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On Idempotent Units in Commutative Group Rings

Ömer KÜSMÜŞ[*1]

Abstract

Special elements as units, which are defined utilizing idempotent elements, have a very crucial place in a commutative group ring. As a remark, we note that an element is said to be idempotent if $r^2 = r$ in a ring. For a group ring RG, idempotent units are defined as finite linear combinations of elements of G over the idempotent elements in R or formally, idempotent units can be stated as of the form $id(RG) = \{\sum_{r_g \in id(R)} r_g g : \sum_{r_g \in id(R)} r_g = 1$ and $r_g r_h = 0$ when $g \neq h\}$ where $id(R)$ is the set of all idempotent elements [3], [4], [5], [6]. Danchev [3] introduced some necessary and sufficient conditions for all the normalized units are to be idempotent units for groups of orders 2 and 3. In this study, by considering some restrictions, we investigate necessary and sufficient conditions for equalities:

$i. V\big(R(G \times H)\big) = id(R(G \times H)),$

$ii. V\big(R(G \times H)\big) = G \times id(RH),$

$iii. V\big(R(G \times H)\big) = id(RG) \times H$

where $G \times H$ is the direct product of groups G and H. Therefore, the study can be seen as a generalization of [3], [4]. Notations mostly follow [12], [13].

Keywords: idempotent, unit, group ring, commutative

1. INTRODUCTION

As widely known, a group ring RG of a given group G over a ring R is defined as the set of finite sums in $\{\sum_{g \in G} r(g)g : r(g) \in R\}$. The sets of all units and normalized units in RG are denoted by $U(RG)$ and $V(RG)$ respectively [8]. Idempotent units are described as elements of the form $\sum_{r_g \in id(R)} r_g g$ such that $\sum_{r_g \in id(R)} r_g = 1$ and $r_g r_h = 0$ when $g \neq h$ [3]. Let $id_C(R)$ display a complete set of orthogonal idempotent elements. For a group G, the p-primary component of G is

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generally shown by G_p and so the maximal torsion part G_0 of G is a co-product of primary components [3], [4], [11]. Since each idempotent unit can generate novel units of the same form, we can utilize the following notation for idempotent units [3], [4]:

$$id(RG) = \langle \sum_{r_g \in id_C(R)} r_g g : g \in G \rangle$$

By the way, we should recall that each element in G is said to be *trivial unit* of RG [1], [2]. Besides, we can observe that every trivial units are also idempotent units. Danchev introduced some necessary and sufficient conditions to be $V(RG) = id(RG)$ for groups of orders 2 and 3 as follows respectively [3]:

Proposition 1. Assume that $|G| = 2$. Then $V(RG) = id(RG)$ if and only if

$$1 - 2r \in U(R) \Leftrightarrow r \in id(R)$$

for all $r \in R$.

Proposition 2. Assume that $|G| = 3$. Then $V(RG) = id(RG)$ if and only if

$$1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R)$$

implies that $r^2 = r$, $f^2 = f$ and $rf = 0$ (Notice that $r^2 = r$, $f^2 = f$ and $rf = 0$ directly implies that $1 + 3r^2 + 3f^2 + 3rf - 3r - 3f = 1$ is a trivial unit).

Group rings over rings of prime characteristic have been classified in terms of the equality $V(RG) = id(RG)$ as follows [3]:

Theorem 3. Let R be a unital and commutative ring of a prime characteristic p. Since G is a non-trivial and Abelian group, $V(RG) = id(RG)$ if and only if the nil-radical of R, $N(R) = 0$ and at most one of the followings is satisfied:

$i.$ Maximal torsion part of G is trivial ($G_t = 1$),

$ii.$ $|G| = p = 2$ and R is a Boolean ring,

$iii.$ $|G| = 2, \forall r \in R, 1 - 2r \in U(R) \Leftrightarrow r^2 = r$,

$iv.$ $|G| = 3$ and

$$1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R)$$

implies that $r^2 = r$, $f^2 = f$ and $rf = 0$.

