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Novel Results based on Generalisation of some Integral Inequalities for Trigonometrically-*P* Function

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Abstract

Trigonometric P-function is defined as a special case of h-convex function. In this article, we used a general lemma that gives trapezoidal, midpoint, Ostrowski, and Simpson type inequalities. With the help of this lemma, we have obtained many integral inequalities and generalisations for trigonometric P-function. We have shown that it goes down to the studies in special cases which are described in our study. Apart from that, we got new results for the trapezoidal, midpoint, Ostrowski, and Simpson type inequalities.

Keywords: Hermite-Hadamard inequality, Simpson-type inequality, Ostrowski-type inequality, Trapezoid-type inequality, Midpoint-type inequality, Trigonometrically-*P* function.

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1. INTRODUCTION

In recent years, many studies on convex functions and integral inequalities have been done and investigated. Firstly, the definition of a convex function is as follows:

Definition 1. *A function* $\Upsilon: I \subset \mathbb{R} \to \mathbb{R}$ *is said to be a convex in the classical sense if, for all* $c, d \in I$ *and* $t \in [0,1]$, *we have*

$$\Upsilon(tc + (1 - t)d) \leq (1 - t)\Upsilon(c) + t\Upsilon(d).$$

In many research fields, the relationship between convexity and inequalities has always been a subject of research. The most important of these is Hermite Hadmard, Ostrowski and Simpson inequalities(see [1, 2, 4, 5, 6, 7, 12, 13, 16, 17, 18, 19, 20, 21, 22]). These famous inequalities are respectively expressed as follows:

$\Upsilon: I \subset \mathbb{R} \to \mathbb{R}$ is the convex function known on the interval I of $\mathbb{R}$ and $c, d \in I$ with $c < d$ as follows:

$$\Upsilon\left(\frac{c + d}{2}\right) \leq \frac{1}{c - d} \int_c^d \Upsilon(x)dx \leq \frac{\Upsilon(c) + \Upsilon(d)}{2}$$

holds. Both inequalities hold in the reversed direction if Υ is concave.

Let $\Upsilon: I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in $I°$, the interior of I, and let c, $d \in I°$ with $c < d$. If $|\Upsilon'(x)| \leq M$, $x \in [c, d]$, then we the following inequality holds

$$\left|\Upsilon(x) - \frac{1}{c - d} \int_c^d \Upsilon(t)dt\right|$$
$$\leq \frac{M}{c - d}\left[\frac{(x - c)^2 + (d - x)^2}{2}\right]$$

for all $x \in [c, d]$. The best possible constant, in the sense that it cannot be replaced by a smaller one, is found $\frac{1}{4}$ in [9].

Let $\Upsilon: [c, d] \to \mathbb{R}$ be a four-times continuously differentiable mapping on (c, d) and $\left\|\Upsilon^{(4)}\right\|_\infty =$

$\sup_{x \in (c,d)} \left|\Upsilon^{(4)}(x)\right| < \infty.$ Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{\Upsilon(c) + \Upsilon(d)}{2} + 2\Upsilon\left(\frac{c + d}{2}\right)\right]\right.$$
$$\left. - \frac{1}{d - c}\int_c^d \Upsilon(x)dx\right|$$
$$\leq \frac{1}{2880}\left\|\Upsilon^{(4)}\right\|_\infty (d - c)^2.$$

[14, 15] and therein.

After convexity became so popular, the researchers worked on new classes of convexity. Thus, they applied known integral inequalities to new convexity classes.

Definition 2. [8] *A non-negative function* $\Upsilon: I \to \mathbb{R}$ *is said to be a P-function if the inequality*

$$\Upsilon(tu + (1 - t)v) \leq \Upsilon(u) + \Upsilon(v)$$

holds for all $u, v \in I$, and $t \in [0,1]$. The set of P-functions on the interval I is denoted by $P(I)$.

