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Obtaining Some Identities With the $\mathrm{n^{th}}$ Power of a Matrix Under the Lorentzian Product

İbrahim Gökcan[1,*] *Ali Hikmet Değer*[2]

[1] *Department of Mathematics, Karadeniz Technical University,Trabzon, Turkey, ORCID: 0000-0002-6933-8494*
[2] *Department of Mathematics, Karadeniz Technical University, Trabzon, Turkey, ORCID: 0000-0003-0764-715X*
* *Corresponding Author E-mail: gokcan4385@gmail.com*

Abstract: The Fibonacci number sequence and related calculations come up in scientific facts in many events that we encounter in daily life. This special number sequence is processed in the occurrence of many events such as calculating the diameter of the equatorial circumference of the Earth, flowers, growth and structures of leaves, trees, reproduction of bees, sunflower and so on [6]. However, in recent years, the relation between the Fibonacci and Lucas Number sequences with continued fractions and matrices has intensively been studied. Many identities have been found by some 2×2 types of special matrices with the n^{th} power that have been associated with the Fibonacci and Lucas numbers. The aim of this study is to examine matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ under the Lorentzian matrix product with the n^{th} power, quadratic equations and characteristic roots unlike the classical matrix product. In addition, we want to acquire some identities with the help of matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ under the Lorentzian matrix product with the n^{th} power in relation to the Fibonacci and Lucas numbers.

Keywords: Characteristic root, Fibonacci and lucas numbers, Lorentzian matrix multiplication, Quadratic equation.

1 Introduction

1.1 Fibonacci and Lucas Number Sequences

Fibonacci number sequence is known as $0, 1, 1, 2, 3, 5, 8, \cdots$ We show with $F_n = F_{n-1} + F_{n-2}$ the n^{th} term of sequence. Lucas number sequence is known as $1, 3, 4, 7, 11, 18, \cdots$. We show with $L_n = L_{n-1} + L_{n-2}$ the n^{th} term of sequence. Lucas number sequence can be obtained from Fibonacci number sequence. There are many identities associated with the Fibonacci and Lucas number sequences.For details, see [1]-[4] and [6]. The k -Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$ and any positive real number k with initial conditions$F_{k,0} = 0$, $F_{k,1} = 1$ [5]. Fibonacci sequence for $k = 1$ and Pell sequence for $k = 2$ are obtained. In [2], generated matrices for Fibonacci and Pell sequences. Using the relationship between Fibonacci and Lucas number sequences, the numerical values of the terms of these sequences and matrix relations of continued fractions has expanded today's working area of Fibonacci and Lucas number sequences. In this study, we aim to examine the n^{th} power under the Lorentzian matrix product that unlike the classical matrix product of $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, quadratic equations and characteristic roots. However, some relations related with Fibonacci and Lucas numbers are obtained with the help of the n^{th} power under the Lorentzian matrix product of $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. Some of these identities obtained from the matrices whose the n^{th} power were taken by the classical product method in the previously studies, where these identities were obtained by Lorentzian product.

1.2 Finding the $\mathrm{n^{th}}$ Power of A Matrix with Help of Classical Matrix Product

Consider the matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. It was studied in 1960 by Charles H. King. It should be noted that $|Q| = -1$. In addition, under the classical matrix product,

$$Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Q^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$
$$Q^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

are available.

If we continue in a similar way, we reach $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ matrix [6].

Theorem 1.2.1 [6] Let $n \geq 1$. Then $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Proposition 1.2.2 [6] Let $n \geq 1$. Then $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

Examples 1.2.3 Some examples are given in this section.