If we consider a cyclic group G of prime order greater than 3, we can construct a unit which is not an idempotent unit using Bass cyclic unit forms as follows [3], [4]:

$$u = (1 + g)^{p-1} - \frac{2^{p-1} - 1}{p} \hat{g}$$

where

$\hat{g} = 1 + g + \cdots + g^{p-1}$ and $G = \langle g : g^p = 1 \rangle$. This means that investigating necessary and sufficient conditions to be $V(RG) = id(RG)$ for cyclic groups G of prime order ≥ 5 is meaningless.

In the next section, we consider non-cyclic groups of order ≥ 6 and construct some necessary and sufficient conditions for all normalized units to be idempotent units. Throughout the paper, R is a commutative ring with unity and D is the direct product of groups.

2. MAIN RESULTS

In this section, we should remember the following notations for the set of orthogonal idempotent elements and a complete set of orthogonal idempotent elements in R are used throughout the paper respectively:

$$id_0(R) = \{e_i \in R : {e_i}^2 = e_i, e_i e_j = 0 \text{ for } i \neq j\},$$

$$id_C(R) = \{e_i \in id_0(R) : \sum e_i = 1\}.$$

Let G and H be two Abelian groups with p-primary and q-primary components G_p and H_q respectively. Using maximal torsion parts of G and H, we indicate the maximal torsion part of the direct product $D = G \times H$ as follows:

$$D_0 = \coprod_p \coprod_q G_p \times H_q = \coprod_q G_p \times \coprod_q H_q$$

where p and q are prime integers.

Owing to the fact that $G_p = 1$ means that G has no p-primary component [3], we intend by the notation $G_p \times H_q = 1$ that G or H has no p-primary or q-primary components respectively. Let $\wp$ denote the set of all prime integers.

Definition 4.

$$supp_C(G \times H) = \{pq: G_p \times H_q \neq 1\}$$

is said to be *support* of $G \times H$.

Example 5. Let $G = \mathbb{Z}_4$ and $H = \mathbb{Z}_9$. Since, $G_2 \neq 1$ and $H_3 \neq 1$, we say $6 \in supp_C(\mathbb{Z}_{36})$.

Definition 6.

$$zd_C(R) = \{pq: \exists 0 \neq r \in R, pqr = 0\}$$

and

$$inv_C(R) = \{pq: pq.1 \in U(R)\}$$

Now, using [4], we can present the first of results in this paper as follows:

Theorem 7. Let G and H be two Abelian groups. Then, since $D = G \times H$,

$$V(RD) = id(RD)$$

if and only if

$N(R) = 0$, $V(RD_0) = id(RD_0)$ and one of the following statements hold:

$a.\ D = D_0$,

$b.\ D \neq D_0$ and

$supp_C(D) \cap [\, inv_C(R) \cup zd_C(R) \,] = \emptyset$.

Proof. Let $V(RD) = id(RD)$. In this situation,

as $D_0 \subseteq D$, we can write the embedding $V(RD_0) \hookrightarrow V(RD) = id(RD)$. Hence,

$$id(RD_0) \subseteq V(RD_0)$$

and

$$V(RD_0) \cap id(RD) = id(RD_0)$$

yield that $V(RD_0) = id(RD_0)$. Now, let r be a nilpotent element in R. Thus, $(rgh)^k = 0$ for some $g \in G, h \in H, k \in \mathbb{N}$. As $1 - (rgh)^k$ is

$$(1 - rgh)(1 + rgh \ldots + r^{k-1}g^{k-1}h^{k-1}) = 1,$$

we conclude that

$$1 + r - rgh \in V(RD) = id(RD)$$

This shows that $r \in N(R) \cap id(R)$ and so $N(R) = 0$.

