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On Some Integral Type Inequality on Time Scales

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Abstract: Time scales have been in the study area of many mathematicians for the last 30 years. Some of these studies are inequalities and dynamic equations. And also, inequalities and dynamic equations, differential calculus, difference calculus and quantum calculus contributed to the solution of many problems in various branches of science. Dynamic equations and inequalities on time scales have many applications in quantum mechanics, neural networks, heat transfer, electrical engineering, optics, economics and population dynamics. It is possible to give an example from the economics, seasonal investments and incomes. In this study, we will prove a special case of inequalities Minkowski's integral type on time scale via the delta integral.

Keywords: Integral type inequality, Time scales, Integral inequalities.

1 Introduction

The concept of dynamic equations in time scales was launched by Stefan Hilger [1]. Recently, the plurality of applications has an accelerating effect on the development of mathematical inequalities and dynamic equations. This caused the attention of researchers in the literature. And they have demonstrated various aspects of integral inequalities [2, 13]-[18]-[19]. The most important examples of time scale studies are differential calculus, difference calculus and quantum calculus [12]. Ozkan et al. [10] demonstrated the extensions of some integral inequalities on time scales. Yang [13] obtained a extension of the diamond alfa integral Holder's inequality. Tuna and Kutukcu [14] have had some general conclusions about Hardy's integral inequalities by using Holder inequalities with delta integral on time scales. In 2013, Chen demonstrated some generalizations of the Minkowski's integral inequality [15].

Our aim of this article is to demonstrate a special case of inequalities Minkowski's integral type on time scale via the delta integral.

2 Auxillary statements and definitions

In this section, some statements will be given that will be necessary for our main result. For more detailed information, the reader can refer to the references [1]-[19].

Definition 2.1. [16] The mappings $\sigma, \rho : T \to T$ defined by $\sigma(t) = \inf s \in T : s > t$, $\rho(t) = \sup s \in T : s > t$, for $t \in T$ (T is a time scale and a nonempty closed subset of R). Respectively, $\sigma(t)$ is forward jump operator and $\rho(t)$ is backward jump operator. $[a, b]$ is an arbitrary interval on time scale T. And $[a, b]_T$ is denoted by $[a, b]T$. If $\sigma(t) > t$, then t is right-scattered and if $\rho(t) < t$, then t is left-scattered and if $\rho(t) = t$, then t is called left-dense.

If $\sigma(t) > t$, then t is right-scattered and if $\sigma(t) = t$, then t is called right-dense. If $\rho(t) < t$, then t is left-scattered and if $\rho(t) = t$, then t is called left-dense. The graininess function μ is defined by $\mu(t) = \sigma(t) - t$. Let $f : T \to R$ be any function. The notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. The constant $t \in T$ and let $\Theta : T \to R$. Define $\Theta^{\Delta}(t)$ to be the number with the property that given any $\varepsilon > 0$ there is a neighborhood V of t with

$$|[\Theta(\sigma(t)) - \Theta(s)] - \theta^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in V$.

A function $f : T \to R$ is said to be right-dense continuous (rd-continuous) provided f is continuous at right-dense points and at left-dense points in T. The set of all rd-continuous functions is denoted by $C_{rd}(T)$.
Assume that $f : T \to R$ and let the constant $s \in T$.
(i) If f is differentiable at s, then f is continuous at s.
(ii) If f is continuous at s and s is right-scattered, then f is differentiable at s with

$$f^{\Delta}(s) = \frac{f(\sigma(s)) - f(s)}{\mu(s)}.$$

(iii) If f is differentiable and s is right-dense, then

$$f^{\Delta}(s)=\lim_{t\to s}\frac{f(s)-f(t)}{s-t}.$$

(iv) If f is differentiable at s, then $f(\sigma(s))=f(s)+\mu(s)f^{\Delta}(s)$ (For details see [16]).

Definition 2.2. [17] If $H:T\to R$ is defined a $\Delta-$ antiderivative of $h:T\to R$, then $H^{\Delta}=h(t)$ holds for $\forall t\in T$. And we define the $\Delta-$ integral of h by $\int_{s^t} h(\tau)\Delta\tau=H(t)-H(s)$ for $s,t\in T$.

Remark 2.3. [17] If $T=R$, then $\sigma(s)=s,\quad \mu(s)=0,\quad g^{\Delta}(s)=g'(s),\quad \int_a^b g(s)\Delta s=\int_a^b g(s)ds.$
If $T=Z$, then $\sigma(s)=s+1,\quad \mu(s)=1,\quad g^{\Delta}(s)=\Delta g(s),\quad \int_a^b g(s)\Delta s=\sum_{s=a}^{b-1} g(s).$
If $T=\lambda Z$, $\lambda>0$, then $\sigma(s)=s+\lambda,\quad \mu(s)=\lambda$, and

$$\theta^{\Delta}(s)=\Delta_\lambda\theta(s)=\frac{\theta(s+\lambda)-\theta(s)}{\lambda},\quad \int_a^b g(s)\Delta s=\sum_{k=0}^{\frac{b-a}{\lambda}-1} g(a+k\lambda)\lambda.$$