1. $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$ [6]
2. $L_{m+n+1} = F_{m+1}L_{n+1} + F_mL_n$ [7]
3. $F_nL_{n+k} - F_{n+k}L_n = 2(-1)^{n+1}F_k$ [7]
4. $L_{n-1}L_{n+1} + F_{n-1}F_{n+1} = 6F_n^2$ [7]
5. $L_nF_{m-n} + F_nL_{m-n} = 2F_m$ [7]
6. $F_{2m}^2 = 5F_m^4 + 4(-1)^mF_m^2$ [7]

1.3 Lorentzian Matrix Product

Let $A = [a_{ij}] \in R_n^m$ and $B = [b_{jk}] \in R_p^n$, where R_n^m is a matrix of type $m \times n$, R_p^n is a matrix of type $n \times p$.
$A._L B = [-a_{i1}b_{1k} + \sum_{j=2}^n a_{ij}b_{jk}]$ matrix product is defined with "$._L$". This product is named with Lorentzian matrix product.
Naturally, $A._L B$ is a matrix of type $m \times p$. L_p^m represents Lorentzian matrices's sets of type $m \times p$.
If we let i^{th} row of A with A_i and j^{th} column of B with B^j, $< A_i, B^j >_L$ inner product (i, j) of $A._L B$.
If we take $x = (x_1, x_2, \cdots, x_n)$ as the first row of A matrix, $y = (y_1, y_2, \cdots, y_n)$ as the first column of B, $(1, 1)$ element of $A._L B$ matrix $< A_1, B^1 >_L = -x_1y_1 + \sum_{i=2}^n x_iy_i$.
Each element of $A._L B \in L_p^m$ is an inner product [3].

Theorem 1.3.1 [3] The following equations are provided.
i. For $\forall A \in L_n^m, B \in L_p^n, C \in L_r^p, A._L(B._L C) = (A._L B)._L C$
ii. For $\forall A \in L_n^m, B, C \in L_p^n, A._L(B + C) = A._L B + A._L C$
iii. For $\forall A, B \in L_n^m, C \in L_p^n, (A + B)._L C = A._L C + B._L C$
iv. For $\forall k \in \mathbb{R}, \forall A \in L_n^m, B \in L_p^n, k(A._L B) = (kA)._L B = A._L(kB)$

Theorem 1.3.2 [3] According to the L -product, the L -unit matrix of type $n \times n$ is denoted by I_n.

For $A \in L_n^m, I_m._L A = A._L I_n = A$
For example for $n = 2$;
Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
$$I._L I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ._L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} < I_1, I^1 > & < I_1, I^2 > \\ < I_2, I^1 > & < I_2, I^2 > \end{pmatrix}$$
$$= \begin{pmatrix} < (1,0), (1,0) > & < (1,0), (0,1) > \\ < (0,1), (1,0) > & < (0,1), (0,1) > \end{pmatrix}$$
$$= \begin{pmatrix} -1.1 + 0.0 & -1.0 + 0.1 \\ -0.1 + 1.0 & -0.0 + 1.1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proposition 1.3.3 [3] L_n^n with $L-$ multiplied is a unit algebra.

Definition 1.3.4 [3] A is a matrix of type $n \times n$, if there is a B matrix of type $n \times n$ such that $A._L B = B._L A = I_n$, is called reversible and denoted by A^{-1}.

Definition 1.3.5 [3] Transpose of $A = [a_{ij}] \in L_n^m$ demonstrations with A^T and define with $A^T = [a_{ji}] \in L_m^n$.

Definition 1.3.6 [3] If $A^{-1} = A^T$ for $A \in L_n^n$ matrix, A is called $L-$ orthogonal.

Definition 1.3.7 [3] Determinant of $A = [a_{ij}] \in L_n^n$ matrix demontrations with $detA$ and define with $detA = \sum_{\sigma \in S_n} a_{\sigma 1 1} a_{\sigma 2 2} \cdots a_{\sigma n n}$. In here, S_n is all permutations's set of $\{1, 2, \cdots, n\}$ and $s(\sigma)$ is sign of σ permutations.

Theorem 1.3.8 [3] For $A, B \in L_n^n$, $det(A._L B) = -(detA)(detB)$.