If D consists only of torsion part, the proof terminates. If not, $(D \neq D_0)$ then assuming

$$supp_C(D) \cap inv_C(R) \neq \emptyset$$

we say that there exists

$$pq \in supp_C(D\ ) \cap inv_C(R)$$

$\exists p, q \in \wp$. This means that $G_p \times H_q \neq 1$ and thus $\exists g \in G_p$ and $\exists h \in H_q$. Applying these torsion elements g and h, we can generate an idempotent

$$e = e(p, q) = \frac{\widehat{(gh)}}{pq}$$

where $\widehat{(gh)} = 1 + gh + \cdots + (gh)^{p-1}$. Using [4], we can compose a unit

$$u = 1 - e + exy \in V(RD) \backslash D$$

where $\exists x \in G \backslash G_0$ ve $\exists y \in H \backslash H_0$ are torsion-free elements. Explicit form of $u = 1 - e + exy$ can be written as

$$u = 1 - p^{-1}q^{-1} - p^{-1}q^{-1}gh - \cdots$$

$$-p^{-1}q^{-1}(gh)^{pq-1} + p^{-1}q^{-1}xy$$

$$+p^{-1}q^{-1}xygh + \cdots + p^{-1}q^{-1}xy(gh)^{pq-1}$$

However, the fact that coefficients -1 and 1 are not orthogonal idempotents displays that

$$u \in V(RD) \backslash id(RD)$$

This contradiction indicates that

$$supp_C(D) \cap inv_C(R) = \emptyset$$

On the other hand, assume that

$$supp_C(D) \cap zd_C(R) \neq \emptyset$$

In this case, for $\exists pq \in supp_C(D) \cap zd_C(R)$ and $\exists 0 \neq r \in R, pqr = 0$. Then we get

$$\begin{aligned} r(1-gh)^{pq} &= r[1 - \binom{pq}{1} gh + \cdots \\ &\quad + \binom{pq}{pq-1} (gh)^{pq-1} - 1] \\ &= pqr \sum n_i(p,q)\,(gh)^i = 0 \end{aligned}$$

where $\exists gh \in G_p \times H_q$ and $n_i(p,q) \in \mathbb{N}$. This gives the unit

$$\omega = 1 + r - rgh \in V(RD)$$

that is not an idempotent unit which is a contradiction as well. This means that

$$supp_C(D) \cap zd_C(R) = \emptyset$$

For the converse of the proof, we assume that $N(R) = 0$ and

$$V(RD_0) = id(RD_0)$$

If $D = D_0$,

$$V(RD) = V(RD_0) = id(RD_0) = id(RD)$$

and thus the proof terminates. If $D \neq D_0$, we define the group epimorphism:

$$\begin{aligned} \phi_C &: D \longrightarrow D/D_0 \\ & gh \longmapsto ghD_0 \end{aligned}$$

Extending linearly ϕ_C yields that

$$\phi_C : RD \longrightarrow R(D/D_0)$$
$$\sum_{(g,h)\in G\times H} \alpha(g,h)gh \mapsto \sum_{(g,h)\in G\times H} \alpha(g,h)gh\,D_0$$

If we restrict ϕ_C to unit groups of RD and $R(D/D_0)$, we see that

$$\phi_C\big(V(RD)\big) \subseteq V(R(D/D_0))$$

Let $V(RD_0) = id(RD_0)$. It is clear that

$$id(R(D/D_0)) \subseteq V(R(D/D_0))$$

For the converse inclusion, assume that

$$\exists u \in V(R(D/D_0)) \backslash id(R(D/D_0))$$

In this case, the augmentation map

$$\varepsilon : V(R(D/D_0)) \longrightarrow V(RD_0) = id(RD_0),$$

$$\varepsilon\left(\sum_{gh\in G\times H} \alpha(g,h)gh\,D_0\right) = \sum_{gh\in G\times H} \alpha(g,h)\,D_0$$

gives the image of u which is not an idempotent unit as

$$\varepsilon(u) \in V(RD_0) \backslash id(RD_0)$$

which contradicts with the assumption. Hence, we conclude that $V(R(D/D_0)) = id(R(D/D_0))$ by inspiring from [7],[9],[10].

