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Research Article

A New Class of Kantorovich-Type Operators

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ABSTRACT. The purpose of the paper called "A new class of Kantorovich-type operators", as the title says, is to introduce a new class of Kantorovich-type operators with the property that the test functions e_1 and e_2 are reproduced. Furthermore, in our approach, an asymptotic type convergence theorem, a Voronovskaja type theorem and two error approximation theorems are given. As a conclusion, we make a comparison between the classical Kantorovich operators and the new class of Kantorovich - type operators.

Keywords: Bernstein polynomials, Kantorovich operators, King operators, fixed points.

2010 Mathematics Subject Classification: 41A36, 41A60.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by e_j the monomial of j degree, $j \in \mathbb{N}_0$, $L_1([0,1]) = \{f | f : [0,1] \longrightarrow \mathbb{R}$ and f integrable Lebesgue on $[0,1]\}$.

In 1930, L. Kantorovich [7] constructed and studied the linear positive operators $K_m : L_1([0,1]) \longrightarrow C([0,1])$, defined for any $f \in L_1([0,1])$, $x \in [0,1]$ and $m \in \mathbb{N}$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt. \tag{1.1}$$

The operators (1.1) are known as Kantorovich operators and they preserve the test function e_0. Following the ideas from [3]-[6], in this paper we introduce a general class which preserves the test functions e_1 and e_2. For our operators a convergence theorem, a Voronovskaja-type theorem and two error approximation theorems are obtained.

The paper is organized as follows: in Section 2 we introduce some preliminary notions which we will use in the construction of the new type of Kantorovich operators, in Section 3 we will construct the new operators and in Section 4 we give an asymptotic type convergence theorem, a Voronovskaja type theorem, two error approximation theorems and a comparison between the classical Kantorovich operators and the new one.

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2. Preliminaries

In this section, we recall some notions and results which we will use in what follows. We consider I, J real intervals with the property $I \cap J \neq \emptyset$, let $E(I), F(J)$ be certain subsets of the space of all real functions defined on I, respectively J,

$$B(I) = \{f | f : I \to \mathbb{R}, f \textit{ bounded on } I\},$$
$$C(I) = \{f | f : I \to \mathbb{R}, f \textit{ continuous on } I\}$$

and

$$C_B(I) = B(I) \cap C(I).$$

For $x \in I$, we consider the function $\psi_x : I \to \mathbb{R}, \psi_x(t) = t - x, t \in I$. For any $m \in \mathbb{N}$, we consider the functions $\varphi_{m,k} : J \to \mathbb{R}$, with the property $\varphi_{m,k}(x) \geq 0$, for any $x \in J, k \in \{0, 1, ..., m\}$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$, $k \in \{0, 1, ..., m\}$. For $m \in \mathbb{N}$, we define the operators $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f). \tag{2.1}$$

Remark 2.1. *The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.*

For any $f \in E(I), x \in I \cap J$ and for $i \in \mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^i). \tag{2.2}$$

In the following, let s be a fixed even natural number and we suppose that the operators $(L_m)_{m\in\mathbb{N}}$ verifies the following conditions:
there exists the smallest $\alpha_s, \alpha_{s+2} \in [0, \infty)$ such that

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.3}$$

for any $x \in I \cap J, j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.4}$$

If $I \subset \mathbb{R}$ is a given interval and $f \in C_B(I)$, then the first order modulus of smoothness of f is the function $\omega_1(f; \cdot) : [0, +\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by $\omega_1(f, \delta) = sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}$.

Theorem 2.1. ([8]) *Let $f : I \longrightarrow \mathbb{R}$ be a function. If $x \in I \cap J$ and f is s times derivable function on I, the function $f^{(s)}$ is continuous on I, then*

$$\lim_{m\longrightarrow\infty} m^{s-\alpha_s} \Big((L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \Big) = 0. \tag{2.5}$$

If f is a s times differentiable function on I, the function $f^{(s)}$ is continuous on I and there exists $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ such that for any natural number $m \geq m(s)$ and for any $x \in I \cap J$ we have

$$\frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{2.6}$$

where $j \in \{s, s+2\}$, then the convergence given in (2.5) is uniformly on $I \cap J$ and

$$m^{s-\alpha_s} \Big| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \Big| \tag{2.7}$$

$$\leq \frac{1}{s!}(k_s + k_{s+2})\omega_1\Big(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\Big)$$

for any $x \in I \cap J$ and $m \geq m(s)$.