Example 1.3.9 Let find the matrix obtained by multiplying $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ under Lorentzian matrix product.

$$\begin{aligned} M._L Q &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} < M_1, Q^1 > & < M_1, Q^2 > \\ < M_2, Q^1 > & < M_2, Q^2 > \end{pmatrix} \end{aligned}$$

$< M_1, Q^1 >= -11 + 1.1. = 0$
$< M_1, Q^2 >= -1.1. + 1.0 = -1$
$< M_2, Q^1 >= -1.1. + 2.1 = 1$
$< M_2, Q^2 >= -1.1. + 2.0 = -1$
$M._L Q = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in L_2^2.$

2 Material and Method

2.1 Finding the $\mathbf{n^{th}}$ Power of a Matrix Under the Lorentzian Matrix Product and Related Identities

Let us find the n^{th} power of $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ matrix under Lorentzian matrix product.

Let $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{1._L} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow det \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = 1$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{2._L} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -F_3 & -F_2 \\ F_2 & F_1 \end{pmatrix} \Rightarrow det \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = -1$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{3._L} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} F_4 & F_3 \\ -F_3 & -F_2 \end{pmatrix} \Rightarrow det \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{4._L} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -F_5 & -F_4 \\ F_4 & F_3 \end{pmatrix} \Rightarrow det \begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix} = -1$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{5._L} = \begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} F_6 & F_5 \\ -F_5 & -F_4 \end{pmatrix} \Rightarrow det \begin{pmatrix} 8 & 5 \\ -5 & -3 \end{pmatrix} = 1$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{6._L} = \begin{pmatrix} 8 & 5 \\ -5 & -3 \end{pmatrix} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -13 & -8 \\ 8 & 5 \end{pmatrix} = \begin{pmatrix} -F_7 & -F_6 \\ F_6 & F_5 \end{pmatrix} \Rightarrow det \begin{pmatrix} -13 & -8 \\ 8 & 5 \end{pmatrix} = -1$$

$\vdots$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} = \begin{pmatrix} (-1)^{n+1} F_{n+1} & (-1)^{n+1} F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix} \Rightarrow det \begin{pmatrix} (-1)^{n+1} F_{n+1} & (-1)^{n+1} F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix} = (-1)^{n+1}$$

$(-1)^{n+1} F_{n+1} (-1)^n F_{n-1} - (-1)^{n+1} F_n (-1)^n F_n = (-1)^{n+1}$

$(-1)^{2n+1} F_{n+1} F_{n-1} - (-1)^{2n+1} F_n^2 = (-1)^{n+1}$

$-F_{n+1} F_{n-1} + F_n^2 = (-1)^{n+1}$
$F_n^2 - F_{n+1} F_{n-1} = (-1)^{n+1}.$

If we examine to be true $F_n^2 - F_{n+1} F_{n-1} = (-1)^{n+1}$ identity with the inductive method

It is true for $n = 1$

$F_n^2 - F_{n+1} F_{n-1} = (-1)^{n+1},$

$F_1^2 - F_2 F_0 = (-1)^2.$

Let we assume that for $n = k$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{k._L} = \begin{pmatrix} (-1)^{k+1} F_{k+1} & (-1)^{k+1} F_k \\ (-1)^k F_k & (-1)^k F_{k-1} \end{pmatrix}$$

Let us show that true for $n = k + 1$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{k+1._L} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{k._L} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} (-1)^{k+1} F_{k+1} & (-1)^{k+1} F_k \\ (-1)^k F_k & (-1)^k F_{k-1} \end{pmatrix} ._L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{k+2}F_{k+1} - (-1)^{k+1}F_k & (-1)^{k+2}F_{k+1} \\ (-1)^{k+1}F_k - (-1)^k F_{k-1} & (-1)^{k+1}F_k \end{pmatrix}$$

$$(-1)^{k+2}F_{k+1} - (-1)^{k+1}F_k = (-1)^{k+1}[-F_{k+1} - F_k] = (-1)^{k+2}[F_{k+1} + F_k] = (-1)^{k+2}F_{k+2}$$

$$(-1)^{k+1}F_k - (-1)^k F_{k-1} = (-1)^k[-F_k - F_{k-1}] = (-1)^{k+1}[F_k + F_{k-1}] = (-1)^{k+1}F_{k+1}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{k+1._L} = \begin{pmatrix} (-1)^{k+2}F_{k+2} & (-1)^{k+2}F_{k+1} \\ (-1)^{k+1}F_{k+1} & (-1)^{k+1}F_k \end{pmatrix}$$

