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Research Paper

Some New Approximate Solutions in Closed-Form to Problems of Nanobars

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Abstract

Following recent technological advancements, a great attention has been paid to the mechanical behaviour of structural elements of nanosize. In this study, some solutions to mechanical problems of bars of nanosize are examined using Eringen's two-phase nonlocal elasticity. Assuming the fraction coefficient of nonlocal part of the material is small, a perturbation expansion with respect to it is performed. With this procedure, the original nonlocal problem is broken into a set of local elasticity problems. Solutions to some example problems of nanobars are provided in closed-form for the first time, and commented on. The new solutions provided herein may well serve for benchmark studies, as well as identification of material parameters of nano-sized structural elements, such as carbon nanotubes.

Keywords: Nanobars; nonlocal elasticity; nanomechanics; closed-form solutions; approximate methods.

1. INTRODUCTION

Rational theories on deformable solids stemmed from the corpuscular models, by which the internal structure of the material was accounted for [1-3]. Oversimplifying the relations between the atoms gave rise to conclusions which were inconsistent with experimental results; about the same time continuum models of solids were proposed and widely accepted. Interested readers are kindly referred to [4] for more detailed information about the evolution of continuum models, and the roots of molecular dynamic analysis.

Cauchy's continuum model is a great approximation to the actual physics of the matter, for most of the engineering materials. However, when the smallest internal organization constituting the material has comparable dimensions with respect to the overall size of the structure; or when the waves of frequency of interest are dispersed due to constituent of the material, more enhanced theories are required to reflect better the behavior of the material [5-10]. For this purpose one may resort to methods which use a model of the smallest unit of the internal structure of the material: molecular dynamic simulations at atomic scale [11], or limit analysis for masonry walls, for example [12]. Being accurate, these models are quite time consuming due to discrete modelling of a very high number of degrees of freedom [4]. As a good alternative, continuum models which account for the characteristics of the internal material organization have been developed, which are called nonlocal continuum theories. These theories either introduces additional kinematic descriptors to those of classical theory of elasticity (classified as "implicit") or assumes a convolution-type constitutive equation, cancelling the axiom of locality (classified "explicit") [13-14]. Possible equivalencies and distinctions between these two classes are recently studied in [15-16].

As one of the simplest, yet, most widely used structural elements, bars are the interest of this study. Herein, bars of nanosize will be examined utilizing Eringen's nonlocal theory of elasticity. In the literature, there is a vast amount of studies dealing with similar problems; yet most of them resort to numerical resolutions of the differential equations; see, for example, [17]. This is obviously for a good reason: the governing equations of nanobars are of integro-differential type, the existence and uniqueness of which requires a great deal of examination. Indeed, the exact solution in [18] was only provided for some special loading and boundary conditions, which satisfy some additional and non-physical conditions, called *constitutive boundary conditions*. Such an additional requirement for an exact solution to exist (albeit in a certain form) induced a debate among the researchers which still continues, and it even led to some strong conclusions indicating the use of strain-driven non-local models must be prevented [18]. The intention here is to stay out of this discussion, and to look for the possibilities of finding approximate solutions to the problems of bars of nanosize, by using a perturbation technique which is recently proposed

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by this author [20], and extending the example problems considered therein.

In addition to those landmark studies cited in up to this point, the interested readers are kindly referred to [21, 22] for relatively recent applications of Eringen's two-phase model, [23-27] for different approaches to the modelling of nanobars, and a review paper [28] for a better insight on classification, limitations, and mathematical aspects of non-local continuum models.

The novel points of the present study are the following: different example problems are examined by using the method proposed in [20], the solutions to them are given in closed-form, quantitative comparison of the results with the literature are provided for further verification of the method, and the convergence of the results are examined. Some benchmark results are provided which can be used for verification purposes of new numerical or analytical techniques to be presented in the future for this very hot topic of solid mechanics.

2. MECHANICS OF NANOBARS

Eringen's nonlocal theory of elasticity is based on the axioms of causality, determinism, equipresence, objectivity, material invariance, neighbourhood, memory, and admissibility. The axiom of locality, which basically leads to the local theory of elasticity, is skipped; and as a result, the stress at a point depends on the strain multiplied by an attenuation function and integrated over the entire domain which consists of material points. Note that this theory keeps the primal fields of local elasticity, but the relation between them is, in the end, qualitatively different. Examinations of structures with finite dimensions provided that there seems to be the need for additional conditions, so-called constitutive boundary conditions, for an exact solution to exist in a certain (differential) form or for the reduction of integro-differential equations to differential equations [29, 30]. Here such requirements will not be looked for; instead, an approximate solution to the integro-differential equation will be pursued.