Definition 3. [23] *Let* $h: J \to \mathbb{R}$ *be a non-negative function,* $h \neq 0$. *We say that* $\Upsilon: I \to \mathbb{R}$ *is an h-convex function, or that* Υ *belongs to the class* $SX(h, I)$, *if* Υ *is non-negative and for all* $u, v \in I, \alpha \in (0,1)$ *we have*

$$\Upsilon(\alpha u + (1 - \alpha)v)$$
$$\leq h(\alpha)\Upsilon(u) + h(1 - \alpha)\Upsilon(v).$$

If this inequality is reversed, then Υ is said to be h-concave, i.e. $\Upsilon \in SV(h, I)$.

In [11], Kadakal gave a different kind of trigonometrically convex function from definition of h-convex function.

Definition 4. [11] *A non-negative function* $\Upsilon: I \to \mathbb{R}$ *is called trigonometrically convex if for every* $u, v \in I$ *and* $t \in [0,1]$,

$$\Upsilon(tu + (1 - t)v) \leq$$

$$\left(sin\frac{\pi t}{2}\right)\Upsilon(u) + cos\left(\frac{\pi t}{2}\right)\Upsilon(v).$$

The class of all trigonometrically convex functions is denoted by $TC(I)$ on interval I. We note that every trigonometrically convex function is a h-convex function for $h(t) = sin\frac{\pi t}{2}$. Moreover, if $\Upsilon(u)$ is a nonnegative function, then every trigonometric convex function is a P-function.

In [3], Bekar obtained the trigonometrically P-function as follows:

Definition 5. [3] *A non-negative function $\Upsilon: I \to \mathbb{R}$ is called trigonometrically P-function if for every $u, v \in I$ and $t \in [0,1]$,*

$$\Upsilon(tu + (1-t)v)$$

$$\leq \left(sin\frac{\pi t}{2}\right) + cos\left(\frac{\pi t}{2}\right)[\Upsilon(u) + \Upsilon(v)].$$

The classes of all trigonometrically P-functions are donated by $TP(I)$ on interval I.

Remark 1. [3] *Clearly, if $\Upsilon(u)$ is a nonnegative function, then every P-function is a trigonometric P-function. Indeed, for every $u, v \in I$ and $t \in [0,1]$ we have*

$$\Upsilon(tu + (1-t)v) \leq \Upsilon(u) + \Upsilon(v)$$

$$\leq \left(sin\frac{\pi t}{2} + cos\frac{\pi t}{2}\right)[\Upsilon(u) + \Upsilon(v)].$$

Example 1. [3] *Non-negative constant functions are trigonometrically P-functions, since $sin\frac{\pi t}{2} + cos\frac{\pi t}{2} \geq 1$ for all $t \in [0,1]$.*

Lemma 1. *Every non-negative trigonometrically convex function is trigonometrically P-function* [3].

In [10], İşcan pointed out the new generalised lemma which is giving many integral inequalities as follows:

Lemma 2. [10] *Let $\Upsilon: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I°$ such that $\Upsilon' \in L[c,d]$, where $c,d \in I$ with $c < d$ and $\theta, \lambda \in [0,1]$. Then the following equality holds:*

$$I(c,d;\theta,\lambda) =$$

$$(d-c)\left[-\lambda^2\int_0^1 (t-\theta)\Upsilon'(tc + (1-t)A_\lambda)dt\right.$$

$$\left. +(1-\lambda)^2\int_0^1 (t-\theta)\Upsilon'(td + (1-t)A_\lambda)dt\right]$$

where $I(c,d;\theta;\lambda) = (1-\theta)(\lambda\Upsilon(c) + (1-\lambda)\Upsilon(d)) + \theta\Upsilon((1-\lambda)c + \lambda d) - \frac{1}{d-c}\int_c^d \Upsilon(x)dx$ and $A_\lambda = (1-\lambda)c + \lambda d$.

We built this study on Lemma 2, where we get different types of integral inequalities. Using this generalised Lemma 2, we have obtained the generalised midpoint, trapezoidal, Simpson and Ostrowski type inequalities for trigonometrically P-function.

2. MAIN RESULTS

It will be referred to $I(c,d;\theta;\lambda) = (1-\theta)(\lambda\Upsilon(c) + (1-\lambda)\Upsilon(d)) + \theta\Upsilon(A_\lambda) - \frac{1}{d-c}\int_c^d \Upsilon(x)dx$, and $A_\lambda = (1-\lambda)c + \lambda d$ where $\theta, \lambda \in [0,1]$, as in this section.