On the other hand

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{2._L} \cdot_L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{3._L} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \cdot_L \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{5._L}$$
$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{1._L} \cdot_L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{5._L} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot_L \begin{pmatrix} 8 & 5 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -13 & -8 \\ 8 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{6._L}$$
$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{2._L} \cdot_L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{4._L} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \cdot_L \begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -13 & -8 \\ 8 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{6._L}$$

For diffrent m and n numbers,

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{m._L} = \begin{pmatrix} (-1)^{m+1}F_{m+1} & (-1)^{m+1}F_m \\ (-1)^m F_m & (-1)^m F_{m-1} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} = \begin{pmatrix} (-1)^{n+1}F_{n+1} & (-1)^{n+1}F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix}$$

Then

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{m._L} \cdot_L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} = \begin{pmatrix} (-1)^{m+1}F_{m+1} & (-1)^{m+1}F_m \\ (-1)^m F_m & (-1)^m F_{m-1} \end{pmatrix} \cdot_L \begin{pmatrix} (-1)^{n+1}F_{n+1} & (-1)^{n+1}F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} (-1)^{m+n+1}F_{m+n+1} & (-1)^{m+n+1}F_{m+n} \\ (-1)^{m+n}F_{m+n} & (-1)^{m+n}F_{m+n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{m+n._L}$$

3 Results

The following equations can be given from the equality of the two matrices.

Proposition 3.1
1. $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$
2. $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$
3. $F_{m+n} = F_mF_{n+1} + F_{m-1}F_n$
4. $F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$
5. $\frac{F_m}{F_n} = \frac{F_{m-1}-F_{m+1}}{F_{n-1}-F_{n+1}}$
6.$F_{m+n+1} + F_{m-1}F_{n-1} = F_{m+n-1} + F_{m+1}F_{n+1}$
7. $F_{m+n} + F_{m+n-1} = F_{m+2}F_n + F_{m+1}F_{n-1}$
8. $2F_{m+n} = L_mF_n + L_nF_m$
9. $F_{2n+1} = F_{n+1}^2 + F_n^2$
10. $F_{2n} = F_nL_n$
11. $F_{2n-1} = F_n^2 + F_{n-1}^2$
12. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$
13. $F_{2n+1} = F_{n+2}F_n + F_{n+1}F_{n-1}$
14. $F_{2n+1} = F_nL_n + F_n^2 + F_{n-1}^2$
15. $2F_{2n} = F_nL_n + F_{n+1}^2 - F_{n-1}^2$
16. $F_{2n+1} + F_{2n-1} = F_{n+1}^2 + 2F_n^2 + F_{n-1}^2$
17. $2F_{2n+1} = F_{n+1}^2 + 2F_n^2 + F_{n-1}^2 + F_nL_n$
18. $2F_{2n+1} = L_{n+2}F_n + L_nF_{n-1}$
Proof:
The Lorentzian matrix product of

