Conference Proceeding Science and Technology, 3(1), 2020, 87–90

Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Hermite Operational Matrix for Solving Fractional Differential Equations

Hatice Yalman Koşunalp[1,*] *Mustafa Gülsu*[2]

[1] *Bayburt University, Social Sciences Vacational School, Bayburt, Turkey, ORCİD: 0000-0001-6313-862X*
[2] *Department of Mathematics, Faculty of Science, Mugla Sitki Kocman University, Mugla, Turkey, ORCİD: 0000-0001-6139-0266*
* *Corresponding Author E-mail: hkosunalp@bayburt.edu.tr, mgulsu@mu.edu.tr*

Abstract: This paper aims to solve the fractional differential equations (FDEs) with operational matrix method by Hermite polynomials in the sense of Caputo derivative. For this purpose, we attempt to re-define the FDEs with a set of algebraic equations with initial conditions which simplifies the complete problem. We achieve either exact or approximated solutions by solving these algebraic equations with the proposed method. To indicate the efficiency of the proposed method, various illustrative examples are solved.

Keywords: Caputo derivative, Fractional Integration, Hermite, Operational Matrix.

1 Introduction

Fractional calculus has been successfully exploited to model a wide range of real-world scientific problems using efficient mathematical models [1]-[2]. The most common application areas are cited in [3]-[5]. Fractional differential equations (FDEs) have been recently gaining a significant interest in science and engineering domain due to its lightweight structure to be directly applied to such a practical problem [6]. Many studies have focused on the fast and efficient solution of the FDEs. A popular method for solving the multi-term FDEs is known as operational matrix with many types of orthogonal polynomials. Typical examples with these polynomials are Chebyshev [7], Jacobi [8], Shifted-Jacobi [9], Shifted-Legendre [10] and Bernoulli [11].
In the operational matrix method, the basic idea is to create a set of algebraic equations to solve the FDEs easily. An operational matrix is created from these equations. This method is easy to be implemented and provides efficient solutions. In this paper, the operational matrix of fractional derivative is derived with Hermite polynomials, in order to solve multi-term FDEs with initial conditions. The method is based on the Caputo derivative. The performance of the proposed method is tested via a number of illustrative examples. The main advantage of the method is its high speed which requires only a few number of step for solution. Therefore, the complexity level of the solution is low which makes it practical. We also consider the FDEs with non-polynomials solution making the proposed method more reliable. Another important feature of this work is that there is a big gap in literature for Hermite operational matrix which is fulfilled by this work. The details of the paper is presented in the following sections.

2 Method of the Solution

2.1 Hermite Polynomials Operational Matrix

In order to solve FDEs, we use Hermite operational matrix [12]. Operational matrix is formed by Caputo derivative on Hermite analytical formula such that:

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k (2x)^{i-2k}}{k!(i-2k)!}, \qquad x \in (-\infty, \infty) \tag{1}$$

and by taking the Caputo derivative it will be:

$$D^v H_i(x) = \sum_{k=0}^{\lfloor i/2 \rfloor} \frac{(-1)^k i! D^v (2x)^{i-2k}}{k!(i-2k)!}. \tag{2}$$

and the formula for $D^v x^{i-2k}$ is

$$D^v(x^k) = \frac{\Gamma(k+1) x^{(k-v)}}{\Gamma(k-v+1)} \tag{3}$$

defined with Gamma functions. For $(x)^{(i-2k-v)}$ we use Hermite truncated series function approximation, then it is:

$$x^{i-2k-v} = \sum_{j=0}^{N} c_j H_j(x). \tag{4}$$

From Eq.4 c_j coefficients can be found with integral formula. Finally we obtain the operational matrix formula as below:

$$\Omega_v(i,j) = \sum_{k=0}^{\lfloor \frac{i-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^(k+r) 2^{(i-2k+j-2r)} i! j! \, \frac{\Gamma(i-2k+j-2r+1)}{2}}{(j-2r)! k! r! \Gamma(i-2k+1-v)}, j = 0, 1, ..., N \tag{5}$$

