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Modified Quasi Boundary Value method for inverse source problem of the bi-parabolic equation

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Abstract

In this study, we study an inverse source problem of the bi-parabolic equation. The problem is severely non-well-posed in the sense of Hadamard, the problem is called well-posed if it satisfies three conditions, such as the existence, the uniqueness, and the stability of the solution. If one of the these properties is not satisfied, the problem is called is non well-posed (ill-posed). According to our research experience, the stability properties of the sought solution are most often violated. Therefore, a regularization method is required. Here, we apply a Modified Quasi Boundary Method to deal with the inverse source problem. Base on this method, we give a regularized solution and we show that the regularized solution satisfies the conditions of the well-posed problem in the sense of Hadarmad. In addition, we present the estimation between the regularized solution and the sought solution by using a priori regularization parameter choice rule.

Keywords: Fractional diffusion equation; Inverse problem; inverse source problem; Regularization.
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1. Introduction

Bi-parabolic equations are frequently used to demonstrate different evolutionary processes in natural sciences, especially describe the unique highlights of the elements of deformed water-saturated porous environments during their filtration fusion the load applied [1]-[3]. For physical motivation and other models,

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the interested reader is referred to [5]-[8, 24, 25] for more details. Let Ω be a bounded domain in $\mathbb{R}$ with the sufficiently smooth boundary $\partial\Omega$. We consider an unknown source issue of deciding the space-subordinate source term $f(x)$ for the accompanying bi-parabolic equation

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = \varphi(t)f(x), & (x,t) \in \Omega \times (0,T), \\ u_t(x,0) = 0, & (x) \in \partial\Omega, t \in (0,T], \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, & x \in \Omega, \\ u(x,0) = 0, & x \in \Omega, \\ u(x,T) = g(x), & x \in \Omega. \end{cases} \tag{1}$$

By the definition of Hadamard [4] about a problem is called well-posed if it satisfies: the existence, the uniqueness, and the stability of the solution. This implies that if one of the three properties is not satisfied, the problem is called is non well-posed. According to our research experience, the stability properties of the sought solution are most often violated. Therefore, to overcome this difficulty, a regularization method is required.

There are many methods to regularized the biparabolic problem. Until recently, to our best knowledge, we have found some research results of the authors as follows: In [9], Tuan and his group consider an inverse initial problem for a biparabolic equation. They apply a filter method for case linear nonhomogeneous problem and Fourier truncation method for the nonlinear bi-parabolic problem. In [10], Tuan et. al surveyed the problem (1) by the Tikhonov method and they show information about the convergent rate between the sought solution and the regularized solution for both the a-priori and the a-posteriori parameter choice rules. In [14], Tuan and his group provided an impressive result of the final value problem for a biparabolic problem with statistical discrete data. In [16], by applying the iteration method, Abdelghani Lakhdari and Nadjib Boussetila give some other convergent rates under *a-priori* bound assumptions on the sought solution.

Besides the method already mentioned above. The quasi-boundary value method is also a commonly used method for examining the ill-posed problem. This method introduced and developed by Showalter, see [11], [12]. It is additionally normally used to tackle some not well presented issues for other equations; for more details, see [13], [18]. In this study, we propose a modified version of quasi-boundary value method to solve the inverse source problem (1), the basic main idea is to use $u^{\epsilon}_{\gamma(\epsilon)}(x,T) = g^{\epsilon}(x) + \gamma(\epsilon)(\mathcal{B}f^{\epsilon}_{\gamma(\epsilon)})(x)$ instead of $u(x,T) = g(x)$. It has been appeared for the modified quasi-boundary value method to manage the unknown source issue (1) that the best convergence rate is $O(\epsilon^{\frac{1}{4}})$ under an a-priori choice of the regularization parameter, inwhich ϵ is the noise level.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we show information about the ill-posedness of problem (1). Next, we propose a regularized solution by the MQBV method and investigate the error estimates are obtained under the a priori parameter choice rule. Finally, we give in Section 4.

2. Preliminaries

Let $\mathcal{K}$ be a real Hilbert space, and let $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset \mathcal{K} \to \mathcal{K}$ be a linear, positive-definite, self-adjoint operator with compact inverse on $\mathcal{K}$. $\mathcal{B}$ has an orthonormal basis of eigenvectors $\phi_k \subset \mathcal{K}$ with real eigenvalues $\xi_k \in \mathbb{R}$.

$$\mathcal{B}\phi_k(x) = \xi_k \phi_k(x), \quad k \in \mathbb{N}, \quad \langle \phi_k, \phi_l \rangle = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}$$
$$\text{and } 0 < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_k \quad \text{with} \quad \xi_k \to \infty \quad \text{for} \quad k \to \infty, \tag{2}$$

and the corresponding eigenelements ϕ_k which form an orthonormal basis in $\mathcal{K}$.