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{m._L} \cdot_L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} = \begin{pmatrix} (-1)^{m+1}F_{m+1} & (-1)^{m+1}F_m \\ (-1)^m F_m & (-1)^m F_{m-1} \end{pmatrix} \cdot_L \begin{pmatrix} (-1)^{n+1}F_{n+1} & (-1)^{n+1}F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} -(-1)^{m+1}F_{m+1}(-1)^{n+1}F_{n+1} + (-1)^{m+1}F_m(-1)^nF_n & -(-1)^{m+1}F_{m+1}(-1)^{n+1}F_n + (-1)^{m+1}F_m(-1)^nF_{n-1} \\ -(-1)^mF_m(-1)^{n+1}F_{n+1} + (-1)^mF_{m-1}(-1)^nF_n & -(-1)^mF_m(-1)^{n+1}F_n + (-1)^mF_{m-1}(-1)^nF_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{m+n+3}F_{m+1}F_{n+1} + (-1)^{m+n+1}F_mF_n & (-1)^{m+n+3}F_{m+1}F_n + (-1)^{m+n+1}F_mF_{n-1} \\ (-1)^{m+n+2}F_mF_{n+1} + (-1)^{m+n}F_{m-1}F_n & (-1)^{m+n+2}F_mF_n + (-1)^{m+n}F_{m-1}F_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{m+n+1}F_{m+n+1} & (-1)^{m+n+1}F_{m+n} \\ (-1)^{m+n}F_{m+n} & (-1)^{m+n}F_{m+n-1} \end{pmatrix}$$

Then;