Theorem 1. *Let $\Upsilon: I \to \mathbb{R}$ be a continuously differentiable function, let $c < d$ in I, $\lambda, \theta \in [0,1]$ and assume that $\Upsilon' \in L[c,d]$. If $|\Upsilon'|$ is a trigonometrically P-function on interval $[c,d]$, then the following inequality holds*

$$|I(c,d;\theta;\lambda)| \qquad\qquad (2.1)$$

$$\leq (d-c)\left(\frac{8}{\pi^2} + \frac{2}{\pi} - \frac{8}{\pi^2}\left[sin\frac{\pi\theta}{2} + cos\frac{\pi\theta}{2}\right]\right)$$

$$[\lambda^2|\Upsilon'(c)| + (\lambda^2 + (1-\lambda)^2)|\Upsilon'((1-\lambda)c + \lambda d)| + (1-\lambda)^2|\Upsilon'(d)|].$$

Proof. If we take the absolute value of both sides of Lemma 2 and then using $|\Upsilon'|$ is trigonometrically P-function, then we have

$$|I(c, d; \theta; \lambda)| \qquad (2.2)$$

$$\leq (d - c)\left[\lambda^2 \int_0^1 |t - \theta||\Upsilon'(tc + (1 - t)A_\lambda)|dt\right.$$

$$\left. + (1 - \lambda)^2 \int_0^1 |\theta - t||\Upsilon'(td + (1 - t)A_\lambda)|dt\right]$$

$$\leq (d - c)\left\{\lambda^2 \int_0^1 |t - \theta|\left(\sin\frac{\pi t}{2}\right.\right.$$

$$\left. + \cos\frac{\pi t}{2}\right) [|\Upsilon'(a)| + |\Upsilon'(A_\lambda)|]dt$$

$$+ (1 - \lambda)^2 \int_0^1 |\theta - t|\left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)[|\Upsilon'(d)|$$

$$\left. + |\Upsilon'(A_\lambda)|]dt\right\}$$

$$= (d - c)\left\{\lambda^2[|\Upsilon'(c)| + |\Upsilon'(A_\lambda)|]\int_0^1 |t\right.$$

$$\left. - \theta|\left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) dt\right.$$

$$+ (1 - \lambda)^2[|\Upsilon'(d)| + |\Upsilon'(A_\lambda)|]\int_0^1 |\theta$$

$$\left. - t|\left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) dt\right\}.$$

We calculate the integrals as follows:

$$\int_0^1 |t - \theta|\left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) dt$$

$$= \frac{8}{\pi^2} + \frac{2}{\pi} - \frac{8}{\pi^2}\left[\sin\frac{\pi \theta}{2} + \cos\frac{\pi \theta}{2}\right]. \qquad (2.3)$$

When the equation (2.3) is written in the inequality, the proof is completed.

Remark 2. *If it is taken* $\sin\frac{\pi \theta}{2} + \cos\frac{\pi \theta}{2} \geq 1, \theta \in [0,1]$, *we get*

$$|I(c, d; \theta; \lambda)| \leq \frac{2(d - c)}{\pi} \qquad (2.4)$$

$$[\lambda^2|\Upsilon'(c)| + (\lambda^2 + (1 - \lambda)^2)|\Upsilon'(A_\lambda)| + (1 - \lambda)^2|\Upsilon'(d)|].$$

Corollary 1. *When θ is taken as 1 in Theorem 1, then we get generalised midpoint-type inequality as follows:*

$$\left|f((1 - \lambda)c + \lambda d) - \frac{1}{d - c}\int_c^d f(x)dx\right|$$

$$\leq \frac{2(d - c)}{\pi}[\lambda^2|f'(c)|$$

$$+ (\lambda^2 + (1 - \lambda)^2)|f'(A_\lambda)| + (1 - \lambda)^2|f'(d)|].$$

Corollary 2. *If θ is taken as 1 and $|\Upsilon'(u)| \leq M$, $u \in [c, d]$ in Theorem 1, then we get the following Ostrowski-type inequality*

$$\left|\Upsilon(u) - \frac{1}{d - c}\int_c^d \Upsilon(v)dv\right| \qquad (2.5)$$

$$\leq \frac{4M}{\pi}\left[\frac{(u - c)^2 + (d - u)^2}{d - c}\right]$$

for each $u \in [c, d]$.