Definition 2.1. *(see [20]) The Hilbert scale space $\mathcal{H}^{2\zeta}(\Omega)$, $(\zeta > 0)$ defined by*

$$\mathcal{H}^{2\zeta}(\Omega) := \Big\{ f \in \mathcal{L}^2(\Omega) : \sum_{k=1}^{\infty} \big(1+\xi_k^2\big)^{2\zeta} \big\langle f, \phi_k \big\rangle^2_{\mathcal{L}^2(\Omega)} \leq \infty \Big\}, \tag{3}$$

with the norm

$$\|f\|^2_{\mathcal{H}^{2\zeta}(\Omega)} = \sum_{k=1}^{\infty} \big(1+\xi_k^2\big)^{2\zeta} \big|\big\langle f, \phi_k \big\rangle_{\mathcal{L}^2(\Omega)}\big|^2 \leq \infty. \tag{4}$$

Lemma 2.2. *Let $\varphi_0, \varphi_1 \geq 0$ satisfy $\varphi_0 \leq |\varphi(t)| \leq \varphi_1$, $\forall t \in [0,T]$, let choose $\epsilon \in \Big(0, \dfrac{\varphi_0}{2}\Big)$, by denoting $\mathcal{A}(\varphi_0, \varphi_1) = \varphi_1 + \dfrac{\varphi_0}{2}$, we get*

$$\frac{\varphi_0}{2} \leq \big|\varphi^{\epsilon}(t)\big| \leq \mathcal{A}(\varphi_0, \varphi_1). \tag{5}$$

Proof. This proof can be found at [10]. □

Lemma 2.3. *Let $\xi_k > \xi_1 > 0$, $\forall k \geq 1$ and $r \in [0,T], \forall t \in [0,T]$, we obtain*

$$\begin{aligned}
&a)\quad \int_0^T e^{-\xi_k(T-r)}(T-r)dr = \xi_k^{-2}\big(1-(1+T\xi_k)e^{-\xi_k T}\big),\\
&b)\quad \frac{1}{1+\xi_k^2} \leq \max\Big\{\frac{3}{T^2}, 1\Big\} \frac{\big(1-(1+T\xi_k)e^{-\xi_k T}\big)}{\xi_k^2},\\
&c)\quad 0 < \frac{\big(1-(1+t\xi_k)e^{-\xi_k t}\big)}{\xi_k^2} < T^2.
\end{aligned} \tag{6}$$

Proof. The proof was proved in [17]. □

3. Main Results

3.1. Uniqueness, ill-posedness and a conditional stability for the unknown source (1)

Taking the inner product of both sides of (1) with $\phi_k(x)$, it gives

$$\begin{cases}
\dfrac{d^2}{dt^2}\big\langle u(t), \phi_k\big\rangle + 2\xi_k \dfrac{d}{dt}\big\langle u(t), \phi_k\big\rangle + \xi_k^2\big\langle u(t), \phi_k\big\rangle = \varphi(t)\big\langle f(x), \phi_k\big\rangle, \quad t \in (0,T),\\
\big\langle u(0), \phi_k\big\rangle = 0, \dfrac{d}{dt}\big\langle u(0), \phi_k\big\rangle = 0,\\
\big\langle u(T), \phi_k\big\rangle = \big\langle g(x), \phi_k\big\rangle.
\end{cases} \tag{7}$$

Using the Lagrange constant variable method, with $u_k(t) = \big\langle u(.,t), \phi_k\big\rangle$, $f_k = \big\langle f(x), \phi_k(x)\big\rangle$, $u_k(0) = \big\langle u(.,0), \phi_k(x)\big\rangle = 0$ and $g_k = \big\langle g(x), \phi_k(x)\big\rangle$.

We obtain

$$\begin{aligned}
u_k(t) = t\, &\Big(\int_0^t e^{-\xi_k(t-r)}\varphi(r)dr\Big)\big\langle f(x), \phi_k(x)\big\rangle\\
&- \Big(\int_0^t e^{-\xi_k(t-r)} r\varphi(r)dr\Big)\big\langle f(x), \phi_k(x)\big\rangle.
\end{aligned} \tag{8}$$

From (8), by letting $t = T$ which leads to

$$\begin{aligned}\big\langle g(x), \phi_k(x)\big\rangle = T\ \Big(\int_0^T e^{-\xi_k(T-r)}\varphi(r)dr\Big)\big\langle f(x), \phi_k(x)\big\rangle \\ - \Big(\int_0^T e^{-\xi_k(T-r)} r\varphi(r)dr\Big)\big\langle f(x), \phi_k(x)\big\rangle.\end{aligned} \tag{9}$$

A simple transformation gives

$$\big\langle f(x), \phi_k(x)\big\rangle = \frac{\big\langle g(x), \phi_k(x)\big\rangle}{\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}, \tag{10}$$

it is shown that

$$f(x) = \sum_{k=1}^{\infty} \frac{\big\langle g(x), \phi_k(x)\big\rangle \phi_k(x)}{\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}. \tag{11}$$

Defining a linear operator : $\mathcal{K} : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ as follows.

$$\mathcal{K}f(x) = \sum_{k=1}^{\infty} \big\langle f(x), \phi_k(x)\big\rangle \int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\ \phi_k(x) = \int_\Omega q(x,\zeta) f(\zeta) d\zeta, \tag{12}$$

whereby $q(x,\zeta) = \sum_{k=1}^{\infty} \int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\ \phi_k(x)\phi_n(\zeta)$. Due to, which leads to $q(x,\zeta) = q(\zeta,x)$ and $\mathcal{K}$ is self-adjoint. Next, we go to prove the compactness of the operator. we define $\mathcal{K}_j$

$$\mathcal{K}_j f(x) = \sum_{k=1}^{j} \big\langle f(x), \phi_k(x)\big\rangle \int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\ \phi_k(x). \tag{13}$$

