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ). Then either M^3 is a manifold of constant scalar curvature or the metric g is shrinking.*

7. Cotton pseudo-symmetric Sasakian 3-manifolds admitting η-Ricci solitons

This section is devoted to study of a Sasakian 3-manifold admitting η-Ricci solitons satisfying the curvature property

$$(R(U,V)\cdot C)(X,Y,Z) = f_C \mathcal{Q}(g,C)(X,Y,Z;U,V), \tag{66}$$

where we assume that $f_C \neq 1$.
From (66) we get

$$\begin{aligned} & -C(R(U,V)X,Y,Z) - C(X,R(U,V)Y,Z) - C(X,Y,C(U,V)Z) \\ = \;& f_C((U\wedge_g V)\cdot C)(X,Y,Z), \end{aligned} \tag{67}$$

from which it follows that

$$\begin{aligned} & C(R(U,V)X,Y,Z) + C(X,R(U,V)Y,Z) + C(X,Y,R(U,V)Z) \\ = \;& f_C[C((U\wedge_g V)X,Y,Z) + C(X,(U\wedge_g V)Y,Z) \\ & +C(X,Y,(U\wedge_g V)Z)]. \end{aligned} \tag{68}$$

In view of (23) and (68) we get

$$\begin{aligned} & C(R(U,V)X,Y,Z) + C(X,R(U,V)Y,Z) + C(X,Y,R(U,V)Z) \\ = \;& f_C[g(V,X)C(U,Y,Z) - g(U,X)C(V,Y,Z) \\ & +g(V,Y)C(X,U,Z) - g(U,Y)C(X,V,Z) \\ & +g(V,Z)C(X,Y,U) - g(U,Z)C(X,Y,V)]. \end{aligned} \tag{69}$$

Replacing X, Z and U by ξ in the preceding equation we find

$$\begin{aligned} & \eta(C(R(\xi,V)\xi,Y)) + \eta(C(\xi,R(\xi,V)Y)) + C(\xi,Y,R(\xi,V)\xi) \\ = \;& f_C[\eta(V)\eta(C(\xi,Y)) - \eta(C(V,Y)) - \eta(Y)\eta(C(\xi,V)) \end{aligned}$$

$$+\eta(V)\eta(C(\xi,Y)) - C(\xi,Y,V)]. \tag{70}$$

Substituting $Y = \phi Y$ and $V = \phi V$ in (70) we obtain

$$\begin{aligned} & -\eta(C(R(\phi V,\xi)\xi,\phi Y)) - \eta(C(R(\xi,\phi V)\phi Y,\xi)) + C(\xi,\phi Y,R(\xi,\phi V)\xi) \\ = \quad & f_C[\eta(C(\phi Y,\phi V)) + C(\phi Y,\xi,\phi V)]. \end{aligned} \tag{71}$$

Using (13) and (35) in (71) we have

$$(1 - f_C)\eta(C(\phi Y,\phi V)) + \frac{1}{4}(R(\xi,\phi V)\phi Y)r + (f_C - 1)C(\xi,\phi Y,\phi V) = 0. \tag{72}$$

From (15), (24) and (25) it follows that

$$\begin{aligned} R(X,Y)Z \quad = \quad & -(2\lambda + \frac{r}{2})[g(Y,Z)X - g(X,Z)Y] \\ & +\mu[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & -g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi]. \end{aligned} \tag{73}$$

The equations (6) and (73) we obtain the followings:

$$R(\xi,\phi V,\phi Y) = -(2\lambda + \frac{r}{2} + \mu)[g(V,Y) - \eta(V)\eta(Y)]\xi \tag{74}$$

and

$$(R(\xi,\phi V)\phi Y)r = 0. \tag{75}$$

Using (32), (38), (73), (74) and (75) in (72), we observe that

$$\mu(f_C - 1)g(Y,\phi V) = 0. \tag{76}$$

Replacing Y by ϕY in (76) and the using (6), we get

$$\mu(f_C - 1)[g(Y,V) - \eta(Y)\eta(V)] = 0. \tag{77}$$

On contraction over Y and V in (77) yields

$$\mu(f_C - 1) = 0, \tag{78}$$

which implies that

$$\mu = 0. \tag{79}$$

In view of (26) and (79), we have

$$\lambda = -2. \tag{80}$$

Thus we can state our next theorem as follows:

Theorem 12. *An η-Ricci soliton (g,ξ,λ,μ) on a Cotton pseudo-symmetric Sasakian 3-manifold is shrinking.*

In view of (79) and (80), from (28) we infer

$$r = 6. \tag{81}$$

Thus we can state the following:

Theorem 13. *A Cotton pseudo-symmetric Sasakian 3-manifold admitting an η-Ricci soliton (g,ξ,λ,μ) is a manifold of constant scalar curvature 6.*

In light of the Theorem 13, from (16) we observe that

$$S(X,Y) = 2g(X,Y) \tag{82}$$

that is, the manifold becomes Einstein. Therefore, we have the following:

Theorem 14. *A Cotton pseudo-symmetric Sasakian 3-manifold admitting an η-Ricci soliton (g, ξ, λ, μ) is an Einstein manifold.*

8. Conclusion

We know that ϕ-sectional curvature (sectional curvature with respect to a plane section orthogonal to ξ) of a 3-dimensional Sasakian manifold M^3 is equal to $\frac{r-4}{2}$. In view of the Theorem **8** and Theorem **12**, we can conclude that r is constant. Hence the ϕ-sectional curvature is constant and so M^3 is a 3-dimensional Sasakian space-form (see Blair [8]). Therefore we can make the following:

Remark 15. *A Sasakian* 3*-manifold admitting an η-Ricci soliton which is Cotton flat or Cotton pseudo-symmetric becomes a Sasakian-space-form.*

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