Consider a bar of length L, along the axis x. The displacement of each point inside the bar will be denoted with u, and the resultant of stress normal to the cross-sections is N. The kinematic relation and balance requirement are independent of its constitution under the assumption of vanishing of nonlocal residuals [10]; therefore,

$$\frac{dN}{dx} = -q(x),\ \varepsilon = \frac{du}{dx} \tag{1}$$

where q is the external distributed load along the axis of the bar, and ε the normal strain along the bar axis and it is the only non-vanishing strain component in case of normal external loads. The constitutive equation of Eringen's two-phase local/nonlocal mixture law is

$$N(x) = B\left[(1-\xi)\varepsilon(x) + \xi\int_0^L K(x,X)\varepsilon(X)dX\right] \tag{2}$$

where B is axial rigidity of the bar, ξ is the mixture parameter denoting the weight of the nonlocal part, and $K(x,X)$ is the attenuation function which represents the interaction between the points depending on the distance between them. Among many alternatives, the exponential kernel is utilized herein:

$$K(x,X) = \frac{1}{2\kappa}\exp\left(\frac{|x-X|}{\kappa}\right) \tag{3}$$

where κ is nonlocal parameter quantifying the zone of interaction between material points. The coefficient of the exponential part stems from the usual normalization of kernel function over infinite domain.

3. SOLUTION PROCEDURE

Formal series expansions of normal force field, N, and axial strain, ε, and axial displacement, u, about a certain value η_0 of a generic parameter η, are as follows.

$$\begin{aligned} N \approx N^n &= \sum_{j=0}^{n}\frac{(\eta-\eta_0)^j}{j!}\left.\frac{d^jN}{d\eta^j}\right|_{\eta=\eta_0} = \sum_{j=0}^{n}\frac{(\eta-\eta_0)^j}{j!}N_j(\eta_0) \\ \varepsilon \approx \varepsilon^n &= \sum_{j=0}^{n}\frac{(\eta-\eta_0)^j}{j!}\left.\frac{d^j\varepsilon}{d\eta^j}\right|_{\eta=\eta_0} = \sum_{j=0}^{n}\frac{(\eta-\eta_0)^j}{j!}\varepsilon_j(\eta_0) \\ u \approx u^n &= \sum_{j=0}^{n}\frac{(\eta-\eta_0)^j}{j!}\left.\frac{d^ju}{d\eta^j}\right|_{\eta=\eta_0} = \sum_{j=0}^{n}\frac{(\eta-\eta_0)^j}{j!}u_j(\eta_0) \end{aligned} \tag{4}$$

Admitting $\eta = \xi$ and $\xi_0 = 0$ provide,

$$\{N,\varepsilon,u\} \approx \{N^n,\varepsilon^n,u^n\} = \sum_{j=0}^{n}\frac{\xi^j}{j!}\left.\frac{d^j\{N,\varepsilon,u\}}{d\xi^j}\right|_{\xi=0} = \sum_{j=0}^{n}\frac{\xi^j}{j!}\{N_j,\varepsilon_j,u_j\} \tag{5}$$

where N_j, ε_j, and u_j are j^{th} derivatives of normal force, axial strain, and axial displacement fields evaluated for ξ = 0.

Inserting Eq. (5) into first of Eq. (1), and Eq. (2) provides,

$$\begin{aligned} &0^{th}\text{ order:} && \frac{dN_0}{dx} = -q(x), && \varepsilon_0 = \frac{N_0}{B}, && \frac{du_0}{dx} = \varepsilon_0. \\ &j^{th}\text{ order:} && \frac{dN_j}{dx} = 0, && \varepsilon_j = \frac{N_j}{B} - j(K*\varepsilon_{j-1} - \varepsilon_{j-1}), && \frac{du_j}{dx} = \varepsilon_j. \end{aligned} \tag{6}$$

where the convolution between kernel function and any generic function, f, is described as below.

$$K*f = \int_0^L K(x,X)f(X)dX \tag{7}$$

This procedure stipulates the nonhomogeneous part to depend on the evolution parameter, providing the dependence of inner actions on stiffness due to the requirement of compatibility equations. Here it is important to note that in case of statically determinate structures, there is no such dependence.

The axial force field and axial displacement field of different orders may be obtained by integration.

$$N_0(x) = N_0(0) - \int_0^x q(\Xi)\, d\Xi, \quad u_0(x) = u_0(0) + \int_0^x \varepsilon_0(\Xi)\, d\Xi.$$

$$N_j(x) = N_j(0), \quad u_j(x) = u_j(0) + \int_0^x \varepsilon_j(\Xi)\, d\Xi. \tag{8}$$

where $N_i(0)$ and $u_i(0)$ $(i = 0,1,2,\ldots)$ are initial values of normal force and axial displacement, to be obtained by imposing only the physical boundary conditions.

4. EXAMPLE PROBLEMS

Some example problems of bars of nano-size with this method have been presented in [20]. Here we will skip those basic examples, and focus on statically indeterminate problems and bars of variable section. Convergence of all examples will be looked for numerically.