Let φ_x be defined by

$$\varphi_x(t) = |t - x|, t \in I, x \in I. \tag{2.8}$$

Theorem 2.2. [9] *Let $L : C(I) \longrightarrow B(I)$ be a linear positive operator. Let φ_x be defined by (2.8).*
(i) If $f \in C_B(I)$, then for every $x \in I$ and $\delta > 0$, one has

$$|(Lf)(x) - f(x)| \leq |f(x)|\,|(Le_0)(x) - 1| + \Big((Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)\cdot(L\varphi_x^2)(x)}\Big)\,\omega_1(f;\delta).$$

(ii) If f is differentiable on I and $f' \in C_B(I)$, then for every $x \in I$ and $\delta > 0$, one has

$$|(Lf)(x) - f(x)| \leq |f(x)|\,|(Le_0)(x) - 1| + |f'(x)||(Le_1)(x) - x(Le_0)(x)| + \sqrt{(L\varphi_x^2)(x)}\,\Big(\sqrt{(Le_0)(x)} + \delta^{-1}\cdot\sqrt{(L\varphi_x^2)(x)}\Big)\,\omega_1(f';\delta).$$

3. A NEW CLASS OF KANTOROVICH-TYPE OPERATORS

Let $a_m, b_m : J \longrightarrow \mathbb{R}$ be functions such that $a_m(x) \geq 0$, $b_m(x) \geq 0$ for any $x \in J$ and $m \in \mathbb{N}_1$, where J and $\mathbb{N}_1 \subset \mathbb{N}$ will be determined later. We define the operators of the following form

$$(K_m^* f)(x) = (m+1)\sum_{k=0}^{m}\binom{m}{k}(a_m(x))^k(b_m(x))^{m-k}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt \tag{3.1}$$

for any $x \in J$, $m \in \mathbb{N}_1$ and $f \in L_1([0,1])$. Then, we get

$$(K_m^* e_0)(x) = \big(a_m(x) + b_m(x)\big)^m, \tag{3.2}$$

$$(K_m^* e_1)(x) = \frac{m}{m+1}a_m(x)\big(a_m(x) + b_m(x)\big)^{m-1} + \frac{1}{2(m+1)}\big(a_m(x) + b_m(x)\big)^m \tag{3.3}$$

and

$$\begin{aligned}(K_m^* e_2)(x) = &\frac{m(m-1)}{(m+1)^2}a_m^2(x)\big(a_m(x) + b_m(x)\big)^{m-2}\\ &+ \frac{2m}{(m+1)^2}a_m(x)\big(a_m(x) + b_m(x)\big)^{m-1}\\ &+ \frac{1}{3(m+1)^2}\big(a_m(x) + b_m(x)\big)^m\end{aligned} \tag{3.4}$$

for any $x \in J$ and $m \in \mathbb{N}_1$.

In what follows, we impose the additional condition to be fulfilled by our operators

$$(K_m^* e_0)(x) = 1 + u_m(x), \tag{3.5}$$

where $x \in J$, $m \in \mathbb{N}_1$ and $u_m : J \longrightarrow \mathbb{R}$.

Remark 3.1. *We want that $K_m^*, m \in \mathbb{N}_1$ be positive operators, then from $(K_m^* e_0) \geq 0$ and (3.5), we have*

$$1 + u_m(x) \geq 0, x \in J, m \in \mathbb{N}_1. \tag{3.6}$$

We will show in Lemma 3.3 that $1+u_m(x)>0, x\in J, m\in\mathbb{N}_1$. From (3.2), we get

$$(a_m(x)+b_m(x))^m = 1+u_m(x), x\in J, m\in\mathbb{N}_1, \tag{3.7}$$

from where

$$a_m(x)+b_m(x) = (1+u_m(x))^{\frac{1}{m}}, x\in J, m\in\mathbb{N}_1. \tag{3.8}$$

The next conditions will be read as follows

$$(K_m^* e_1)(x) = x \tag{3.9}$$

and

$$(K_m^* e_2)(x) = x^2 \tag{3.10}$$

for any $x\in J$ and $m\in\mathbb{N}_1$.

From (3.3), (3.8) and (3.9), we get

$$a_m(x) = \frac{m+1}{m}(1+u_m(x))^{\frac{1-m}{m}}\left(x-\frac{1}{2(m+1)}(1+u_m(x))\right), \tag{3.11}$$

$x\in J, m\in\mathbb{N}_1$.