1. $(-1)^{m+n+1}F_{m+n+1} = (-1)^{m+n+3}F_{m+1}F_{n+1} + (-1)^{m+n+1}F_mF_n$

If both sides of equality are simplified with $(-1)^{m+n+1}$

$F_{m+n+1} = (-1)^2F_{m+1}F_{n+1} + F_mF_n$

$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$

2.$(-1)^{m+n+1}F_{m+n} = (-1)^{m+n+3}F_{m+1}F_n + (-1)^{m+n+1}F_mF_{n-1}$

If both sides of equality are simplified with $(-1)^{m+n+1}$

$F_{m+n} = (-1)^2F_{m+1}F_n + F_mF_{n-1}$

$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

3. $(-1)^{m+n}F_{m+n} = (-1)^{m+n+2}F_mF_{n+1} + (-1)^{m+n}F_{m-1}F_n$

If both sides of equality are simplified with $(-1)^{m+n}$

$F_{m+n} = (-1)^2F_mF_{n+1} + F_{m-1}F_n$

$F_{m+n} = F_mF_{n+1} + F_{m-1}F_n$

4. $(-1)^{m+n}F_{m+n-1} = (-1)^{m+n+2}F_mF_n + (-1)^{m+n}F_{m-1}F_{n-1}$

If both sides of equality are simplified with $(-1)^{m+n}$

$F_{m+n-1} = (-1)^2F_mF_n + F_{m-1}F_{n-1}$

$F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$

5. From (2) and (3);

$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

$F_{m+n} = F_mF_{n+1} + F_{m-1}F_n$

$F_{m+1}F_n + F_mF_{n-1} = F_mF_{n+1} + F_{m-1}F_n$

$F_mF_{n-1} - F_mF_{n+1} = F_{m-1}F_n - F_{m+1}F_n$

$F_m(F_{n-1} - F_{n+1}) = F_n(F_{m-1} - F_{m+1})$

$\frac{F_m}{F_n} = \frac{F_{m-1}-F_{m+1}}{F_{n-1}-F_{n+1}}$

6. From (1) and (4);

$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$

$F_mF_n = F_{m+n+1} - F_{m+1}F_{n+1}$

$F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$

$F_mF_n = F_{m+n-1} - F_{m-1}F_{n-1}$

$F_{m+n+1} - F_{m+1}F_{n+1} = F_{m+n-1} - F_{m-1}F_{n-1}$

$F_{m+n+1} + F_{m-1}F_{n-1} = F_{m+n-1} + F_{m+1}F_{n+1}$

7. From (2) and (4);

$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

$F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$

$F_{m+n} + F_{m+n-1} = F_{m+1}F_n + F_mF_{n-1} + F_mF_n + F_{m-1}F_{n-1}$

$= (F_{m+1} + F_m)F_n + (F_m + F_{m-1})F_{n-1}$

$= F_{m+2}F_n + F_{m+1}F_{n-1}$

8. From (2) and (3);

$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

$F_{m+n} = F_mF_{n+1} + F_{m-1}F_n$

$2F_{m+n} = F_{m+1}F_n + F_mF_{n-1} + F_mF_{n+1} + F_{m-1}F_n$

$= (F_{m+1} + F_{m-1})F_n + (F_{n+1} + F_{n-1})F_m$

$= L_mF_n + L_nF_m$

$2F_{m+n} = L_mF_n + L_nF_m$

9. If $m = n$ is taken in (1) identity

$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$

$F_{2n+1} = F_{n+1}F_{n+1} + F_nF_n = F_{n+1}^2 + F_n^2$

$F_{2n+1} = F_{n+1}^2 + F_n^2$

10. If $m = n$ is taken in (2) identity

$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

$F_{2n} = F_{n+1}F_n + F_nF_{n-1}$

$= F_n(F_{n+1} + F_{n-1})$

$= F_nL_n$

$F_{2n} = F_nL_n$

11. If $m = n$ is taken in (4)

$F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$

$F_{2n-1} = F_nF_n + F_{n-1}F_{n-1} = F_n^2 + F_{n-1}^2$

$F_{2n-1} = F_n^2 + F_{n-1}^2$

12. If $m = n$ is taken in (6) identity

$F_{m+n+1} + F_{m-1}F_{n-1} = F_{m+n-1} + F_{m+1}F_{n+1}$

$F_{n+n+1} + F_{n-1}F_{n-1} = F_{n+n-1} + F_{n+1}F_{n+1}$

$F_{2n+1} + F_{n-1}^2 = F_{2n-1} + F_{n+1}^2$

$F_{n+1}^2 - F_{n-1}^2 = F_{2n+1} - F_{2n-1} = F_{2n}$

13. If $m = n$ is taken in (7) identity

$$F_{m+n} + F_{m+n-1} = F_{m+2}F_n + F_{m+1}F_{n-1}$$

$$F_{2n} + F_{2n-1} = F_{n+2}F_n + F_{n+1}F_{n-1}$$

$$F_{2n+1} = F_{n+2}F_n + F_{n+1}F_{n-1}$$

14. From (10) and (11);

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$

$$F_{2n} = F_nL_n$$

$$F_{2n} + F_{2n-1} = F_nL_n + F_n^2 + F_{n-1}^2$$

$$F_{2n+1} = F_nL_n + F_n^2 + F_{n-1}^2$$

15. From (10) and (12);

$$F_{2n} = F_nL_n$$

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$

$$F_{2n} + F_{2n} = F_nL_n + F_{n+1}^2 - F_{n-1}^2$$

$$2F_{2n} = F_nL_n + F_{n+1}^2 - F_{n-1}^2$$

16. From (9) and (11);

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$