4.1. Doubly-fixed uniform bar under uniform distributed load

This is an important example to show numerically the possible convergence of the solution technique for statically indeterminate problems. Boundary conditions are,

$$u(0) = u(L) = 0 \tag{9}$$

which, with the formal series expansion, becomes,

$$u_i(0) = u_i(L) = 0, \quad i = 0,1,2,\ldots,n \tag{10}$$

In the case of uniform distributed load,

$$q(x) = q_0 \Rightarrow N_0 = N_0(0) - q_0 x$$
$$\frac{du_0}{dx} = \frac{N_0(0) - q_0 x}{B} \Rightarrow u_0 = u_0(0) + \frac{N_0(0)}{B} x - \frac{q_0 x^2}{2B} \tag{11}$$

Applying the boundary conditions, $u(0) = 0$, $u(L) = 0$ provide,

$$N_0 = \frac{1}{2} q_0 (L - 2x), \quad u_0 = \frac{q_0 L}{2B} x - \frac{q_0 x^2}{2B} \tag{12}$$

which are very well-known solutions of local elasticity.

Using the 0^{th}-order solutions, and boundary conditions of each order, the normal force field of higher-order turns out to be identical to zero. Higher-order displacement fields are obtained as below.

$$u_1 = \frac{\left(1 - e^{x/\kappa}\right) e^{-\frac{L+z}{\kappa}} \left(e^{z/\kappa} - e^{L/\kappa}\right)(2\kappa + L)\kappa q_0}{4B} \tag{13}$$

$$\begin{aligned} u_2 = {} & \frac{q_0 e^{\frac{x-2L}{\kappa}}}{B} \left(e^{L/\kappa} \left(-\frac{\kappa^2}{4} + \frac{L^2}{4} + \kappa\left(\frac{3L}{8} - \frac{x}{2}\right) - \frac{Lx}{4} \right) + \kappa\left(\frac{\kappa}{4} + \frac{L}{8}\right) \right) + \frac{q_0 \kappa e^{-\frac{L+x}{\kappa}}}{B} \left(\frac{\kappa}{4} + \frac{L}{8}\right) \\ & + \frac{q_0 \kappa}{B} \left(1 - e^{-2L/\kappa} - 2\frac{L}{\kappa} e^{-L/\kappa}\right)\left(\frac{\kappa}{4} + \frac{L}{8}\right) + \frac{q_0 e^{\frac{L}{\kappa} - \frac{L+x}{\kappa}}}{B} \left(-\frac{\kappa^2}{4} + \kappa\left(\frac{x}{2} - \frac{L}{8}\right) + \frac{Lx}{4} \right) \end{aligned} \tag{14}$$

$$\begin{aligned} u_3 = {} & \frac{q_0 e^{-\frac{2L}{\kappa}} \left(-\frac{3\kappa^2}{8} + \frac{3L^2}{8} + \frac{9\kappa L}{16} \right)}{B} + \frac{q_0 e^{-\frac{L}{\kappa}}}{B} \left(-\frac{3\kappa^2}{16} + \frac{3L^3}{16\kappa} - \frac{3L^2}{16} - \frac{39\kappa L}{32} \right) \\ & + \frac{q_0 e^{L/\kappa}}{B} \left[e^{-\frac{2L+x}{\kappa}} \left(\frac{9\kappa^2}{16} - \frac{3L^2}{16} + \kappa\left(-\frac{3L}{32} - \frac{3x}{8} \right) - \frac{3Lx}{16} \right) \right. \\ & \left. + e^{\frac{x-3L}{\kappa}} \left(\frac{9\kappa^2}{16} - \frac{3L^2}{8} + \kappa\left(\frac{3x}{8} - \frac{15L}{32} \right) + \frac{3Lx}{16} \right) \right] + \frac{3q_0\kappa}{16B} \left(2 + e^{-\frac{3L}{\kappa}} - e^{\frac{x-3L}{\kappa}} - e^{-\frac{2L+x}{\kappa}} \right)\left(\kappa + \frac{L}{2} \right) \\ & + \frac{q_0 e^{\frac{2L}{\kappa}}}{B} \left[e^{\frac{x-3L}{\kappa}} \left(-\frac{3\kappa^2}{8} + \frac{3L^2}{8} - \frac{3L^3}{16\kappa} + \frac{3L^2 x}{8\kappa} - \frac{3Lx^2}{16\kappa} + \kappa\left(\frac{21L}{16} - \frac{3x}{2} \right) - \frac{3x^2}{8} \right) \right. \\ & \left. + e^{-\frac{2L+x}{\kappa}} \left(-\frac{3\kappa^2}{8} - \frac{3Lx^2}{16\kappa} + \kappa\left(\frac{3x}{2} - \frac{3L}{16} \right) + \frac{3Lx}{4} - \frac{3x^2}{8} \right) \right] \end{aligned} \tag{15}$$