It is obvious that $\phi_C\big(id(RD)\big) = id(R(D/D_0))$ and $\phi_C\big(id(RD)\big) \subseteq \phi_C\big(V(RD)\big)$. Since

$$\phi_C\big(id(RD)\big) = id(R(D/D_0))$$

and

$$\phi_C\big(V(RD)\big) \subseteq V(R(D/D_0)) = id(R(D/D_0)),$$

we attain the inclusion:

$$\phi_C\big(V(RD)\big) \subseteq \phi_C\big(id(RD)\big)$$

Applying the first isomorphism theorem serves that

$$\begin{aligned} \frac{V(RD)}{Ker\phi_C \subseteq V(RD_0)} &\simeq \phi_C\big(V(RD)\big) \\ &= \phi_C\big(id(RD)\big) \end{aligned}$$

Remember that $\phi_C\big(id(RD)\big) = id(R(D/D_0))$. Thus,

$$V(RD) = Ker\phi_C.id(R(D/D_0))$$
$$\subseteq V(RD_0).id(R(D/D_0))$$

By the hypothesis $V(RD_0) = id(RD_0)$, we can write

$$V(RD) \subseteq id(RD_0).id(R(D/D_0))$$

so $V(RD) \subseteq id(RD)$. Thus, $V(RD) = id(RD)$. ■

Theorem 8. Let $D = K_4$ (Klein 4-Group). Then, $V(RD) = id(RD)$ if and only if

$$1 - 4rs - 4rf - 4sf - 16rsf \in U(R)$$

implies that $r, s, f \in id_C(R)$. One can

notice that if $r, s, f \in id_C(R)$,

$$1 - 4rs - 4rf - 4sf - 16rsf = 1$$

is already a unit in R.

Proof. Since

$$D = K_4 = \langle g, h: g^2 = h^2 = 1, gh = hg\rangle$$

the group ring RD can be seen as an R-module as $RD = \langle 1, g, h, gh\rangle_R$. As the normalized units have augmentation one [12], we can state the normalized unit group as

$V(RD) =$

$\{1 - (r + s + f) + rg + sh + fgh: r, s, f \in R\}$.

Assume that $V(RD) = id(RD)$. Then, parameters of units in $V(RD)$ are idempotent elements in R. Let us consider a unit in $V(RD)$ as

$$u = 1 - (r_1 + s_1 + f_1) + r_1 g + s_1 h + f_1 gh$$

with the inverse

$$u^{-1} = 1 - (r_2 + s_2 + f_2) + r_2 g + s_2 h + f_2 gh.$$

Then,

$$uu^{-1} = 1 - X + Yg + Zh + Tgh = 1$$

so $X = Y = Z = T = 0$ where

$X = (r_1 + s_1 + f_1 + r_2 + s_2 + f_2) - (r_1 + s_1 + f_1)(r_2 + s_2 + f_2) - r_1 r_2 - s_1 s_2 - f_1 f_2 = 0,$

$Y = r_2(1 - r_1 - s_1 - f_1) + r_1(1 - r_2 - s_2 - f_2) + s_1 f_2 + s_2 f_1 = 0,$

$Z = s_2(1 - r_1 - s_1 - f_1) + s_1(1 - r_2 - s_2 - f_2) + r_1 f_2 + r_2 f_1 = 0,$

$T = f_2(1 - r_1 - s_1 - f_1) + f_1(1 - r_2 - s_2 - f_2) + r_1 s_2 + r_2 s_1 = 0.$

Arranging X, Y, Z and T, we get a system of linear equations as follows:

$I.\ r_2(2r_1 + s_1 + f_1 - 1) + s_2(r_1 + 2s_1 + f_1 - 1) + f_2(r_1 + s_1 + 2f_1 - 1) = r_1 + s_1 + f_1,$