From (12), (13) and using the Lemma 2.3 Part a), we know that

$$\big\|\mathcal{K}_j f - \mathcal{K}f\big\|^2_{\mathcal{L}^2(\Omega)} \le \varphi_1^2 \sum_{k=j+1}^{\infty} \frac{1}{\xi_k^2} \big|\big\langle f, \phi_k\big\rangle\big|^2 \le \frac{\varphi_1^2}{\xi_j^2}\big\|f\big\|^2_{\mathcal{L}^2(\Omega)}. \tag{14}$$

Therefore, $\|\mathcal{K}_j - \mathcal{K}\|_{\mathcal{L}^2(\Omega)} \longmapsto 0$ when $j \to \infty$. $\mathcal{K}$ is also a compact operator. The SVsD for the self-adjoint are

$$\omega_k = \int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr, \tag{15}$$

and corresponding eigenvectors are ϕ_k. From (12), the problem of finding the source function can be rewritten as an operator equation

$$(\mathcal{K}f)(x) = g(x). \tag{16}$$

3.2. The non well-posed problem (1)

Problem (1) is well known to be severely ill-posed. For making the purpose of the ill-posedness of problem (1), illustrative example will be used. By choosing $\widehat{g}_k$ be as follows $\widetilde{g}_k := \xi_k^{-1/2}\phi_k$. If we choose $g = 0$ then from (11), we will have $f = 0$. In here, we have

$$\|\widetilde{g}_k - g\|_{\mathcal{L}^2(\Omega)} = \|\xi_k^{-1/2}\phi_k(\cdot)\|_{\mathcal{L}^2(\Omega)} = \xi_k^{-1/2}. \tag{17}$$

From (17), we get

$$\lim_{k\to+\infty} \|\widetilde{g}_k - g\|_{\mathcal{L}^2(\Omega)} = \lim_{k\to+\infty} \xi_k^{-1/2} = 0. \tag{18}$$

With the final condition $\widetilde{g}_k$, then we get

$$\begin{aligned}
\widetilde{f}_k(x) &= \sum_{k=1}^{\infty} \frac{\big\langle \widetilde{g}_k(x), \phi_k(x)\big\rangle \phi_k(x)}{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr} \\
&= \sum_{k=1}^{\infty} \frac{\big\langle \xi_k^{-1/2}\phi_k(x), \phi_k(x)\big\rangle \phi_k(x)}{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr} \\
&= \frac{\xi_k^{-1/2}\phi_k(x)}{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}.
\end{aligned} \tag{19}$$

We get $\|\widetilde{f}_k(\cdot) - f(\cdot)\|_{\mathcal{L}^2(\Omega)}$ as follows

$$\begin{aligned}
\big\|\widetilde{f}_k(\cdot) - f(\cdot)\big\|_{\mathcal{L}^2(\Omega)} &= \left\| \frac{\xi_k^{-1/2}\ \phi_k(\cdot)}{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr} \right\|_{\mathcal{L}^2(\Omega)} \\
&= \frac{\xi_k^{-1/2}}{\int\limits_0^T e^{-\xi_k(T-\tau)}(T-\tau)\varphi(\tau)d\tau}.
\end{aligned} \tag{20}$$

From (20), we have to estimate

$$\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr \le \varphi_1 \int_0^T e^{-\xi_k(T-r)}(T-r)dr \le \frac{\varphi_1 T}{\xi_k}\big(e^{-\xi_k T} + 1\big). \tag{21}$$

Combining (20) and (21), we know that

$$\big\|\widetilde{f}_k(\cdot) - f(\cdot)\big\|_{\mathcal{L}^2(\Omega)} \ge \frac{\xi_k^{1/2}}{\varphi_1 T\big(e^{-\xi_k T} + 1\big)}. \tag{22}$$

From (22) leads to

$$\lim_{k\to+\infty} \big\|\widetilde{f}_k(\cdot) - f(\cdot)\big\|_{\mathcal{L}^2(\Omega)} > \lim_{k\to+\infty} \frac{\xi_k^{1/2}}{\varphi_1 T\big(e^{-\xi_k T} + 1\big)} = +\infty. \tag{23}$$

Combining (18) and (23), we can conclude that problem (1) is ill-posed in the Hadamard sense in $\mathcal{L}^2(\Omega)$-norm.

3.3. Conditional stability of source term f

In this section, we present restrictive steadiness by the accompanying hypothesis.