2.2 *Method with Operational Matrix*

The fractional differential equations form we solve is:

$$D^v u(x) = \sum_{i=1}^{k} \gamma_i D^{\beta_j} u(x) + \gamma_{k+1} u(x) + g(x) \tag{6}$$

which is combined with the initial conditions given below

$$u^{(i)}(0) = d_i, i = 0, 1, ..., m-1. \tag{7}$$

where i takes real consecutive constants ranging from 0 to 1 , ... , k, $m-1 < v < m$ and $0 < \beta_1 < \beta_2 < ... < \beta_k < v$ and $g(x)$ is the source function given. For solving the initial value problem (6)-(7), $u(x)$ and $g(x)$ functions approximated by Hermite polynomials as below:

$$u(x) = \sum_{i=0}^{N} c_i H_i(x) = C^T \phi(x) \tag{8}$$

and

$$g(x) = \sum_{i=0}^{N} g_i H_i(x) = G^T \phi(x) \tag{9}$$

These functions matrix forms is used for solving the problem. When each matrix form is substituted in Eq.6 the final form of the equation is

$$\left(C^T \mathbf{D}^{(v)} - C^T \sum_{j=1}^{k} \gamma_j \mathbf{D}^{\beta_j} - \gamma_{k+1} C^T - G^T \right) \phi(x) = 0 \tag{10}$$

obtained. Then by solving this equation by Tau method we have $N-m+1$ equations. The conditions matrix forms is given

$$u^i(0) = C^T D^i(0) \tag{11}$$

and by considering the equations $m+1$ equations obtained. Together with the equations from Eq. 10 and the equations from Eq.11 $N+1$ equations obtained at the end. By solving this equation system the unknown coefficients c_j is obtained. By putting c_j into the solutions form given in Eq. 8 the approximate solution is obtained.

3 Numerical Solutions

Example 1. First example is an initial value problem with these conditions

$$D^2 u(x) + \frac{8}{5} D^{3/2} u(x) - \frac{1}{4} u(x) = \frac{1}{4} x^2 - \frac{1}{4} x - \frac{8}{5} \frac{\sqrt{x}}{\sqrt{\pi}}, \qquad u(0) = 0, u'(0) = 1 \tag{12}$$

This problem is solved exactly with

$$u(x) = x(1-x).$$

The matrix of derivatives are given below:

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}^{(3/2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3.1205 & 2.3081 & 0.3901 \end{pmatrix}$$

By putting them into the Eq.12 we get the unknown coefficients as:

$$\begin{bmatrix} c_0 & c_1 & c_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & -0.25 \end{bmatrix}$$

Then the approximate solution offered in this method and the exact solution will be found the same as

$$u(x) = C^T.\Phi(x) = x(1-x). \tag{13}$$

Example 2. Second example is given as:

$$D^{\alpha}u(x) + u(x) = 0, 1 < \alpha < 2, \qquad u(0) = 1, u'(0) = 0 \tag{14}$$

with conditions and this problem is solved exactly when $\alpha = 1$ and it is

$$u(x) = \sum_{k=0}^{\infty} \frac{-x^{\alpha k}}{\Gamma(\alpha k + 1)}.$$

This example is solved for $N = 4$ and $\alpha = 1.5$ we get the operational matrix as

$$\mathbf{D}^{(3/2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 3.1205 & 2.3081 & 0.3901 & -0.0962 & -0.0244 \\ 2.3081 & 7.0211 & 4.3278 & 0.6826 & -0.1623 \\ -7.4892 & 9.2325 & 14.0422 & 5.7703 & 0.6826 \end{pmatrix}$$