$II.\ r_2(1 - 2r_1 - s_1 - f_1) + s_2(-r_1 + f_1) + f_2(-r_1 + s_1) = -r_1,$

$III.\ r_2(-s_1 + f_1) + s_2(1 - r_1 - 2s_1 - f_1) + f_2(r_1 - s_1) = -s_1,$

$IV.\ r_2(s_1 - f_1) + s_2(r_1 - f_1) + f_2(1 - r_1 - s_1 - 2f_1) = -f_1.$

Since $x := [r_2, s_2, f_2]^T$, $A =$

$$\begin{bmatrix} 1 - 2r_1 - s_1 - f_1 & -r_1 + f_1 & -r_1 + s_1 \\ -s_1 + f_1 & 1 - r_1 - 2s_1 - f_1 & r_1 - s_1 \\ s_1 - f_1 & r_1 - f_1 & 1 - r_1 - s_1 - 2f_1 \end{bmatrix}$$

and $B = [-r_1, -s_1, -f_1]^T$, we know that the existence of x in the system $Ax = B$ depends on whether $det(A) \in R$ is a unit.

On behalf of the simplicity, let us make the substitutions: $r_1 = r, s_1 = s$ and $f_1 = f$ while we compute $det(A)$. A straightforward computation introduces that

$$\begin{aligned} det(A) = 1 - 4r + 4r^2 - 4s + 12rs - 8r^2 s \\ + 4s^2 - 8rs^2 - 4f + 12rf \\ - 8r^2 f + 12sf - 16rsf - 8s^2 f \\ + 4f^2 - 8rf^2 - 8sf^2 \end{aligned}$$

Considering $r, s, f \in id(R)$ simplifies $det(A)$ as

$$det(A) = 1 - 4rs - 4rf - 4sf - 16rsf$$

which is a unit in R. This actually implies that $r, s, f \in id_C(R)$ because of the assumption $V(RD) = id(RD)$. For the reverse direction of the proof, the reader can notice that the assumption that $det(A) = 1 - 4rs - 4rf - 4sf - 16rsf$ implies that $r, s, f \in id_C(R)$ directly displays that any normalized unit in

$$V(RD) = \{1 - (r + s + f) + rg + sh + fgh$$

$$: r, s, f \in id_C(R)\}$$

is in $id(RD)$. ■

Theorem 9. Let $D = K_4$ (Klein 4-Group). If $V(RD) = id(RD)$, then $V(RD) =$

$$\{1 - r - s - f + rg + sh + fgh : r, s, f \in R\}$$

implies that $r + f = 0$ and

$$1 - 2(r + s) \in U(R) \Leftrightarrow r + s \in id(R)$$

Proof. Let $G = \langle g \rangle, H = \langle h \rangle, o(g) = o(h) = 2$.

Define a group homomorphism as

$f: G \times H \to \langle \omega, h \rangle$ with $f(g, h) = (\omega, h)$ where $\omega = e^{i\pi}$. Extending linearly f to group rings over the ring R and restricting it to unit groups give $f: R(G \times H) \to R\langle \omega, h \rangle$ and

$$f: V\big(R(G \times H)\big) \to V(R\langle h \rangle) = V(RC_2)$$

respectively. One can easily observe that

$$Ker\ f = \langle 1 + g, h^2 \rangle_R$$

and then,

$$\frac{RD}{\langle 1 + g,\ h^2 \rangle_R} \simeq R\langle h \rangle$$

Thus, when we choose a unit u from the above definition of $V(RD)$ as

$$u = 1 - r - s - f + rg + sh + fgh,$$

we sight that

$$f(u) = 1 - 2r - s - f + (s - f)h \in V(RH)$$

Since $V(RH) = id(RH)$ and the augmentation of $f(u)$ is 1, we conclude that $r + f = 0$ and

$$1 - 2(r + s) \in U(R) \Leftrightarrow r + s \in id(R) \qquad ■$$

Furthermore, with the help of $\exists v = f(u)^{-1}$ as $v = k + lh$, we can observe that

$$f(u)v = [1 + r - s + (s - r)h][k + lh] = 1$$

if and only if the system of linear equations

$$k(1 - s + r) + l(s - r) = 1$$
$$k(s - r) + l(1 - s + r) = 0$$

has a unique solution pair (r, s) in R. This unique solution depends on

$$(1 - s + r)^2 - (s - r)^2 = (1 - 2s)(1 + 2r)$$

Then, we can deduce that $f(u)v = 1$ if and only if $(1 - 2r)$ and $(1 - 2s)$ are units in R. Due to the fact that $V(RC_2) = id(RC_2)$ if and only if Proposition 1. hold, we can conclude that

$$V(RD) = id(RD) \Leftrightarrow V(RC_2) = id(RC_2)$$

where $D = K_4$ and C_2 is a group of order 2.