Theorem 3.1. *Since* $\|f\|_{\mathcal{H}^{2m}(\Omega)} \le E$ *for* $E > 0$ *then the norm of* $\big\|f\big\|_{\mathcal{L}^2(\Omega)}$ *is well defined*

$$\|f\|_{\mathcal{L}^2(\Omega)} \le \overline{C}(m,E)\|g\|^{\frac{m}{m+1}}_{\mathcal{L}^2(\Omega)},$$

whereby

$$\overline{C}(m,E) = \frac{E^{\frac{1}{m+1}}}{|\varphi_0|^{\frac{m}{m+1}}\big|1-(1+T\xi_1)e^{-\xi_1 T}\big|^{\frac{m}{m+1}}}. \tag{24}$$

Proof. From (11) and Hölder inequality, we know that:

$$\begin{aligned}
\|f\|^2_{\mathcal{L}^2(\Omega)} &= \sum_{k=1}^{\infty}\left|\frac{\langle g,\phi_k\rangle}{\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}\right|^2 \\
&= \sum_{k=1}^{\infty}\frac{\big|\langle g,\phi_k\rangle\big|^{\frac{2}{m+1}}\big|\langle g,\phi_k\rangle\big|^{\frac{2m}{m+1}}}{\left|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\right|^2} \\
&\le \left(\sum_{k=1}^{\infty}\frac{\big|\langle g,\phi_k\rangle\big|^2}{\left|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\right|^{2m+2}}\right)^{\frac{1}{m+1}}\left(\sum_{k=1}^{\infty}\big|\langle g,\phi_k\rangle\big|^2\right)^{\frac{m}{m+1}} \\
&\le \left(\sum_{k=1}^{\infty}\frac{\Big|\langle f,\phi_k\rangle\Big|^2}{\left|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\right|^{2m}}\right)^{\frac{1}{m+1}}\|g\|^{\frac{2m}{m+1}}_{\mathcal{L}^2(\Omega)}.
\end{aligned} \tag{25}$$

Using the inequality $\left|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\right|^m \ge |\varphi_0|^{2m}\left(\frac{\big|1-(1+T\xi_k)e^{-\xi_k T}\big|}{\xi_k^2}\right)^{2m}$ and using the Lemma 2.3 Part b), it gives

$$\sum_{k=1}^{\infty}\frac{\big|\langle f,\phi_k\rangle\big|^2}{\left|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\right|^{2m}} \le \left(\frac{\big(\max\big\{\frac{3}{T^2},1\big\}\big)}{\varphi_0}\right)^{2m}\|f\|^2_{\mathcal{H}^{2m}(\Omega)}. \tag{26}$$

Combining (25) and (26), we get

$$\|f\|^2_{\mathcal{L}^2(\Omega)} \le \left(\frac{\big(\max\big\{\frac{3}{T^2},1\big\}\big)}{\varphi_0}\right)^{\frac{2m}{m+1}}\|f\|^{\frac{2}{m+1}}_{\mathcal{H}^{2m}(\Omega)}\|g\|^{\frac{2m}{m+1}}_{\mathcal{L}^2(\Omega)}. \tag{27}$$

□

3.4. Modified Quasi Boundary Value regularization method and convergence rates

Let $\big\{u^{\epsilon}_{\gamma(\epsilon)}(x,t), f^{\epsilon}_{\gamma(\epsilon)}(x)\big\}$ be the solution of the following regularized problem as follows:

$$\begin{cases}
u^{\epsilon}_{tt,\gamma(\epsilon)}(x,t) + 2\Delta u^{\epsilon}_{t,\gamma(\epsilon)}(x,t) + \Delta^2 u^{\epsilon}_{\gamma(\epsilon)}(x,t) = \varphi^{\epsilon}(t)f^{\epsilon}_{\gamma(\epsilon)}(x), & (x,t)\in\Omega\times(0,T), \\
u^{\epsilon}_{t,\gamma(\epsilon)}(x,0) = 0, & (x)\in\partial\Omega, t\in(0,T], \\
u^{\epsilon}_{\gamma(\epsilon)}|_{\partial\Omega} = \Delta u^{\epsilon}_{\gamma(\epsilon)}|_{\partial\Omega} = 0, & x\in\Omega, \\
u^{\epsilon}_{\gamma(\epsilon)}(x,0) = 0, & x\in\Omega, \\
u^{\epsilon}_{\gamma(\epsilon)}(x,T) = g^{\epsilon}(x) + \gamma(\epsilon)(\mathcal{B}f^{\epsilon}_{\gamma(\epsilon)})(x), & x\in\overline{\Omega}
\end{cases} \tag{28}$$

If the observed data $(\varphi^\epsilon(t), g^\epsilon(x))$ of $(\varphi(t), g(x))$ with a noise level of ϵ and satisfied

$$\big\|g(\cdot) - g^\epsilon(\cdot)\big\|_{\mathcal{L}^2(\Omega)} \le \epsilon, \quad \|\varphi^\epsilon - \varphi\|_{\mathcal{L}^\infty(0,T)} \le \epsilon, \tag{29}$$

then we can present a regularized solution as follows

$$f^\epsilon_{\gamma(\epsilon)}(x) = \sum_{k=1}^{\infty} \frac{\big\langle g^\epsilon(x), \phi_k(x)\big\rangle \phi_k(x)}{\gamma(\epsilon)\xi_k + \int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi^\epsilon(r)dr}. \tag{30}$$

and

$$f_{\gamma(\epsilon)}(x) = \sum_{k=1}^{\infty} \frac{\big\langle g(x), \phi_k(x)\big\rangle \phi_k(x)}{\gamma(\epsilon)\xi_k + \int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}. \tag{31}$$