To look for the (weak) convergence, L_2-norm of the balance residual, R^{m}, will be utilized.

$$R^m = \frac{d}{dx}\left[(1 - \xi)\frac{du^m}{dx} + \xi K * \frac{du^m}{dx} \right] + q_0 \tag{16}$$

Fig. 1 shows the variation of the residual of balance equation for different orders of approximate displacement fields, nonlocal parameter, and fraction coefficient. As expected, they are null for the full local model, designated with $\boldsymbol{\xi} = 0$, and tend to grow as the fraction coefficient increases. This is, again, expected since the approximation is made on the value of the fraction coefficient; hence, increasing values of it simply requires the consideration of more terms in the series expansion. A similar outcome is reported in [20], but in the absence of distributed load and for statically determinate problems of nanobars.

Fig.2 provides the visual representation of Eqs. $(12)_2$, (13), (14), and (15). Higher-order functions of displacement grow with increasing nonlocal parameter which gives us a hint on a possible limit for $\boldsymbol{\kappa}$ when looking for formal proof of convergence of the method. Note that these functions are to be modulated with the fraction coefficient when obtaining the final displacement field, see Eq. (5). The increasing effect of nonlocality close to boundaries become more visible as the order of function increases.

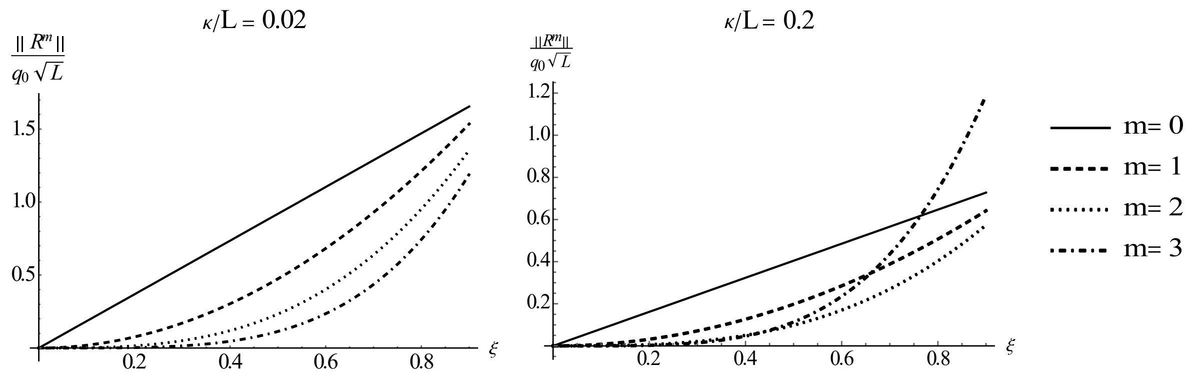


Figure 1. L_2-norm of balance residual for different orders of approximation

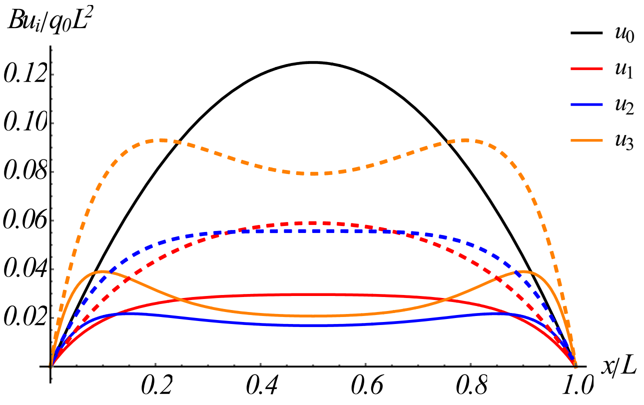


Figure 2. Variations of displacement functions of different orders. Solid lines: **κ**/L = 0.1, Dashed lines. **κ**/L = 0.2

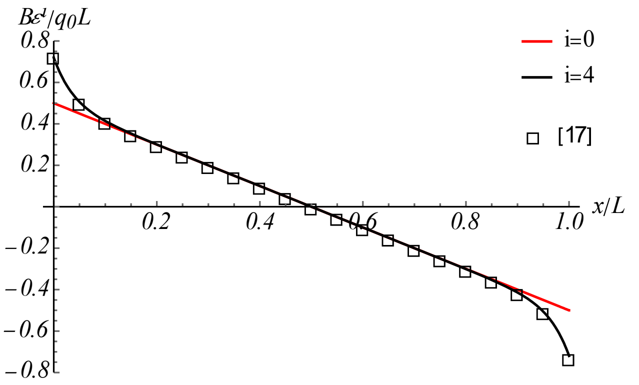


Figure 3. Variation of strain along doubly-fixed bar under uniform distributed load. **ξ** = 0.5, **κ**/L = 0.05

A similar problem is considered in [17], where a numerical technique with considerable computational expense is utilized. Fig.3 shows the comparison of the present results with [17], where an excellent agreement is observed. It strengthens the argument that the closed-form expressions presented herein are very effective.