From (3.8) and (3.11) we obtain

$$\begin{aligned} b_m(x) =&(1+u_m(x))^{\frac{1}{m}}\left(1-\frac{m+1}{m}\cdot\frac{1}{1+u_m(x)}\cdot\right.\\ &\left.\left(x-\frac{1}{2(m+1)}(1+u_m(x))\right)\right), \end{aligned} \tag{3.12}$$

$x\in J, m\in\mathbb{N}_1$.

Because $a_m(x)\geq 0, b_m(x)\geq 0, x\in J, m\in\mathbb{N}_1$, from (3.7), (3.11) and (3.12) we get

$$x-\frac{1}{2(m+1)}(1+u_m(x))\geq 0$$

and

$$1-\frac{m+1}{m}\cdot\frac{1}{1+u_m(x)}\left(x-\frac{1}{2(m+1)}(1+u_m(x))\right)\geq 0,$$

$x\in J, m\in\mathbb{N}_1$, from where we obtain

$$\frac{2(m+1)}{2m+1}x-1\leq u_m(x)\leq 2(m+1)x-1, \tag{3.13}$$

$x\in J, m\in\mathbb{N}_1$.

From (3.4), (3.8), (3.10) and (3.11) it follows

$$\begin{aligned} &(-5m-3)u_m^2(x)+\\ &(-12m(m+1)^2x^2+12(m+1)^2x-2(5m+3))u_m(x)+\\ &(-12(m+1)^2x^2+12(m+1)^2x-(5m+3))=0. \end{aligned} \tag{3.14}$$

The relation (3.14) is an equation in $u_m(x)$ with the discriminant

$$\Delta_m(x) = 48(m+1)^2x^2\Big(3(m+1)^2(mx-1)^2+(5m+3)(m-1)\Big). \tag{3.15}$$

The discriminant $\Delta_m(x)$, after some calculation, has the following form

$$\Delta_m(x) = \Big(12m(m+1)^2x^2-12(m+1)^2x\Big)^2+4(5m+3)12(m+1)^2x^2(m-1), \tag{3.16}$$

so for $x\neq 0$ and $m\in\mathbb{N}$ we obtain that $\Delta_m(x)>0$.

Then, in the above conditions, we have the solutions of the equation (3.14)

(3.17)
$$u_{m,1}(x) = \frac{-6m(m+1)^2x^2 + 6(m+1)^2x - (5m+3)}{5m+3} - \frac{2(m+1)x\sqrt{9(m+1)^2(mx-1)^2 + 3(5m+3)(m-1)}}{5m+3}$$

and

(3.18)
$$u_{m,2}(x) = \frac{-6m(m+1)^2x^2 + 6(m+1)^2x - (5m+3)}{5m+3} + \frac{2(m+1)x\sqrt{9(m+1)^2(mx-1)^2 + 3(5m+3)(m-1)}}{5m+3}$$

for any $x \in J, m \in \mathbb{N}_1$.

For $u_{m,1}(x)$, we have $\lim\limits_{m\longrightarrow\infty} u_{m,1}(x) = -\infty$ then $u_{m,1}(x)$ does not satisfy the relation (3.6), so from the relation (3.18) follows that $u_m(x) = u_{m,2}(x)$.

Lemma 3.1. *The relation (3.13) happens for any $x \in J, m \in \mathbb{N}_1$ if and only if*

(3.19)
$$\frac{2}{3(m+1)} \le x \le \frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}.$$

Proof. After some calculation, it follows from the relations (3.13) and (3.18). □

Remark 3.2. *(i) The following inequalities state*

$$\frac{2}{3(m+1)} > 0$$

and

$$\frac{2(3m^2+3m+1)}{3(m+1)(2m+1)} < 1$$

for $m \in \mathbb{N}$.

(ii) The sequence $\left(\frac{2}{3(m+1)}\right)_{m\in\mathbb{N}}$ is decreasing and the sequence $\left(\frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}\right)_{m\in\mathbb{N}}$ is increasing.

(iii) From (ii), the following relations state

$$\frac{2}{3(m+1)} \le \frac{1}{3}$$

and

$$\frac{7}{9} \le \frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}, m \in \mathbb{N}.$$

(iv) From (3.19) and (iii) follows $\frac{1}{3} \le x \le \frac{7}{9}$, so the operators K_m^* are positive for $m \in \mathbb{N}$.

(v) If $c \in \left(0, \frac{1}{3}\right)$, because $\lim\limits_{m\longrightarrow\infty} \dfrac{2}{3(m+1)} = 0$ it follows that there exists $m(c) \in \mathbb{N}$ such that $\frac{2}{3(m+1)} \le c$, for any $m \in \mathbb{N}$ and $m \ge m(c)$.