Theorem 3.2. *Suppose that* $\|g^\epsilon(\cdot) - g(\cdot)\|_{\mathcal{L}^2(\Omega)} \le \epsilon$ *and* $\|\varphi(\cdot) - \varphi^\epsilon(\cdot)\|_{\mathcal{L}^\infty(0,T)} \le \epsilon$*, and assume that* $\|f\|_{\mathcal{H}^{2m}(\Omega)} \le E$ *for* $m > 0$*, then we have the following estimate*

i) If $0 < m \le \dfrac{3}{4}$*, and choosing* $\gamma(\epsilon) = \left(\dfrac{\epsilon}{E}\right)^{\frac{3}{3+4m}}$*, then the following estimate*

$$\big\|f(\cdot) - f^\epsilon_{\gamma(\epsilon)}(\cdot)\big\|_{\mathcal{L}^2(\Omega)} \text{ is of order } \epsilon^{\frac{2m}{3+4m}}. \tag{32}$$

ii) If $m > \dfrac{3}{4}$*, by choosing* $\gamma(\epsilon) = \left(\dfrac{\epsilon}{E}\right)^{\frac{1}{2}}$*, then the following estimate*

$$\big\|f(\cdot) - f^\epsilon_{\gamma(\epsilon)}(\cdot)\big\|_{\mathcal{L}^2(\Omega)} \text{ is of order } \epsilon^{\frac{1}{4}}. \tag{33}$$

Proof. The proof of based on the concept of triangle inequality, we get

$$\|f(\cdot) - f^\epsilon_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)} \le \|f(\cdot) - f_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)} + \|f_{\gamma(\epsilon)}(\cdot) - f^\epsilon_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)}. \tag{34}$$

First of all, we notice that

$$\begin{aligned} & f_{\gamma(\epsilon)}(x) - f^\epsilon_{\gamma(\epsilon)}(x) \\ &= \underbrace{\sum_{k=1}^{\infty} \frac{\big\langle g(x), \phi_k(x)\big\rangle \phi_k(x)}{\gamma(\epsilon)\xi_k + \int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr} - \frac{\big\langle g(x), \phi_k(x)\big\rangle \phi_k(x)}{\gamma(\epsilon)\xi_k + \int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi^\epsilon(r)dr}}_{:=\mathcal{P}_1} \\ &+ \underbrace{\sum_{k=1}^{\infty} \frac{\big\langle g(x), \phi_k(x)\big\rangle \phi_k(x)}{\gamma(\epsilon)\xi_k + \int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi^\epsilon(r)dr} - \frac{\big\langle g^\epsilon(x), \phi_k(x)\big\rangle \phi_k(x)}{\gamma(\epsilon)\xi_k + \int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi^\epsilon(r)dr}}_{:=\mathcal{P}_2} \end{aligned} \tag{35}$$

We need estimate $\big\|f_{\gamma(\epsilon)}(\cdot) - f^\epsilon_{\gamma(\epsilon)}(\cdot)\big\|_{\mathcal{L}^2(\Omega)}$ only consider two steps:

Step 1: Let us evaluate $\|\mathcal{P}_1\|^2_{\mathcal{L}^2(\Omega)}$.

$$\begin{aligned}
&\|\mathcal{P}_1\|^2_{\mathcal{L}^2(\Omega)} \\
&\le \sum_{k=1}^{\infty}\Bigg(\frac{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)(\varphi(r)-\varphi^\epsilon(r))dr\ \langle g,\phi_k\rangle\phi_k}{\big[\gamma(\epsilon)\xi_k+\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi^\epsilon(r)dr\big]\times\big[\gamma(\epsilon)\xi_k+\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\big]}\Bigg)^2 \\
&\le \sum_{k=1}^{\infty}\Bigg(\frac{\|\varphi-\varphi^\epsilon\|_{\mathcal{L}^\infty(0,T)}\int\limits_0^T e^{-\xi_k(T-r)}(T-r)dr}{\gamma(\epsilon)\xi_k}\Bigg)^2\Bigg|\frac{\langle g,\phi_k\rangle\phi_k(\cdot)}{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}\Bigg|^2
\end{aligned}\tag{36}$$

Using the Lemma 2.3 Part a) and Part c), we can know that

$$\begin{aligned}
&\|\mathcal{P}_1\|^2_{\mathcal{L}^2(\Omega)} \\
&\le \Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)^2\sum_{k=1}^{\infty}\frac{\big(1-(1+T\xi_k)e^{-\xi_kT}\big)^2}{\xi_k^6}\Bigg|\frac{\langle g,\phi_k\rangle\phi_k}{\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}\Bigg|^2 \\
&\le \Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)^2T^4\sum_{k=1}^{\infty}\frac{1}{\xi_k^{2+4m}}\big(1+\xi_k^2\big)^{2m}\big|\langle f,\phi_k\rangle\big|^2 \\
&\le \Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)^2T^4\|f\|^2_{\mathcal{H}^{2m}(\Omega)}\sum_{k=1}^{\infty}\frac{1}{\xi_k^{2+4m}}.
\end{aligned}\tag{37}$$

Step 2: Estimate for $\|\mathcal{P}_2\|^2_{\mathcal{L}^2(\Omega)}$.

$$\begin{aligned}
\|\mathcal{P}_2\|_{\mathcal{L}^2(\Omega)} &\le \sum_{k=1}^{\infty}\Bigg(\frac{\big|\langle g-g^\epsilon,\phi_k\rangle\big|^2}{\gamma(\epsilon)\xi_k+\int\limits_0^T e^{-\xi_k(T-r)}(T-r)\varphi^\epsilon(r)dr}\Bigg)^2 \\
&\le \Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)^2\sum_{k=1}^{\infty}\frac{1}{\xi_k^2}.
\end{aligned}\tag{38}$$