4.2. Bar with exponentially varying section

This example is to illustrate the behavior of nanobars of variable section, and the possible convergence characteristics of the present solution procedure. The axial stiffness of the bar is assumed to vary exponentially.

$$B = B_0 \exp\left(-\beta x / L\right) \tag{17}$$

In the case of uniform normal force, which results from a concentrated load at the free end (with other end fixed), displacement fields of different orders are as reported below.

$$u_0 = \frac{N_0 L\left(e^{\frac{\beta x}{L}} - 1\right)}{\beta B_0} \tag{18}$$

$$u_1 = \frac{N_0}{2B_0(L-\beta\kappa)(\beta\kappa+L)}\Bigg[-\kappa L e^{\beta-\frac{L+x}{\kappa}+\frac{x}{\kappa}}\left(L+\beta\kappa\right)+\kappa L e^{\beta-\frac{L+x}{\kappa}+\frac{2x}{\kappa}}\left(L+\beta\kappa\right)$$
$$+\kappa L e^{\frac{L}{\kappa}-\frac{L+x}{\kappa}}\left(-L+\left(\beta\kappa+L\right)e^{x/\kappa}+\beta\kappa\left(1-2e^{\frac{\beta x}{L}+\frac{x}{\kappa}}\right)\right)\Bigg] \tag{19}$$

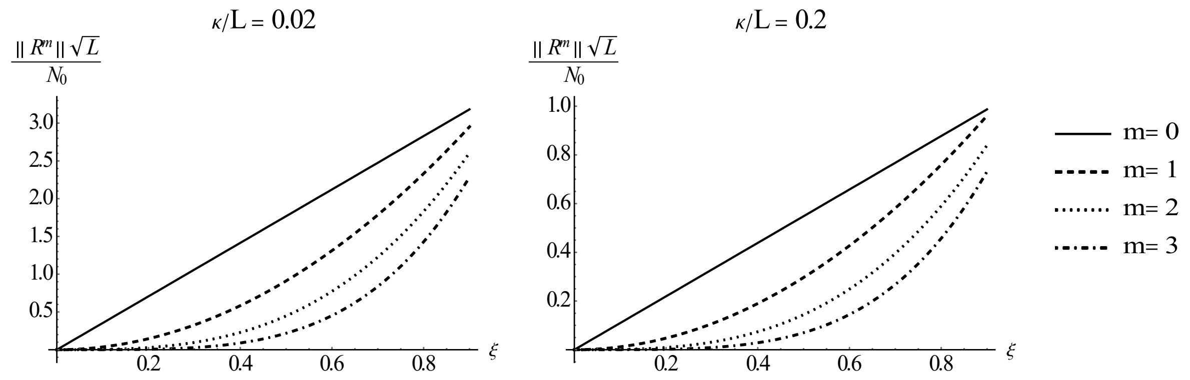
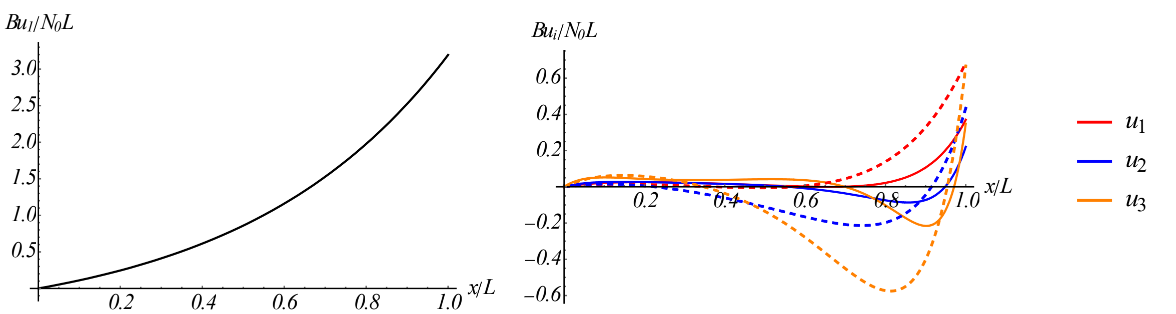


Figure 4. L_2-norm of balance residual for bar of variable section (**β** = 1) for different orders of approximation.