Proof. For each $u \in [c, d]$, there exist $\lambda_u \in [0,1]$ such that $u = (1 - \lambda_u)c + \lambda_u d$. Hence we have $\lambda_u = \frac{u - c}{d - c}$ and $1 - \lambda_u = \frac{d - u}{d - c}$. Therefore for each $u \in [c, d]$, from the inequality (2.1) we obtain the inequality (2.5).

Corollary 3. *When θ is taken as 0 in Theorem 1, then we get generalised trapezoid type inequality as follows:*

$$\left| \lambda \Upsilon(c) + (1 - \lambda)\Upsilon(d) - \frac{1}{d - c} \int_c^d \Upsilon(x)dx \right|$$

$$\leq \frac{2(d - c)}{\pi}$$

$$[\lambda^2 |\Upsilon'(c)| + (\lambda^2 + (1 - \lambda)^2)|\Upsilon'(A_\lambda)| + (1 - \lambda)^2 |\Upsilon'(d)|].$$

Corollary 4. When θ, λ are taken as $\frac{2}{3}$, $\frac{1}{2}$, respectively in Theorem 1, then we get Simson-type inequality as follows:

$$\left| \frac{1}{6} \left[\Upsilon(c) + 4\Upsilon\left(\frac{c + d}{2}\right) + \Upsilon(d) \right] \right.$$

$$\left. - \frac{1}{d - c} \int_c^d \Upsilon(x)dx \right|$$

$$\leq \frac{(d - c)}{2} \left(\frac{4(1 - \sqrt{3})}{\pi^2} + \frac{2}{\pi} \right)$$

$$\left\{ \frac{|\Upsilon'(c)| + |\Upsilon'(d)|}{2} + \left| \Upsilon'\left(\frac{c + d}{2}\right) \right| \right\}.$$

Corollary 5. When θ, λ are taken as 1, $\frac{1}{2}$, respectively in Theorem 1, then we get midpoint-type inequality as follows:

$$\left| \Upsilon\left(\frac{c + d}{2}\right) - \frac{1}{d - c} \int_c^d \Upsilon(x)dx \right|$$

$$\leq \frac{d - c}{\pi} \left\{ \frac{|\Upsilon'(c)| + |\Upsilon'(d)|}{2} + \left| \Upsilon'\left(\frac{c + d}{2}\right) \right| \right\}.$$

Corollary 6. When θ, λ are taken as 0, $\frac{1}{2}$, respectively in Theorem 1, then we get Trapezoidal-type inequality as follows:

$$\left| \frac{\Upsilon(c) + \Upsilon(d)}{2} - \frac{1}{d - c} \int_c^d \Upsilon(x)dx \right|$$

$$\leq \frac{d - c}{\pi} \left\{ \frac{|\Upsilon'(c)| + |\Upsilon'(d)|}{2} + \left| \Upsilon'\left(\frac{c + d}{2}\right) \right| \right\}.$$

Theorem 2. Let $\Upsilon: I \to \mathbb{R}$ be a continuously differentiable function, let $c < d$, $c, d \in I$ and $\lambda, \theta \in [0,1]$ assume that $q > 1$. If $|\Upsilon'|^q$ is a trigonometrically P-function on the interval $[c, d]$, then the following inequality holds

$$|I(c, d; \theta; \lambda)| \leq \qquad (2.6)$$

$$(d - c) \left(\frac{\theta^{p+1} + (1 - \theta)^{p+1}}{p + 1} \right)^{\frac{1}{p}} \left(\frac{4}{\pi} \right)^{1/q}$$