If $\xi_k\ge\widetilde{C}k^{\frac{2}{d}}$ for $k\in\mathbb{N}$, where $\widetilde{C}$ do not depend of k, see [19]. For $0<d<4$ and denoting $\widetilde{M_2}:=\frac{1}{\widetilde{C}^2}\sum\limits_{k=1}^{\infty}\frac{1}{k^{\frac{4}{d}}}$. Next, if $d>0$ and $m>\frac{1}{2}\big(\frac{d}{4}-1\big)$, then by letting $\widetilde{M_1}=\frac{1}{\widetilde{C}^{2+4m}}\sum\limits_{k=1}^{\infty}\frac{1}{k^{\frac{4+8m}{d}}}$. Therefore, form (37) and (38), it gives

$$\|\mathcal{P}_1\|_{\mathcal{L}^2(\Omega)}\le\Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)T^2E\sqrt{\widetilde{M_1}},\ \text{and}\ \|\mathcal{P}_2\|_{\mathcal{L}^2(\Omega)}\le\Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)\sqrt{\widetilde{M_2}}.\tag{39}$$

Combining (35), (39) and (38), we obtain

$$\big\|f_{\gamma(\epsilon)}(\cdot)-f^\epsilon_{\gamma(\epsilon)}(\cdot)\big\|_{\mathcal{L}^2(\Omega)}\le\Big(\frac{\epsilon}{\gamma(\epsilon)}\Big)\Big(T^2E\sqrt{\widetilde{M_1}}+\sqrt{\widetilde{M_2}}\Big).\tag{40}$$

The proof is completed by showing that (11), we notice that

$$
\begin{aligned}
&\|f(\cdot)-f_{\gamma(\epsilon)}(\cdot)\|^2_{\mathcal{L}^2(\Omega)}\\
&\le \sum_{k=1}^{\infty}\Big(\Big|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\Big|^{-1}-\Big|\gamma(\epsilon)\xi_k+\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\Big|^{-1}\Big)^2\\
&\times|\langle g,\phi_k\rangle|^2\\
&\le \sum_{k=1}^{\infty}\frac{[\gamma(\epsilon)\xi_k]^2}{\Big(\gamma(\epsilon)\xi_k+\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\Big)^2}\frac{\big|\langle g,\phi_k\rangle\big|^2}{\big|\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr\big|^2}\\
&\le \sum_{k=1}^{\infty}\frac{\gamma(\epsilon)\xi_k\big(1+\xi_k^2\big)^{-2m}}{4\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}\|f\|^2_{\mathcal{H}^{2m}(\Omega)}\le\sum_{k=1}^{\infty}\big|\mathbb{E}(\xi_k)\big|^2\|f\|^2_{\mathcal{H}^{2m}(\Omega)}. \qquad (41)
\end{aligned}
$$

From (41), we continue to estimate $\mathbb{E}(\xi_k)$. Denoting $\widetilde{Q}(\xi_1,T,\varphi_0)=(4\varphi_0\big|1-(1+T\xi_k)e^{-\xi_kT}\big|)^{-1}$, we obtain

$$
\begin{aligned}
\big|\mathbb{E}(\xi_k)\big|^2&\le\frac{\gamma(\epsilon)\xi_k(1+\xi_k^2)^{-2m}}{4\int_0^T e^{-\xi_k(T-r)}(T-r)\varphi(r)dr}\le\frac{\gamma(\epsilon)\xi_k^{3-4m}}{4\varphi_0\big|1-(1+T\xi_k)e^{-\xi_kT}\big|}\\
&\le\frac{\gamma(\epsilon)\xi_k^{3-4m}}{4\varphi_0\big|1-(1+T\xi_1)e^{-\xi_1T}\big|}\le\gamma(\epsilon)\xi_k^{3-4m}\widetilde{Q}(\xi_1,T,\varphi_0). \qquad (42)
\end{aligned}
$$

It can happened that

<u>Case 1</u>: $m>\frac{3}{4}$, it is easy to see that $\xi_k^{3-4m}\le\xi_1^{3-4m}$ and combining this with (41), we deduce that

$$\big\|f(\cdot)-f_{\gamma(\epsilon)}(\cdot)\big\|_{\mathcal{L}^2(\Omega)}\le[\gamma(\epsilon)]^{\frac12}\ \xi_1^{\frac32-2m}\ E\ \big[\widetilde{Q}(\xi_1,T,\varphi_0\big]^{\frac12}. \qquad (43)$$

<u>Case 2</u>: $0<m\le\frac{3}{4}$, with $m\in\Big(0,\frac{3}{4}\Big)$. We rewrite $\mathbb{N}$ by $\mathbb{N}=\mathcal{W}_1\cup\mathcal{W}_2$ whereby

$$\mathcal{W}_1=\Big\{k\in\mathbb{N},\xi_k^{3-4m}\le[\gamma(\epsilon)]^{-\ell}\Big\},\mathcal{W}_2=\Big\{k\in\mathbb{N},\xi_k^{3-4m}>[\gamma(\epsilon)]^{-\ell}\Big\}. \qquad (44)$$