And if $T=\{s:s=p^k,k\in N_0,p>1\}$, then $\sigma(s)=ps,\mu(s)=(p-1)s$,

$$\Theta^{\Delta}(s)=\Delta\Theta(s)=\frac{\theta(ps)-\theta(s)}{(p-1)s},\quad \int_{s_0}^{\infty} g(s)\Delta s=\sum_{k=n_0}^{\infty} g(p^k)\mu(p^k),$$

where $s_0=p^{n_0}$, and if $T=N_0^2=\{n^2:n\in N_0\}$, then $\sigma(s)=(\sqrt{s}+1)^2$,

$$\mu(s)=1+2\sqrt{s},\quad \Delta_N\theta(s)=\frac{\theta(\sigma(s))-\theta(s)}{1+2\sqrt{s}}.$$

If $G^{\Delta}(s)=g(s)$, then delta integral of g is defined by $\int_a^s g(t)\Delta t=G(s)-G(a)$. It can be shown (see [17]) that if $g\in C_{rd}(T)$, then delta integral $G(s)=\int_{s_0}^s g(t)\Delta t$ exists, $s_0\in T$, and satisfies $G^{\Delta}(s)=g(s)$, $s\in T$. We will make use of the following product gf and quotient g/f rules for the derivative (where $ff^{\sigma}\neq 0$, here $f^{\sigma}=f\circ\sigma$ of two differentiable functions g,f (for details see [16, 17]).

$$(gf)^{\Delta}=g^{\Delta}f+g^{\sigma}f^{\Delta}=gf^{\Delta}+g^{\Delta}f^{\sigma},\quad and\quad \left(\frac{g}{f}\right)=\frac{g^{\Delta}f-gf^{\Delta}}{ff^{\sigma}}.$$

A function $\pi:T\to R$ is regressive provided $1+\mu(s)\pi(s)\neq 0,\quad s\in T$.
The Keller's chain rule [17, Theorem 1.90] defined by

$$(\Theta^{\delta}(s))^{\Delta}=\delta\int_0^1[g\Theta^{\sigma}+(1-g)\Theta]^{\delta-1}dg\Theta^{\Delta}(s),$$

Using $f^{\sigma}(s)=f(s)+\mu(s)f^{\Delta}(s)$, we obtain

$$(\Theta^{\delta}(s))^{\Delta}=\delta\int_0^1[\Theta+g\mu(s)\Theta^{\Delta}(s)]^{\delta-1}dg\Theta^{\Delta}(s).$$

The integration is given by

$$\int_a^b x(s)y^{\Delta}(s)\Delta s=[x(s)y(s)]_a^b-\int_a^b x^{\Delta}(s)y^{\sigma}(s)\Delta s.$$

The inverse Holder inequality to help us with our results is defined as follows. Let $a,b\in T$. For $x,y\in C_{rd}(T,R)$, we have

$$\left[\int_a^b|x(s)|^q\Delta s\right]^{\frac{1}{q}}\left[\int_a^b|y(s)|^p\Delta s\right]^{\frac{1}{p}}\leq C_p\int_a^b x(s)y(s)\Delta s,$$

where $p>1$ and $1/p+1/q=1$.

3 Main Result

Theorem 3.1. *If h is $\Delta-$ integrable on [a,b], then $|h|$ is $\Delta-$integrable on [a,b] and we have*

$$|\int_a^b h(\gamma)\Delta\gamma|\leq\int_a^b|h(\gamma)|\Delta\gamma.$$

Proof. For details of proof see [Theorem 2, 14]

Theorem 3.2. *Two mappings $g, h : I \to R$ are $\Delta-$ integrable functions on $I = [a, b] \in T$ with $1 < l \leq g^p, \quad h^p \leq L < \infty$. If $p > 1$, then we have*

$$\left(\int_a^b |g(\gamma)|^p \Delta\gamma\right)^{1/p} + \left(\int_a^b |h(\gamma)|^p \Delta\gamma\right)^{1/p} \leq 2\,(L/l)^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \Delta\gamma\right)^{1/p}. \tag{1}$$

Proof. We know that if $0 < l \leq g^p \leq L < \infty$, then we have

$$l^{1/p} \leq g \leq L^{1/p} \tag{2}$$

Similarly, if $0 < l \leq h^p \leq L < \infty$, then we have

$$l^{1/p} \leq h \leq L^{1/p} \tag{3}$$

Respectively, if we multiply both sides of (2) and (3) by $\left(\int_a^b |g(\gamma)|^p \Delta\gamma\right)^{1/p}$ and $\left(\int_a^b |h(\gamma)|^p \Delta\gamma\right)^{1/p}$, then we have

$$l^{1/p} \left(\int_a^b |g(\gamma)|^p \Delta\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |g(\gamma)|^p \Delta\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \Delta\gamma\right)^{1/p} \tag{4}$$

and

$$l^{1/p} \left(\int_a^b |h(\gamma)|^p \Delta\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |h(\gamma)|^p \Delta\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \Delta\gamma\right)^{1/p} \tag{5}$$

Now, if we add (4) and (5) inequalities to each other, then we have

$$l^{1/p} \left[\left(\int_a^b |h(\gamma)|^p \Delta\gamma\right)^{1/p} + \left(\int_a^b |g(\gamma)|^p \Delta\gamma\right)^{1/p}\right] \leq 2L^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \Delta\gamma\right)^{1/p} \tag{6}$$

Thus, we have proved (1) inequality.