$$[\lambda^2 (|\Upsilon'(c)|^q + |\Upsilon'(A_\lambda)|^q)^{1/q}$$

$$+ (1 - \lambda)^2 (|\Upsilon'(d)|^q + |\Upsilon'(A_\lambda)|^q)^{1/q}].$$

Proof. From Lemma 2 and by Hölder's integral inequality, we have

$$|I(c, d; \theta; \lambda)| \leq (d - c) \qquad (2.7)$$

$$\left[\lambda^2 \left(\int_0^1 |t - \theta|^p \right)^{1/p} \left(\int_0^1 |\Upsilon'(tc + (1 - t)A_\lambda)|^q dt \right)^{1/q} \right.$$

$$\left. + (1 - \lambda)^2 \left(\int_0^1 |t - \theta|^p \right)^{1/p} \left(\int_0^1 |\Upsilon'(td + (1 - t)A_\lambda)|^q dt \right)^{1/q} \right].$$

Since $|\Upsilon'|^q$ is trigonometrically P-function on $[c, d]$, and by simple computation, we get

$$\int_0^1 |\Upsilon'(tc + (1 - t)[(1 - \lambda)c + \lambda d])|^q dt \quad (2.8)$$

$$\leq \int_0^1 \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2} \right) [|\Upsilon'(c)|^q$$

$$+ |\Upsilon'(A_\lambda)|^q] dt$$

$$= \frac{4}{\pi} [|\Upsilon'(c)|^q + |\Upsilon'(A_\lambda)|^q]$$

$$\int_0^1 |\Upsilon'(td + (1$$

$$- t)A_\lambda)|^q dt \qquad (2.9)$$

$$\leq \int_0^1 \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)[|\Upsilon'(d)|^q$$

$$+ |\Upsilon'(A_\lambda)|^q] dt$$

$$= \frac{4}{\pi}[|\Upsilon'(d)|^q + |\Upsilon'(A_\lambda)|^q]$$

and

$$\int_0^1 |t - \theta|^p dt = \frac{\theta^{p+1} + (1 - \theta)^{p+1}}{p + 1}. \qquad (2.10)$$

Thus, substitute (2.8)-(2.10) in (2.7), we obtain the inequality (2.6). This completes the proof.

Corollary 7. *When θ is taken as 1 in Theorem 2, then we get generalised midpoint-type inequality as follows:*

$$\left|\Upsilon((1 - \lambda)c + \lambda d) - \frac{1}{d - c}\int_a^b \Upsilon(x)dx\right|$$

$$\leq (b - a)\left(\frac{1}{p + 1}\right)^{\frac{1}{p}}\left(\frac{4}{\pi}\right)^{1/q}$$

$$[\lambda^2(|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q)^{1/q}$$

$$+(1 - \lambda)^2(|\Upsilon'(b)|^q + |\Upsilon'(A_\lambda)|^q)^{1/q}].$$

Corollary 8. *When θ is taken as 0 in Theorem 2, then we get generalised trapezoidal-type inequality as follows:*

$$\left|\lambda\Upsilon(a) + (1 - \lambda)\Upsilon(b) - \frac{1}{b - a}\int_a^b \Upsilon(x)dx\right|$$

$$\leq (b - a)\left(\frac{1}{p + 1}\right)^{\frac{1}{p}}\left(\frac{4}{\pi}\right)^{1/q}$$

$$[\lambda^2(|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q)^{1/q}$$

$$+(1 - \lambda)^2(|\Upsilon'(b)|^q + |\Upsilon'(A_\lambda)|^q)^{1/q}].$$

Corollary 9. *If θ is taken as 1 and $|\Upsilon'(x)| \leq M$, $x \in [a, b]$ in Theorem 2, then we get the following Ostrowski-type inequality*

$$\left|\Upsilon(x) - \frac{1}{b - a}\int_a^b \Upsilon(u)du\right| \qquad (2.11)$$

$$\leq \frac{M}{(p + 1)^{\frac{1}{p}}}\left(\frac{8}{\pi}\right)^{1/q}\left[\frac{(x - a)^2 + (b - x)^2}{b - a}\right]$$

for each $x \in [a, b]$.

Proof. For each $x \in [a, b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1 - \lambda_x)a + \lambda_x b$. Hence we have $\lambda_x = \frac{x - a}{b - a}$ and $1 - \lambda_x = \frac{b - x}{b - a}$. Therefore for each $x \in [a, b]$, from the inequality (2.6) we obtain the inequality (2.11).