Hence, from (41) and the condition $\|f\|_{\mathcal{H}^{2m}(\Omega)}\le E$, we can find that

$$
\begin{aligned}
\big\|f(\cdot)-f_{\gamma(\epsilon)}(\cdot)\big\|^2_{\mathcal{L}^2(\Omega)}&=\sup_{k\in\mathcal{W}_1}\gamma(\epsilon)\xi_k^{3-4m}\widetilde{Q}(\xi_1,T,\varphi_0)E^2+\sum_{k=1}^{\mathcal{W}_2}\xi_k^{-4m}E^2\\
&\le\widetilde{Q}(\xi_1,T,\varphi_0)E^2[\gamma(\epsilon)]^{1-\ell}+\sup_{k\in\mathcal{W}_2}\xi_k^{-4m}E^2\\
&\le\widetilde{Q}(\xi_1,T,\varphi_0)E^2[\gamma(\epsilon)]^{1-\ell}+[\gamma(\epsilon)]^{\frac{4m\ell}{3-4m}}E^2. \qquad (45)
\end{aligned}
$$

We choice $\ell=1-\frac{4m}{3}$, it gives

$$\big\|f(\cdot)-f_{\gamma(\epsilon)}(\cdot)\big\|^2_{\mathcal{L}^2(\Omega)}\le\widetilde{Q}(\xi_1,T,\varphi_0)E^2[\gamma(\epsilon)]^{\frac{4m}{3}}+[\gamma(\epsilon)]^{\frac{4m}{3}}E^2. \qquad (46)$$

Using the inequality $(a^2+b^2)\le\sqrt{a+b},\forall a,b\ge0$, one has

$$\big\|f(\cdot)-f_{\gamma(\epsilon)}(\cdot)\big\|_{\mathcal{L}^2(\Omega)}\le\big[\gamma(\epsilon)\big]^{\frac{2m}{3}}\Big(\big[\widetilde{Q}(\xi_1,T,\varphi_0)\big]^{\frac12}E+E\Big). \qquad (47)$$

a) If $0 < m \leq \dfrac{3}{4}$, combining (34), (40) and (46), we conclude that

$$\begin{aligned}\|f(\cdot) - f^{\epsilon}_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)} &\leq \frac{\epsilon}{\gamma(\epsilon)}\Big(T^2 E\sqrt{\widetilde{M}_1} + \sqrt{\widetilde{M}_2}\Big) \\ &+ [\gamma(\epsilon)]^{\frac{2m}{3}}\Big(\big[\widetilde{Q}(\xi_1, T, \varphi_0)\big]^{\frac{1}{2}} E + E\Big).\end{aligned} \tag{48}$$

b) If $m > \dfrac{3}{4}$, combining (34), (40) and (43), we conclude that

$$\begin{aligned}\|f(\cdot) - f^{\epsilon}_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)} &\leq \frac{\epsilon}{\gamma(\epsilon)}\Big(T^2 E\sqrt{\widetilde{M}_1} + \sqrt{\widetilde{M}_2}\Big) \\ &+ [\gamma(\epsilon)]^{\frac{1}{2}}\ \xi_1^{\frac{3}{2}-2m}\ E\ \big[\widetilde{Q}(\xi_1, T, \varphi_0\big]^{\frac{1}{2}}.\end{aligned} \tag{49}$$

Therefore, we conclude that

a) If $0 < m \leq \dfrac{3}{4}$, by choosing $\gamma(\epsilon) = \Big(\dfrac{\epsilon}{E}\Big)^{\frac{3}{3+4m}}$, then we get

$$\begin{aligned}\|f(\cdot) - f^{\epsilon}_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)} &\leq \epsilon^{\frac{4m}{3+4m}} E^{\frac{3}{3+4m}}\Big(T^2 E\sqrt{\widetilde{M}_1} + \sqrt{\widetilde{M}_2}\Big) \\ &+ \epsilon^{\frac{2m}{3+4m}} E^{\frac{2m}{3+4m}}\Big(\big[\widetilde{Q}(\xi_1, T, \varphi_0)\big]^{\frac{1}{2}} E + E\Big).\end{aligned} \tag{50}$$

b) If $m > \dfrac{3}{4}$, by choosing $\gamma(\epsilon) = \Big(\dfrac{\epsilon}{E}\Big)^{\frac{1}{2}}$, then we get

$$\begin{aligned}\|f(\cdot) - f^{\epsilon}_{\gamma(\epsilon)}(\cdot)\|_{\mathcal{L}^2(\Omega)} &\leq \epsilon^{\frac{1}{2}} E^{\frac{1}{2}}\Big(T^2 E\sqrt{\widetilde{M}_1} + \sqrt{\widetilde{M}_2}\Big) \\ &+ \epsilon^{\frac{1}{4}} E^{\frac{1}{4}}\Big(\big[\widetilde{Q}(\xi_1, T, \varphi_0)\big]^{\frac{1}{2}} E + E\Big).\end{aligned} \tag{51}$$

This completes the proof of Theorem 3.2. □

4. Conclusion

In this study, we investigate an reconstruct source function $f(x)$ for a bi-parabolic equation. The conditional stability is given in Theorem 3.1. We used a MQBV method for obtaining a regularized solution. Based on an a priori assumption for (11), the error estimate is obtained under an a priori regularization parameter choice rule. The authors is greatly indebted to professor Nguyen Huy Tuan for suggesting the problem and for many stimulating conversations.