Theorem 10. Let two distinct cyclic groups be $G = \langle g : g^3 = 1 \rangle$ and $H = \langle h : h^2 = 1 \rangle$. Then,

$$V(RD) = G \times id(RH)$$

if and only if the followings hold:

$i.$ $1 + 3(r^2 + f^2 + rf + r + f) \in U(RH)$

implies that $(r, f) \in \{(0,0), (0,-1), (-1,0)\}$,

$ii.$ $1 - 2r \in U(R) \Leftrightarrow r \in id(R)$.

Proof. Let us define a group epimorphism

$$\rho_G: D \to H$$
$$(g, h) \mapsto h$$

Extending it to group rings as

$$\rho_G: RD \to RH$$

with $\rho_G\left(\sum_{gh\in D}\alpha_{gh}gh\right)=\sum_{gh\in D}\alpha_{gh}h$, the kernel of ρ_G is obtained as

$$\kappa_G := Ker\,\rho_G = \langle 1-g, 1-g^2\rangle_{RH}$$

One can establish a short exact sequence as

$$\kappa_G \overset{i}{\rightarrow} RD \overset{\rho_G}{\rightarrow} RH$$

with inclusion i. Moving it to unit groups, we can construct

$$K_G \overset{i}{\rightarrow} V(RD) \overset{\rho_G}{\rightarrow} V(RH)$$

with $K_G := (1+\kappa_G) \cap V(RD)$. Using the embedding $V(RH) \hookrightarrow V(RD)$, we can write

$$V(RD) = K_G \times V(RH)$$

Since $|H|=2$, using [3] for $V(RH)=id(RH)$, we obtain the latter condition in phrase of the theorem. Now, we investigate the necessary and sufficient condition to be $K_G = G$. Due to $K_G =$

$\{u = 1 + r(1-g) + f(1-g^2): r, f \in RH\}$, choose an inverse of a unit u in K_G as

$$\exists v = 1 + r'(1-g) + f'(1-g^2)$$

Then, a straightforward computation shows that $uv = 1 + A(1-g) + B(1-g^2) = 1$ where

$$A = r + r' + 2rr' + rf' + fr' - r'f'$$

and

$$B = r' + f' - rr' + rf' + r'f + 2r'f'$$

Then the system

$$\begin{aligned} r + r' + 2rr' + rf' + fr' - r'f' &= 0 \\ r' + f' - rr' + rf' + r'f + 2r'f' &= 0 \end{aligned}$$

or equivalently

$$\begin{bmatrix} 1+2r+r' & r-r' \\ -r+r' & 1+r+2r' \end{bmatrix}\begin{bmatrix} f \\ f' \end{bmatrix} = \begin{bmatrix} -r \\ -f \end{bmatrix}$$

has a unique solution if and only if

$$1 + 3(r^2 + f^2 + rf + r + f) \in V(RH)$$

and by the fact that u must be a trivial unit,

$$(r,f) \in \{(0,0),(0,-1),(-1,0)\}$$

as required. ■

By exchanging the types of direct components of $V(RD)$ in the previous theorem, we state and prove the following one as well:

Theorem 11. Let $G = \langle g: g^3 = 1\rangle$ and $H = \langle h: h^2 = 1\rangle$. Then, $V(RD) = id(RG) \times H$ if and only if the following statements are satisfied:

$a.$ $1 + 2r \in U(RG) \Leftrightarrow r = 0$ or $r = -1$,

$b.$ $1 + 3(r^2 + f^2 + rf - r - f) \in U(R)$ implies that $r, f \in id_0(R)$.