Corollary 10. *When θ, λ are taken as $\frac{2}{3}, \frac{1}{2}$, respectively in Theorem 2, then we get Simson-type inequality as follows:*

$$\left|\frac{1}{6}\left[\Upsilon(a) + 4\Upsilon\left(\frac{a + b}{2}\right) + \Upsilon(b)\right]\right.$$

$$\left. - \frac{1}{b - a}\int_a^b \Upsilon(x)dx\right|$$

$$\leq \frac{b - a}{4}\left(\frac{2^{p+1} + 1}{3^{p+1}(p + 1)}\right)^{1/p}\left(\frac{4}{\pi}\right)^{1/q}$$

$$\left\{ \left[|\Upsilon'(a)|^q + \left| \Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[|\Upsilon'(b)|^q + \left| \Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right\}.$$

Corollary 11. *When θ, λ are taken as 1, $\frac{1}{2}$, respectively in Theorem 2, then we get midpoint-type inequality as follows:*

$$\left| \Upsilon\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \Upsilon(x)dx \right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}$$

$$\left\{ \left[|\Upsilon'(a)|^q + \left| \Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[|\Upsilon'(b)|^q + \left| \Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right\}.$$

Corollary 12. *When θ, λ are taken as 0, $\frac{1}{2}$, respectively in Theorem 2, then we trapezoid-type inequality as follows:*

$$\left| \frac{\Upsilon(a) + \Upsilon(b)}{2} - \frac{1}{b-a}\int_a^b \Upsilon(x)dx \right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{ \left[|\Upsilon'(a)|^q + \left| \Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[|\Upsilon'(b)|^q + \left| \Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right\}.$$

Theorem 3. *Let $\Upsilon: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function, $\Upsilon' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0,1]$. If $|\Upsilon'|^q$ is trigonometrically P-function on $[a,b]$, $q > 1$, then the following inequality holds*

$$|I(a,b;\theta;\lambda)| \qquad (2.12)$$

$$\leq (b-a)\left[\theta^2 - \theta + \frac{1}{2}\right]^{1-\frac{1}{q}}\left(\frac{8}{\pi^2} + \frac{2}{\pi}\right.$$

$$\left. - \frac{8}{\pi^2}\left(sin\frac{\pi\theta}{2} + cos\frac{\pi\theta}{2}\right)\right)^{1/q}$$

$$\{\lambda^2[|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}$$

$$+ (1-\lambda)^2[|\Upsilon'(b)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}\}.$$

Proof. We proceed similarly as in the proof Theorem 2. Since $|\Upsilon'|^q$ is trigonometrically $P-$function on $[a,b]$ and using the power mean inequality, we get

$$|I(a,b;\theta;\lambda)| \leq (b-a) \qquad (2.13)$$

$$\left[\lambda^2\left(\int_0^1 |t-\theta|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-\theta| \right.\right.$$

$$\left. |\Upsilon'(ta + (1-t)[(1-\lambda)a + \lambda b])|^q dt\right)^{1/q}$$

$$(1-\lambda)^2\left(\int_0^1 |t-\theta|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-\theta| \right.$$

$$\left.\left. |\Upsilon'(tb + (1-t)[(1-\lambda)a + \lambda b])|^q dt\right)^{1/q}\right]$$

$$\leq (b-a)\left(\theta^2 - \theta + \frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{8}{\pi^2} + \frac{2}{\pi}\right.$$

$$\left. - \frac{8}{\pi^2}\left(sin\frac{\pi\theta}{2} + cos\frac{\pi\theta}{2}\right)\right)^{1/q}$$

$$\{\lambda^2[|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}$$

$$(1-\lambda)^2[|\Upsilon'(b)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}\}.$$

This completes the proof.