After applying the method proposed we get c_j as:

$$C^T = \begin{bmatrix} 0.5464 & 0.1391 & -0.1692 & 0.0232 & 0.0096 \end{bmatrix}$$

$$u_N(x) = 1 - 1.1376x^2 + 0.1856x^3 + 0.1536x^4 \tag{15}$$

For $\alpha = 1.5$ the

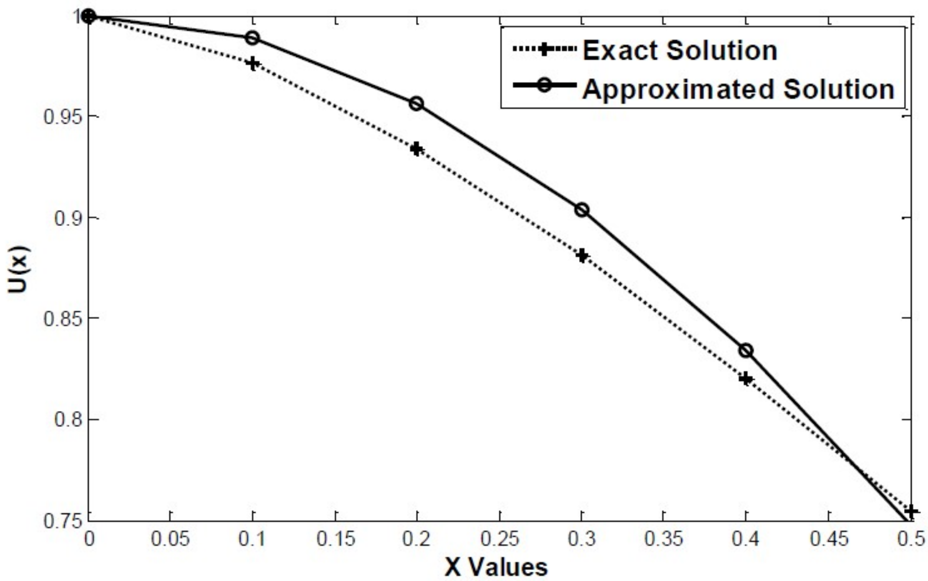

Fig. 1: for $\alpha = 1.5$ the approximate and exact solutions

Example 3. Third example is

$$D^{\frac{3}{2}}u(x) + u(x) = \sigma^{\frac{3}{2}}e^{\sigma x} + 3e^{\sigma x}, \qquad u(0) = 1, u'(0) = \sigma \tag{16}$$

The exact solution of the example is

$$u(x) = e^{\sigma x}$$

The example is solved for $N = 2$ and after some steps of proposed method done, the approximate solution is found as:

$$u(x) = 1 + 0.2x + 0.04x^2 \tag{17}$$

And the figure is given for the approximate and exact solution below:

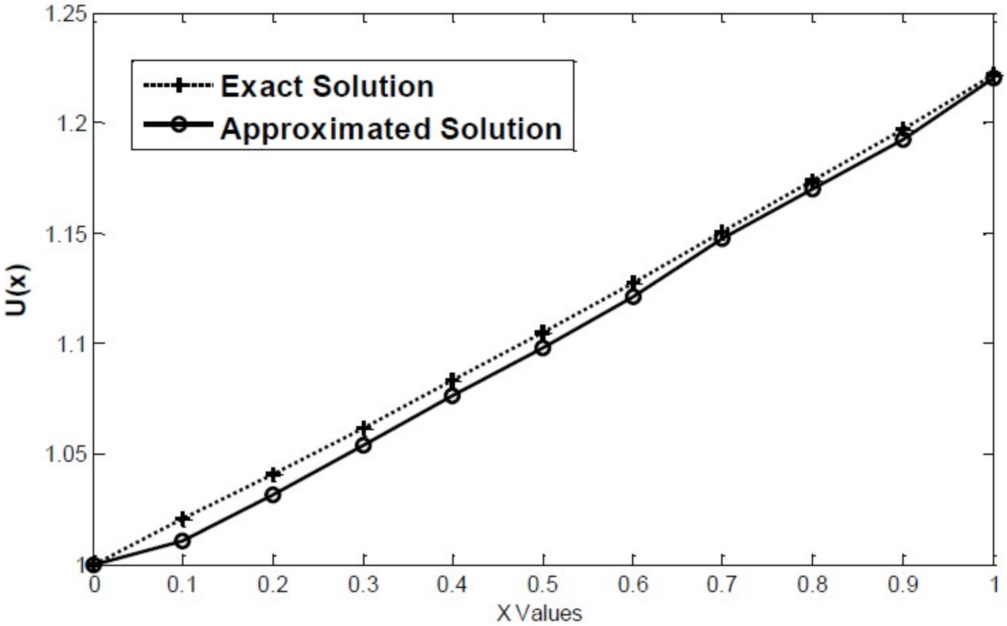

Fig. 2: For $\sigma = 0.2$ the exact and approximate solutions

4 Conclusion

In this paper, an explicit derivation of operational matrix of fractional derivation by Hermite polynomials is presented. It aims to solve fractional differential equations with Caputo sense. Linear form of fractional differential equations is considered which is subject to initial conditions. We simplify the whole problem by converting the fractional differential equations into a group of algebraic equations. By solving these algebraic equations, we achieve exact or approximated solutions. We test the performance of the proposed method by solving some numerical examples, presenting high accurate results.

5 References

1 K.S. Miller, B. Ross, (Eds.), *Introduction to the Fractional Calculus and Fractional Differential Equations,*, John Wiley and Sons, Inc., New York, 1993.
2 K.B. Oldham, J. Spanier, *The Fractional Calculus, Theory and Appilcations of Differentiation and Integration to Arbitrary Order.*, Dover Publication, Mineola, 2006.
3 A. Plonka , J. Spanier, *Recent Developments in dispersive kinetics.*, Progr. React. Kinet. Mech., **25**(2)(2000), 109-127.
4 P. Allegrini, M. Buiatti, P. Grinolini, B.L. West *Fractional Brownian Motion As a Nonstationary Process: Analternative Paradigm for dNA Sequences.*, Phys. Rev. E, **57**(4)(1998), 558-567.
5 J. Bisquert, *Fractional Diffusion in the Multiple-Trapping Regime and Revision of the Equivalence with the Continuous time Random Walk*, Phys. Rev. Lett., **91**(2003).
6 A. A. Kilbas, H. M. Srivastava, *On matrix transformations between some sequence spaces and the hausdorff measure of noncompactness*, Theory and Applications of Fractional Differential Equations, Elsevier, San Diego, 2006.
7 A. H. Bhrawy, A. S. Alofi, *The Operational Matrix of Fractional Integration for Shifted Chebyshev Polynomials*, Appl. Math. Lett., **26**(2013), 25-31.
8 E. H. Doha, A. H. Bhrawy, S. S. Ezz-Eldien, *A New Jacobi Operational Matrix: An Application for Solving Fractional Differential Equations*, Appl. Math. Modell, **36**(2013), 4931-4943.
9 A. H. Bhrawy, M. A. Alghamdi, *A Shifted Jacobi –Gauss-Lobatto Collocation Method for Solving Nonlinear Fractional Langevin Equation*, Bound. Value Probl. **62**(2012).
10 M. H. Akrami, M. H, Atabekzadeh, G. H. Erjaee, *The Operational Matrix of Fractional Integration for Shifted Legendre Polynomials*, Iran. J. Sci. Technol., **37**(4)(2013), 439-444.
11 R. Belgacem, A. Bokhari, A. Amir, *Bernoulli Operational Matrix of Fractional Derivative for Solution of Fractional Differential Equations*, Gen. Lett. Math., **5**(1)(2018), 32-46.
12 F. Dusunceli, E. Celik, *Numerical Solution for High-Order Linear Complex Differential Equations By Hermite Polynomials*, Iğdır Univ. J. Inst. Sci. Tech., **7**(4)(2017), 189-201.