(a) (b)

Figure 5. Variations of displacement functions of different orders. Solid lines: **κ**/L = 0.1, Dashed lines. **κ**/L = 0.2, **β** = 2.

$$
\begin{aligned}
u_2 = {} & \frac{N_0}{B_0(L-\beta\kappa)^2(\beta\kappa+L)^2}\Bigg[\kappa L^4\left(\frac{1}{4}-\frac{1}{4}e^{-\frac{2L}{\kappa}}\right)+\kappa^2L^3\left(\frac{1}{4}\beta e^{-\frac{2L}{\kappa}}-\frac{\beta}{4}\right) \\
& +\kappa^3L^2\left(\frac{1}{4}\beta^2e^{-\frac{2L}{\kappa}}-\frac{5\beta^2}{4}\right)+\frac{1}{4}e^{-\frac{2(L-x)}{\kappa}-\frac{x}{\kappa}}\left(\kappa L^4-\beta\kappa^2L^3-\beta^2\kappa^3L^2+\beta^3\kappa^4L\right) \\
& +e^{-\frac{x}{\kappa}}\left(\frac{L^4x}{2}+\kappa^3\left(\frac{5\beta^2L^2}{4}+\frac{\beta^3Lx}{2}\right)-\kappa\left(\frac{L^4}{4}+\frac{\beta L^3x}{2}\right)+\kappa^2\left(\frac{\beta L^3}{4}-\frac{\beta^2L^2x}{2}\right)\right. \\
& \left.\left.+\kappa^4L\left(2\beta^3e^{x\left(\frac{1}{\kappa}+\frac{\beta}{L}\right)}-\frac{5\beta^3}{4}\right)\right)\right)+e^{\beta+\frac{L}{\kappa}-\frac{2(L-x)}{\kappa}-\frac{x}{\kappa}}\left(-\frac{L^5}{2}+\frac{L^4x}{2}+\kappa^3\left(\left(\frac{\beta^3}{2}-\frac{5\beta^2}{4}\right)L^2-\frac{1}{2}\beta^3Lx\right)\right. \\
& \left.+\kappa\left(\left(\frac{1}{4}-\frac{\beta}{2}\right)L^4+\frac{1}{2}\beta L^3x\right)+\kappa^2\left(\left(\frac{\beta^2}{2}+\frac{\beta}{4}\right)L^3-\frac{1}{2}\beta^2L^2x\right)-\frac{5}{4}\beta^3\kappa^4L\right) \\
& +e^{\beta-\frac{L}{\kappa}}\left(\frac{L^5}{2}+\frac{1}{2}\beta\kappa L^4-\frac{1}{2}\beta^2\kappa^2L^3+\left(\beta^2-\frac{\beta^3}{2}\right)\kappa^3L^2+\beta^3\kappa^4L\right. \\
& \left.\left.+e^{-\frac{x}{\kappa}}\left(-\frac{\kappa L^4}{4}-\frac{1}{4}\beta\kappa^2L^3+\frac{1}{4}\beta^2\kappa^3L^2+\frac{1}{4}\beta^3\kappa^4L\right)\right)\right)+\kappa^4L\left(-\frac{3\beta^3}{4}-\frac{1}{4}\beta^3e^{-\frac{2L}{\kappa}}\right)\Bigg]
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
u_3 = {} & \frac{N_0}{(\beta\kappa-L)^3(\beta\kappa+L)^3B_0}\Bigg[C_0+C_1\mathrm{e}^{-\frac{3L+x}{\kappa}+\frac{2L+2x+\beta\kappa}{\kappa}}+C_2\mathrm{e}^{-\frac{2L}{\kappa}}+C_3\mathrm{e}^{\beta+\frac{2x}{\kappa}-\frac{3L+x}{\kappa}}+C_4\mathrm{e}^{\frac{L}{\kappa}-\frac{3L+x}{\kappa}} \\
& C_5\mathrm{e}^{\beta-\frac{3L}{\kappa}}+C_6\mathrm{e}^{\beta+\frac{2L}{\kappa}-\frac{3L+x}{\kappa}}C_7\mathrm{e}^{-\frac{3L+x}{\kappa}+\frac{L+2x}{\kappa}}C_8\mathrm{e}^{\beta-\frac{L}{\kappa}}+C_9\mathrm{e}^{\frac{3L}{\kappa}-\frac{3L+x}{\kappa}}\Bigg]
\end{aligned}
\tag{21}
$$

The constants C_i in Eq.(21) are provided in the appendix. Similar to Eq. (16), a balance residual is defined. Figure 4 provides its variation with the fraction coefficient. The figures are provided for β = 1, but note that different values of this parameter do not alter the balance residual appreciably. Similar to the case of the uniform section, a smaller value of nonlocal parameter requires the consideration of more terms in the series expansion as it provides a sharper change of field functions providing higher gradients to be represented by higher-order terms.

Displacement functions of different orders, the expressions of which are given in Eqs. (18-21) in closed-form, are illustrated in Fig. 5. Zeroth-order, and higher-order terms are provided separetely as they differ in terms of the order of magnitude in this case. It is observed that higher-order displacement functions provide corrections which are more appreciable closer to the free end, where the *boundary effect* is present. Then again it is important to note that even if the magnitudes of higher-order terms seem to be quite high, they are to be modulated with the fraction coefficient before the final (approximate) displacement field is calculated.