Remark 3. *If it is taken $sin\frac{\pi\theta}{2} + cos\frac{\pi\theta}{2} \geq 1$, $\theta \in [0,1]$ in the inequality (2.12), we get*

$$|I(a,b;\theta;\lambda)| \qquad (2.14)$$

$$\leq (b-a)\left(\theta^2 - \theta + \frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2}{\pi}\right)^{1/q}$$

$$\left\{\lambda^2[|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}\right.$$

$$\left. + (1-\lambda)^2[|\Upsilon'(b)| + |\Upsilon'(A_\lambda)|^q]^{1/q}\right\}.$$

Corollary 13. *When θ is taken as 0 in Theorem 3, then we get generalised trapezoidal-type inequality as follows:*

$$\left|\lambda\Upsilon(a) + (1-\lambda)\Upsilon(b) - \frac{1}{b-a}\int_a^b \Upsilon(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left\{\lambda^2[|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}\right.$$

$$\left. + (1-\lambda)^2[|\Upsilon'(b)|^q + |\Upsilon'((A_\lambda)|^q]^{1/q}\right\}.$$

Corollary 14. *When θ is taken as 1 in Theorem 3, then we get generalised midpoint-type inequality as follows:*

$$\left|\Upsilon((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b \Upsilon(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left\{\lambda^2[|\Upsilon'(a)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}\right.$$

$$\left. + (1-\lambda)^2[|\Upsilon'(b)|^q + |\Upsilon'(A_\lambda)|^q]^{1/q}\right\}.$$

Corollary 15. *If θ is taken as 1 and $|\Upsilon'(x)| \leq M$, $x \in [a,b]$ in Theorem 3, then we get the following Ostrowski-type inequality*

$$\left|\Upsilon(x) - \frac{1}{b-a}\int_a^b \Upsilon(u)du\right|$$

$$\leq \frac{M}{2}\left(\frac{8}{\pi}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right]. \qquad (2.15)$$

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1 - \lambda_x = \frac{b-x}{b-a}$. Therefore for each $x \in [a,b]$, from the inequality (2.12) we obtain the inequality (2.15).

Corollary 16. *When θ, λ are taken as 0, $\frac{1}{2}$, respectively in Theorem 3, then we trapezoid-type inequality as follows:*

$$\left|\frac{\Upsilon(a) + \Upsilon(b)}{2} - \frac{1}{b-a}\int_a^b \Upsilon(x)dx\right|$$

$$\leq \frac{b-a}{8}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left\{\left[|\Upsilon'(a)|^q + \left|\Upsilon'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$

$$\left. + \left[|\Upsilon'(b)|^q + \left|\Upsilon'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}.$$

Corollary 17. *When θ, λ are taken as 1, $\frac{1}{2}$, respectively in Theorem 3, then we get midpoint-type inequality as follows:*

$$\left|\Upsilon\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \Upsilon(x)dx\right| \leq \frac{b-a}{8}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}$$

$$\left\{\left[|\Upsilon'(a)|^q + \left|\Upsilon'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.$$

$$\left. + \left[|\Upsilon'(b)|^q + \left|\Upsilon'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 18. *When θ, λ are taken as $\frac{2}{3}$, $\frac{1}{2}$, respectively in Theorem 3, then we get Simson-type inequality as follows:*

$$\left| \frac{1}{6} \left[\Upsilon(a) + 4\Upsilon\left(\frac{a+b}{2}\right) + \Upsilon(b) \right] \right.$$

$$\left. - \frac{1}{b-a} \int_a^b \Upsilon(x)dx \right|$$

$$\leq \frac{b-a}{4} \left(\frac{5}{18}\right)^{1-\frac{1}{q}} \left[\frac{4(1-\sqrt{3})}{\pi^2} + \frac{2}{\pi} \right]^{\frac{1}{q}}$$

$$\left\{ \left[|\Upsilon'(a)|^q + \left|\Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[|\Upsilon'(b)|^q + \left|\Upsilon'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right\}.$$

3. CONCLUSION

In this study, we applied the trapezoidal, midpoint, Ostrowski, and Simpson type inequalities for Trigonometrically *P*-function by using a general lemma given by İ. İşcan [10]. As a result, we obtain integral inequalities of type trapezoidal, midpoint, Ostrowski and Simpson for Trigonometrically P-function. Our results can be applied to different types of convexity.

Research and Publication Ethics

This paper has been prepared within the scope of international research and publication ethics**.**

Ethics Committee Approval

This paper does not require any ethics committee permission or special permission.

Conflict of Interests

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this paper.

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