$$F_{2n+1} + F_{2n-1} = F_{n+1}^2 + F_n^2 + F_n^2 + F_{n-1}^2$$

$$L_{2n} = F_{n+1}^2 + 2F_n^2 + F_{n-1}^2$$

17. From (10) and (16);

$$F_{2n} = F_nL_n$$

$$F_{2n+1} + F_{2n-1} = F_{n+1}^2 + 2F_n^2 + F_{n-1}^2$$

$$F_{2n+1} + F_{2n} + F_{2n-1} =$$

$$= F_{n+1}^2 + 2F_n^2 + F_{n-1}^2 + F_nL_n$$

$$2F_{2n+1} = F_{n+1}^2 + 2F_n^2 + F_{n-1}^2 + F_nL_n$$

18. From (13) and (14);

$$F_{2n+1} = F_{n+2}F_n + F_{n+1}F_{n-1}$$

$$F_{2n+1} = F_nL_n + F_n^2 + F_{n-1}^2$$

$$2F_{2n+1} = F_{n+2}F_n + F_{n+1}F_{n-1} + F_nL_n + F_n^2 + F_{n-1}^2$$

$$= (F_{n+2} + F_n)F_n + (F_{n+1} + F_{n-1})F_{n-1} + F_nL_n$$

$$= L_{n+1}F_n + L_nF_{n-1} + F_nL_n$$

$$= (L_{n+1} + L_n)F_n + L_nF_{n-1}$$

$$= L_{n+2}F_n + L_nF_{n-1}$$

$$2F_{2n+1} = L_{n+2}F_n + L_nF_{n-1}$$

Let us examine the quadratic equation of the matrix and its characteristic roots. Let A and I are matrices of type $n \times n$. The quadratic equation of A is $|A - xI| = 0$. Roots of equation are charecteristic roots of A. If we examine the quadratic equation and characteristic roots of the n^{th} power of the $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ matrix under the Lorentzian product, then

$$\left|\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} - xI\right| = \left|\begin{pmatrix} (-1)^{n+1}F_{n+1} & (-1)^{n+1}F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} (-1)^{n+1}F_{n+1} & (-1)^{n+1}F_n \\ (-1)^n F_n & (-1)^n F_{n-1} \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} (-1)^{n+1}F_{n+1} - x & (-1)^{n+1}F_n \\ (-1)^n F_n & (-1)^n F_{n-1} - x \end{pmatrix}\right|$$

If we take the determinant

$$\left|\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} - xI\right| =$$
$$= ((-1)^{n+1}F_{n+1} - x)((-1)^n F_{n-1} - x) - (-1)^{n+1}F_n(-1)^n F_n$$
$$= (-1)^{n+1}F_{n+1}(-1)^n F_{n-1} - x((-1)^{n+1}F_{n+1} + (-1)^n F_{n-1}) + x^2 - (-1)^{n+1}F_n(-1)^n F_n$$
$$= (-1)^{2n+1}F_{n+1}F_{n-1} - x((-1)^n F_{n-1} - (-1)^n F_{n+1}) + x^2 - (-1)^{2n+1}F_n^2$$
$$= -F_{n+1}F_{n-1} - x((-1)^n F_{n-1} - (-1)^n F_{n+1}) + x^2 + F_n^2$$
$$= -F_{n+1}F_{n-1} + F_n^2 - x((-1)^n F_{n-1} - (-1)^n F_{n+1}) + x^2$$
$$= (-1)^{n+1} - (-1)^n x(F_{n-1} - F_{n+1}) + x^2$$
$$= (-1)^{n+1} + (-1)^n x(F_{n+1} - F_{n-1}) + x^2$$
$$= (-1)^{n+1} + x(-1)^n F_n + x^2$$
$$= x^2 + x(-1)^n F_n + (-1)^{n+1}$$

So, we can reach the characteristic roots using the quadratic equation as

$$x = \frac{-(-1)^n F_n \mp \sqrt{F_n^2 - 4(-1)^{n+1}}}{2}$$
$$x = \frac{-(-1)^n F_n \mp \sqrt{F_n^2 + 4(-1)^n}}{2}$$
$x_1 = \frac{-(-1)^n F_n + \sqrt{F_n^2 + 4(-1)^n}}{2}$ and $x_2 = \frac{-(-1)^n F_n - \sqrt{F_n^2 + 4(-1)^n}}{2}$.

In this study, we examine unlike the classical matrix product, the n^{th} power of the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ was found using the Lorentzian matrix product. Quadratic equation $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$ is found with the n^{th} power of a matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.
Let m and n are positive integers,

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{m._L} \cdot_L \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{n._L} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{m+n._L}$$

equation is obtained under Lorentzian product and identities related to Fibonacci and Lucas numbers are obtained from the equations of two matrices. Some of these identities obtained with the help of the n^{th} powers of the matrices in the aforementioned studies. As a result, by the n^{th} power of a special matrix under Lorentzian product were achieved identities that accuracy previously known and different identities.

4 References

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