Proof. Define a group and a ring epimorphisms as in the previous theorem such as

$$\begin{aligned} \rho_H &: D \rightarrow G \\ (g,h) &\mapsto g \end{aligned}$$

and $\rho_H: RD \rightarrow RG$ with

$$\rho_H\left(\sum_{gh\in D}\alpha_{gh}gh\right) = \sum_{gh\in D}\alpha_{gh}g$$

respectively. We can view that

$$\kappa_H := Ker\,\rho_H = \langle 1-h\rangle_{RG}$$

Besides, we can construct the short exact sequence $\kappa_H \overset{i}{\rightarrow} RD \overset{\rho_H}{\rightarrow} RG$. Restricting the last sequence to unit groups, we deduce that the following short exact sequence can be established:

$$K_H \overset{i}{\rightarrow} V(RD) \overset{\rho_H}{\rightarrow} V(RG)$$

where $K_H := (1+\kappa_H) \cap V(RD)$. Because of the embedding $V(RG) \hookrightarrow V(RD)$, the last sequence splits as $V(RD) = K_H \times V(RG)$. On account of $|G| = 3$, we already know that $V(RG) = id(RG)$ if and only if

$$1 + 3(r^2 + f^2 + rf - r - f) \in U(R)$$

implies that $r, f \in id_0(R)$ [3]. On the other hand, since $|H| = 2$, one can observe that

$$K_H = \{1 + r(1 - h): r \in RG\} = H$$

if and only if $r = 0$ or $r = -1$. ■

3. DISCUSSIONS AND SUGGESTIONS

To sum up, we have attained some necessary and sufficient conditions for normalized unit group $V(RD)$ to be idempotent unit group $id(RD)$ or direct product of trivial unit group and idempotent unit group as $id(RG) \times H$ (or $G \times id(RH)$) in this paper. Consequently, as originality of the paper, we can say that the paper has been both extended some results in [3], [4] and defined novel types of units which are combined with both idempotent units and trivial units. As an open problem and future work, necessary and sufficient conditions for

$$V(RD) = id(RG) \times id(RH)$$

may be studied for Abelian groups.

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Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

Ethics Committee Approval

This paper does not require any ethics committee permission or special permission.

Conflicts of Interests

No potential conflict of interest was reported by the author.

Author's Contributions

ÖK performed both the theoretical results and applicational calculations with final version of the paper.

REFERENCES

[1] P. Danchev, "Trivial units in commutative group algebras," Extr. Math., vol. 23, pp. 49-60, 2008.

[2] P. Danchev, "Trivial units in abelian group algebras," Extr. Math., vol. 24, pp. 47-53, 2009.

[3] P. Danchev, "Idempotent units in commutative group rings," Kochi J. Math., vol. 4, pp. 61-68, 2009.

[4] P. Danchev, "Idempotent units of commutative group rings," Commun. Algebra, vol. 38, pp. 4649-4654, 2010.

[5] P. Danchev, "On some idempotent torsion decompositions of normed units in commutative group rings," J. Calcutta Math. Soc., vol. 6, pp. 31-34, 2010.

[6] P. Danchev, "Idempotent-torsion normalized units in abelian group rings," Bull Calcutta Math. Soc., vol. 1(103), pp. 31-34, 2011.

[7] G. Karpilovsky, "On units in commutative group rings," Arch. Math. (Basel), vol. 38, pp. 420–422, 1982.

[8] G. Karpilovsky, "On finite generation of unit groups of commutative group rings," Arch. Math. (Basel), vol. 40, pp. 503–508, 1983.

[9] G. Karpilovsky, "Unit groups of group rings," Harlow: Longman Sci. and Techn., 1989.

[10] G. Karpilovsky, "Units of commutative group algebras," Expo. Math., vol. 8, pp. 247-287, 1990.

[11] W. May, "Group algebras over finitely generated rings," J. Algebra, vol. 39 pp. 483–511, 1976.

[12] C. Polcino Milies and S. K. Sehgal, "An introduction to group rings," Kluwer, North-Holland, Amsterdam, 2002.

[13] S. K. Sehgal, "Topics in group rings," Marcel Dekker, New York, 1978.