5. CONCLUSIONS

Example problems of nanobars under different loading and boundary conditions are considered herein. It is the extension of the work by this author providing an approximate solution procedure based on formal series expansion of field functions in terms of so-called fraction coefficient providing the contribution of long-range interactions in constitutive relation. In particular, demonstration of the convergence of solutions in different scenarios is of great importance as a general convergence theorem for this method and this problem has not been proven yet. Moreover, a verification study is performed to compare the solutions presented in closed-form to that of an existing study which provides a numerical technique. The very good agreement between the results basically indicates the applicability of the closed-form expressions presented in this study, which provides great simplifications especially when it comes to material identification procedures where the solutions of a certain mathematical model are to be calculated repetitively. The results of this work basically enlarges the area of applicability of this series solution technique, and therefore may well be used as benchmark solutions.

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Appendix:

The constants C_i in 3^{rd}-order displacement field of nanobar with exponentially varying section.

$$C_0=-\frac{3\kappa L^6}{8}+\frac{3}{8}\beta\kappa^2L^5+\frac{3}{2}\beta^2\kappa^3L^4-\frac{3}{2}\beta^3\kappa^4L^3-\frac{33}{8}\beta^4\kappa^5L^2-\frac{15}{8}\beta^5\kappa^6L \tag{A.1}$$

$$\begin{aligned}C_1=&-\frac{3L^8}{8\kappa}-\frac{3\beta L^7}{8}+\frac{3L^7x}{4\kappa}+\frac{3L^7}{2}+\frac{3}{4}\beta^2\kappa L^6+\frac{3}{2}\beta\kappa L^6-\frac{3\kappa L^6}{8}-\frac{3L^6x^2}{8\kappa}+\frac{3}{4}\beta L^6x\\&-\frac{3L^6x}{2}+\frac{3}{4}\beta^3\kappa^2L^5-\frac{9}{2}\beta^2\kappa^2L^5-\frac{3}{8}\beta\kappa^2L^5-\frac{3}{8}\beta L^5x^2-\frac{3}{2}\beta^2\kappa L^5x-\frac{3}{2}\beta\kappa L^5x\\&-\frac{3}{8}\beta^4\kappa^3L^4-\frac{9}{2}\beta^3\kappa^3L^4+\frac{3}{2}\beta^2\kappa^3L^4+\frac{3}{4}\beta^2\kappa L^4x^2-\frac{3}{2}\beta^3\kappa^2L^4x+\frac{9}{2}\beta^2\kappa^2L^4x\\&-\frac{3}{8}\beta^5\kappa^4L^3+3\beta^4\kappa^4L^3+\frac{3}{2}\beta^3\kappa^4L^3+\frac{3}{4}\beta^3\kappa^2L^3x^2+\frac{3}{4}\beta^4\kappa^3L^3x+\frac{9}{2}\beta^3\kappa^3L^3x\\&+3\beta^5\kappa^5L^2-\frac{33}{8}\beta^4\kappa^5L^2-\frac{3}{8}\beta^4\kappa^3L^2x^2+\frac{3}{4}\beta^5\kappa^4L^2x-3\beta^4\kappa^4L^2x-\frac{33}{8}\beta^5\kappa L\\&-\frac{3}{8}\beta^5\kappa^4Lx^2-3\beta^5\kappa^5Lx\end{aligned} \tag{A.2}$$

$$\begin{aligned}C_2=&-\frac{3L^7}{4}+\frac{3L^6\kappa}{8}+\frac{3}{4}L^6\beta\kappa-\frac{3}{8}L^5\beta\kappa^2+\frac{3}{2}L^5\beta^2\kappa^2-\frac{3}{2}L^4\beta^2\kappa^3-\frac{3}{2}L^4\beta^3\kappa^3\\&+\frac{3}{2}L^3\beta^3\kappa^4-\frac{3}{4}L^3\beta^4\kappa^4+\frac{9}{8}L^2\beta^4\kappa^5+\frac{3}{4}L^2\beta^5\kappa^5-\frac{9}{8}L\beta^5\kappa^6\end{aligned} \tag{A.3}$$

$$C_3=-\frac{3L^6\kappa}{16}-\frac{3}{16}L^5\beta\kappa^2+\frac{3}{8}L^4\beta^2\kappa^3+\frac{3}{8}L^3\beta^3\kappa^4-\frac{3}{16}L^2\beta^4\kappa^5-\frac{3}{16}L\beta^5\kappa^6 \tag{A.4}$$

$$C_4=\frac{3L^6\kappa}{16}-\frac{3}{16}L^5\beta\kappa^2-\frac{3}{8}L^4\beta^2\kappa^3+\frac{3}{8}L^3\beta^3\kappa^4+\frac{3}{16}L^2\beta^4\kappa^5-\frac{3}{16}L\beta^5\kappa^6 \tag{A.5}$$