References

[1] V.M. Bulavatsky, Biparabolic mathematical model of the filtration consolidation problem, Dopov. NAN Ukrainy, No.**8**, 13–17 (1997).

[2] V.M. Bulavatsky, Mathematical modeling of filtrational consolidation of soil under motion of saline solutions on the basis of biparabolic model, J. Autom. Inform. Sci., 35, No.**8**, 13–22 (2003).

[3] V.M. Bulavatsky and V.V. Skopetsky, Generalized mathematical model of the dynamics of consolidation processes with relaxation, Cybern. Syst. Analysis, 44, No.**5**, 646–654 (2008).

[4] J. Hadamard, Lectures on the Cauchy Problem in Linear Differential Equations, Yale University Press, New Haven, CT, 1923.

[5] K.A. Ames, B. Straughan, Non-Standard and Improperly Posed Problems, Academic Press, New York (1997).

[6] A.S. Carasso, Bochner Subordination logarithmic diffusion equations, and blind deconvolution of Hubble space telescope imagery and other scientific data, SIAM J. Imaging Sci. 3(4), 954–980 (2010).
[7] L.E. Payne, On a proposed model for heat conduction, IMA J. Appl. Math. 71, 590-599 (2006).
[8] L. Wang, X. Zhou, X. Wei, Heat Conduction: Mathematical Models and Analytical Solutions, Springer, Berlin (2008).
[9] Huy Tuan Nguyen, Mokhtar Kirane, Nam Danh Hua Quoc and Van Au Vo, Approximation of an Inverse Initial Problem for a Biparabolic Equation, Mediterranean Journal of Mathematics (2018) 15:18, https://doi.org/10.1007/s00009-017-1053-0.
[10] Danh Hua Quoc Nam, Le Dinh Long, Donal ORegand, Tran Bao Ngoc and Nguyen Huy Tuan, Identification of the right-hand side in a bi-parabolic equation with final data, Applicable Analysis https://doi.org/10.1080/00036811.2020.1775817
[11] R.E. Showalter, The final value problem for evolution equations, J.Math.Anal.Appl., 1974, 47, 563-572.
[12] R.E. Showalter, Quasi-reversibility of first and second order parabolic evolution equations, Improperly Posed Boundary Value Problems. Pitman, London, 1975, 76–84.
[13] N. Boussetila, F. Rebbani, Optimal regularization method for ill-posed Cauchy problems, Electron. J. Differential Equations, 2006, 147, 1-15.
[14] Nguyen Huy Tuan, Tran Ngoc Thach, Hoan Luu Vu Cam and Nguyen Huu Can, On a final value problem for a biparabolic equation with statistical discrete data, Applicable Analysis, DOI: 10.1080/00036811.2020.1723554.
[15] K. Besma, B. Nadjib, and R. Faouzia, A modified quasi-boundary value method for an abstract ill-posed biparabolic problem, Open Mathematics, Research Article, https://doi.org/10.1515/math-2017-0140.
[16] A. Lakadari, N. Boussetila, An iterative regularization method for an abstract ill-posed biparabolic problem, Boundary Value Problem, 2015, 55, 1-17, https://10.1186/s13661-015-0318-4.
[17] F. Zouyed, S. Djemoui, An Iterative Regularization Method for Identifying the Source Term in a Second Order Differential Equation, Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume 2015, 9 pages.
[18] Z. Chen, Y. Lu, Y. Xu, H. Yang, Multi-parameter Tikhonov regularization for linear ill-posed operator equations, J. Comput. Math., 2008, 26, 37 – 55.
[19] R. Courant, D. Hilbert, Methods of mathematical physics. New York (NY): Interscience; 1953.
[20] M.T. Nair, S.V. Pereverzev, U. Tautenhahn, URegularization in Hilbert scales under general smoothing conditions, Inverse Prob. 21(6) (2005), 1851-1869.
[21] V.M. Bulavatsky, Fractional differential analog of biparabolic evolution equation and some its applications, Cybernetics and Systems Analysis, 2016, 52(5), 337–347.
[22] L. Joseph, D.D. Preziosi, Heat Waves, Rev.Mod. Physics, 1989, 41–73.
[23] D. Coltony, M. Pianayand, R. Potthast, A simple method using Morozovs discrepancy principle for solving inverse scattering problems, Inverse Problems, **13** (1997), 1477 – 1493.
[24] Said Abbas, Mouffak Benchohra, Johnny Henderson and Jamal E. Lazreg, *Weak Solutions for a Coupled System of Partial Pettis Hadamard Fractional Integral Equations*, Advances in the Theory of Nonlinear Analysis and its Applications Volume 1, Issue 2, Pages:136-146, Year: 2017, Article Id: 2017:9
[25] Samira Hamani, Wafaa Benhamida, *Johnny Henderson, Boundary Value Problems For Caputo-Hadamard Fractional Differential Equations*, Advances in the Theory of Nonlinear Analysis and its Applications Volume 2, Issue 3, Pages:138-145, Year: 2018, Article Id: 2018:13