(vi) If $d \in \left(\frac{7}{9}, 1\right)$, because $\lim\limits_{m\longrightarrow\infty} \dfrac{2(3m^2+3m+1)}{3(m+1)(2m+1)} = 1$ follows that there exists $m(d) \in \mathbb{N}$ such that $d \le \frac{2(3m^2+3m+1)}{3(m+1)(2m+1)}$, for any $m \in \mathbb{N}$ and $m \ge m(d)$.

(vii) Let $\mathbb{N}_1 = \{m \in \mathbb{N} \mid m \ge \max(m(c), m(d)) = m(c,d)\}$.

Lemma 3.2. *If $0 < c < d < 1$, then exists $m(c,d) \in \mathbb{N}$ such that the operators K_m^* are positive on $[c,d]$, for $m \in \mathbb{N}, m \ge m(c,d)$.*

Proof. It follows from Lemma 3.1 and Remark 3.2. □

Lemma 3.3. *The inequality*

$$1 + u_m(x) > 0 \tag{3.20}$$

holds for any $x \in [c, d]$ *and* $m \in \mathbb{N}_1$.

Proof. We take into account the relation (3.18). □

Let c and d be real numbers with $0 < c < d < 1$, then $I = [0, 1]$, $J = [c, d]$,

$$\begin{aligned}\varphi_{m,k}(x) = {} & (m+1)(1+u_m(x))^{1-k} \\ & \times \left(\frac{m+1}{m} \left(x - \frac{1}{2(m+1)}(1+u_m(x)) \right) \right)^k \\ & \times \left(1 - \frac{m+1}{m(1+u_m(x))} \left(x - \frac{1}{2(m+1)}(1+u_m(x)) \right) \right)^{m-k}\end{aligned}$$

and

$$A_{m,k}(f) = \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt,$$

$f \in L_1([0, 1])$, $x \in [c, d]$, $m \in \mathbb{N}_1$.

Then the operators (3.1) become

$$\begin{aligned}(K^*_m f)(x) = {} & (m+1) \sum_{k=0}^{m} \binom{m}{k} (1+u_m(x))^{1-k} \\ & \times \left(\frac{m+1}{m} \left(x - \frac{1}{2(m+1)}(1+u_m(x)) \right) \right)^k \\ & \times \left(1 - \frac{m+1}{m(1+u_m(x))} \left(x - \frac{1}{2(m+1)}(1+u_m(x)) \right) \right)^{m-k} \\ & \times \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt,\end{aligned} \tag{3.21}$$

$x \in [c, d]$, $m \in \mathbb{N}_1$.

Lemma 3.4. *For* $x \in [c, d]$ *and* $m \in \mathbb{N}_1$, *the following identities*

$$(T_{m,0} K^*_m)(x) = 1 + u_m(x), \tag{3.22}$$

$$(T_{m,1} K^*_m)(x) = -mxu_m(x), \tag{3.23}$$

$$(T_{m,2} K^*_m)(x) = m^2 x^2 u_m(x) \tag{3.24}$$

hold.

Proof. We take (2.2), (3.9) and (3.10) into account. □

Lemma 3.5. *For* $x \in [c, d]$, $m \in \mathbb{N}_1, m \geq m_*, m_* = \max\big(m(0), m(2)\big)$, *we have*

$$\begin{aligned} &\alpha_0 = 0, && \alpha_2 = 1, \\ &B_0(x) = 1, && B_2(x) = x(1-x), \\ &k_0 = 1, && k_2 = \frac{1}{4}. \end{aligned}$$

Proof. We have that

$$(T_{m,0}K_m^*)(x) = 1 + u_m(x),$$

then

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,0}K_m^*)(x)}{m^0} = 1,$$

so from relations (2.3), (2.4) and (2.6) we get $\alpha_0 = 0, B_0(x) = 1$ and $k_0 = 1$ for $x \in [c,d]$, $m \in \mathbb{N}_1, m \geq m(0)$.

We have that

$$(T_{m,2}K_m^*)(x) = m^2x^2u_m(x).$$

Because

$$\lim_{m\longrightarrow\infty} mu_m(x) = \frac{1-x}{x},$$

we obtain

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,2}K_m^*)(x)}{m^1} = x(1-x).$$

Then from relations (2.3), (2.4) and (2.6), we get $\alpha_2 = 1, B_2(x) = x(1-x)$ and $k_2 = \frac{1}{4}$ for $x \in [c,d]$, $m \in \mathbb{N}_1, m \geq m(2)$. □

4. Properties for the new class of Kantorovich type operators

In this section, we present some properties of the new class of Kantorovich type operators, where c and d are real fixed numbers, $0 < c < d < 1$.