$$C_5=\frac{3L^6\kappa}{16}+\frac{3}{16}L^5\beta\kappa^2-\frac{3}{8}L^4\beta^2\kappa^3-\frac{3}{8}L^3\beta^3\kappa^4+\frac{3}{16}L^2\beta^4\kappa^5+\frac{3}{16}L\beta^5\kappa^6 \tag{A.6}$$

$$\begin{aligned}C_6=&-\frac{3L^7}{8}-\frac{3L^6x}{8}+\frac{9L^6\kappa}{16}-\frac{3}{8}L^6\beta\kappa-\frac{3}{8}L^5x\beta\kappa+\frac{9}{16}L^5\beta\kappa^2+\frac{3}{4}L^5\beta^2\kappa^2\\&+\frac{3}{4}L^4x\beta^2\kappa^2-\frac{15}{8}L^4\beta^2\kappa^3+\frac{3}{4}L^4\beta^3\kappa^3+\frac{3}{4}L^3x\beta^3\kappa^3-\frac{15}{8}L^3\beta^3\kappa^4-\frac{3}{8}L^3\beta^4\kappa^4\\&-\frac{3}{8}L^2x\beta^4\kappa^4+\frac{21}{16}L^2\beta^4\kappa^5-\frac{3}{8}L^2\beta^5\kappa^5-\frac{3}{8}Lx\beta^5\kappa^5+\frac{21}{16}L\beta^5\kappa^6\end{aligned} \tag{A.7}$$

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$$C_7 = \frac{3L^7}{4} - \frac{3L^6 x}{8} - \frac{9L^6\kappa}{16} - \frac{3}{4}L^6\beta\kappa + \frac{3}{8}L^5 x\beta\kappa + \frac{9}{16}L^5\beta\kappa^2 - \frac{3}{2}L^5\beta^2\kappa^2 + \frac{3}{4}L^4 x\beta^2\kappa^2$$
$$+\frac{15}{8}L^4\beta^2\kappa^3 + \frac{3}{2}L^4\beta^3\kappa^3 - \frac{3}{4}L^3 x\beta^3\kappa^3 - \frac{15}{8}L^3\beta^3\kappa^4 + \frac{3}{4}L^3\beta^4\kappa^4 - \frac{3}{8}L^2 x\beta^4\kappa^4$$
$$-\frac{21}{16}L^2\beta^4\kappa^5 - \frac{3}{4}L^2\beta^5\kappa^5 + \frac{3}{8}Lx\beta^5\kappa^5 + \frac{21}{16}L\beta^5\kappa^6 \tag{A.8}$$

$$C_8 = -\frac{9L^7}{8} + \frac{3L^7\beta}{8} + \frac{3L^8}{8\kappa} - \frac{3L^6\kappa}{16} - \frac{9}{8}L^6\beta\kappa - \frac{3}{4}L^6\beta^2\kappa - \frac{3}{16}L^5\beta\kappa^2 + \frac{15}{4}L^5\beta^2\kappa^2$$
$$-\frac{3}{4}L^5\beta^3\kappa^2 + \frac{3}{8}L^4\beta^2\kappa^3 + \frac{15}{4}L^4\beta^3\kappa^3 + \frac{3}{8}L^4\beta^4\kappa^3 + \frac{3}{8}L^3\beta^3\kappa^4 - \frac{21}{8}L^3\beta^4\kappa^4 + \frac{3}{8}L^3\beta^5\kappa^4$$
$$+\frac{45}{16}L^2\beta^4\kappa^5 - \frac{21}{8}L^2\beta^5\kappa^5 + \frac{45}{16}L\beta^5\kappa^6 \tag{A.9}$$

$$C_9 = -\frac{3L^6 x}{2} - \frac{3}{8}L^5 x^2\beta + \frac{3L^6 x^2}{8\kappa} + \frac{3L^6\kappa}{8} + \frac{3}{2}L^5 x\beta\kappa - \frac{3}{4}L^4 x^2\beta^2\kappa$$
$$-\frac{3}{8}L^5\beta\kappa^2 + \frac{9}{2}L^4 x\beta^2\kappa^2 + \frac{3}{4}L^3 x^2\beta^3\kappa^2 - \frac{3}{2}L^4\beta^2\kappa^3 - \frac{9}{2}L^3 x\beta^3\kappa^3$$
$$+\frac{3}{8}L^2 x^2\beta^4\kappa^3 + \frac{3}{2}L^3\beta^3\kappa^4 - 3L^2 x\beta^4\kappa^4 - \frac{3}{8}Lx^2\beta^5\kappa^4 + \frac{33}{8}L^2\beta^4\kappa^5$$
$$+3Lx\beta^5\kappa^5 - \frac{33}{8}L\beta^5\kappa^6 + 6\mathrm{e}^{\frac{x\beta}{L}+\frac{x}{\kappa}}L\beta^5\kappa^6 \tag{A.10}$$