Theorem 4.3. *If $f \in C([0,1])$, then*

$$\lim_{m\longrightarrow\infty} (K_m^*f)(x) = f(x) \tag{4.1}$$

uniformly on $[c,d]$ and

$$|(K_m^*f)(x) - f(x)| \leq |f(x)| \cdot |u_m(x)| + \frac{5}{4} \cdot \omega_1\left(f; \frac{1}{\sqrt{m}}\right), \tag{4.2}$$

for any $x \in [c,d]$ and $m \in \mathbb{N}_1$.

Proof. From (2.7), for $\alpha_0 = 0$, $\alpha_2 = 2$, $k_0 = 0$ and $k_2 = \frac{1}{4}$, we get

$$|(K_m^*f)(x) - f(x)(1+u_m(x))| \leq \frac{5}{4} \cdot \omega_1\left(f; \frac{1}{\sqrt{m}}\right), \tag{4.3}$$

for any $x \in [c,d]$, $m \in \mathbb{N}_1$, $m \geq m_*$ which is equivalent with (4.2).

□

Theorem 4.4. *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function. If f is two times differentiable on $[0,1]$, the function $f^{(2)}$ is continuous on $[0,1]$ and $x \in [c,d]$, then*

$$\begin{aligned}\lim_{m\longrightarrow\infty} m((K_m^*f)(x) - f(x)) &= \frac{1-x}{x} f(x) + (x-1)f^{(1)}(x) \\ &+ \frac{x(1-x)}{2} f^{(2)}(x),\end{aligned} \tag{4.4}$$

for any $x \in [c,d]$, $m \in \mathbb{N}_1$.

Proof. Using the relation (2.5) and Lemma (3.1), the relation (4.4) follows. □

The relation (4.4) is a Voronovskaja type theorem.

Theorem 4.5. *If* $f \in C([0,1])$, *then*

$$|(K_m^* f)(x) - f(x)| \leq |f(x)| \cdot |u_m(x)| + 3 \cdot \omega_1(f;\delta_1) \tag{4.5}$$

for any $x \in [c,d]$, $m \in \mathbb{N}_1$, *where* $\delta_1 = \sqrt{\frac{mx+1}{m^2}}$.

Proof. Using Theorem 2.2 (i), from relation (3.2) for $\delta = \sqrt{(K_m^* e_0)(x) \cdot (K_m^* \varphi_x^2)(x)}$, we have

$$|(K_m^* f)(x) - f(x)| \leq |f(x)| \cdot |u_m(x)| + 3 \cdot \omega_1(f;\delta_1) \tag{4.6}$$

for any $x \in [c,d], m \in \mathbb{N}_1$.

After some calculus, we get $\delta = \sqrt{(1+u_m(x)) \cdot x^2 \cdot u_m(x)}$. Because $\lim\limits_{m \longrightarrow \infty} m u_m(x) = \dfrac{1-x}{x}$, we have that there exists $m(1) \in \mathbb{N}_1$ such that $u_m(x) < \frac{1}{mx}$ for any $x \in [c,d], m \geq m(1), m(1) \in \mathbb{N}_1$. Then $\delta < \sqrt{\frac{mx+1}{m^2}} = \delta_1$ and from (4.6) we obtain (4.5).

We observe that for the genuine Kantorovich operators we have the relation $|(K_m f)(x) - f(x)| \leq 2 \cdot \omega_1\left(f; \frac{1}{2\sqrt{m+1}}\right)$ and for our operators we have the relation (4.5) and if we make a comparison between this two results, we remark that $\delta_1 < \frac{1}{2\sqrt{m+1}}$, for any $x \in [c,h]$, $m \geq m_1$, $m_1 \in \mathbb{N}_1$, where h is a real number that has the following properties:
(i) $0 < c < h < d$ and $h < \frac{1}{4}$;
(ii) there exists $m(h) \in \mathbb{N}$ such that for any $m \geq m(h)$, the inequality $h < \frac{m^2-4m-4}{4m^2+4m}$ holds, where $\delta_1 < \frac{1}{2\sqrt{m+1}}$ is equivalent with $x < \frac{m^2-4m-4}{4m^2+4m}$;
(iii) $m_1 = max\Big(m(c), m(h), m(d)\Big), m_1 \in \mathbb{N}_1$.

□

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