Theorem 3.3. *Two mappings $g, h : I \to R$ are $\nabla-$ integrable functions on $I = [a, b] \in T$ with $1 < l \leq g^p, \quad h^p \leq L < \infty$. If $p > 1$, then we have*

$$\left(\int_a^b |g(\gamma)|^p \nabla\gamma\right)^{1/p} + \left(\int_a^b |h(\gamma)|^p \nabla\gamma\right)^{1/p} \leq 2\,(L/l)^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \nabla\gamma\right)^{1/p}. \tag{7}$$

Proof. The proof of the theorem can be made analogous to the proof of Theorem 3.2 by using the properties of the $\nabla-$derivative.

We know that if $0 < l \leq g^p \leq L < \infty$, then we have

$$l^{1/p} \leq g \leq L^{1/p} \tag{8}$$

Similarly, if $0 < l \leq h^p \leq L < \infty$, then we have

$$l^{1/p} \leq h \leq L^{1/p} \tag{9}$$

Respectively, if we multiply both sides of (8) and (9) by $\left(\int_a^b |g(\gamma)|^p \nabla\gamma\right)^{1/p}$ and $\left(\int_a^b |h(\gamma)|^p \nabla\gamma\right)^{1/p}$, then we have

$$l^{1/p} \left(\int_a^b |g(\gamma)|^p \nabla\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |g(\gamma)|^p \nabla\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \nabla\gamma\right)^{1/p} \tag{10}$$

and

$$l^{1/p} \left(\int_a^b |h(\gamma)|^p \nabla\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |h(\gamma)|^p \nabla\gamma\right)^{1/p} \leq L^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \nabla\gamma\right)^{1/p} \tag{11}$$

Now, if we add (10) and (11) inequalities to each other, then we have

$$l^{1/p} \left[\left(\int_a^b |h(\gamma)|^p \nabla\gamma\right)^{1/p} + \left(\int_a^b |g(\gamma)|^p \nabla\gamma\right)^{1/p}\right] \leq 2L^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \nabla\gamma\right)^{1/p} \tag{12}$$

Thus, we have proved (7) inequality.

Theorem 3.4. *Two mappings $g, h : I \to R$ are $\Diamond_\alpha-$ integrable functions on $I = [a, b] \in T$ with $1 < l \leq g^p, \quad h^p \leq L < \infty$. If $p > 1$, then we have*

$$\left(\int_a^b |g(\gamma)|^p \Diamond_\alpha\gamma\right)^{1/p} + \left(\int_a^b |h(\gamma)|^p \Diamond_\alpha\gamma\right)^{1/p} \leq 2\,(L/l)^{1/p} \left(\int_a^b |g(\gamma) + h(\gamma)|^p \Diamond_\alpha\gamma\right)^{1/p}. \tag{13}$$

Proof. The proof of the theorem can be made analogous to the proof of Theorem 3.3 by using the properties of the $\Diamond_\alpha-$derivative.

Let $f(t)$ be differentiable on T for $\forall\alpha, t\in T$. Then, we define $f^{\Diamond_\alpha}(t)$ by

$$f^{\Diamond_\alpha}(t)=\alpha f^{\Delta}(t)+(1-\alpha)f^{\nabla}(t)$$

for $0\leq\alpha\leq 1$ (for details see [17]).

If we get $\alpha, b, t\in T$ and $f: T\longrightarrow R$, then we have

$$\int_b^t f(\gamma)\Diamond_\alpha\gamma=\alpha\int_b^t f(\gamma)\Delta\gamma+(1-\alpha)\int_b^t f(\gamma)\nabla\gamma$$

for $0\leq\alpha\leq 1$ (for details see [17]).

We know that if $0<l\leq g^p\leq L<\infty$, then we have

$$l^{1/p}\leq g\leq L^{1/p} \tag{14}$$

Similarly, if $0<l\leq h^p\leq L<\infty$, then we have

$$l^{1/p}\leq h\leq L^{1/p} \tag{15}$$

Respectively, if we multiply both sides of (14) and (15) by $\left(\int_a^b|g(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}$ and $\left(\int_a^b|h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}$, then we have

$$l^{1/p}\left(\int_a^b|g(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}\leq L^{1/p}\left(\int_a^b|g(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}\leq L^{1/p}\left(\int_a^b|g(\gamma)+h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p} \tag{16}$$

and

$$l^{1/p}\left(\int_a^b|h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}\leq L^{1/p}\left(\int_a^b|h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}\leq L^{1/p}\left(\int_a^b|g(\gamma)+h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p} \tag{17}$$

Now, if we add (16) and (17) inequalities to each other, then we have

$$l^{1/p}\left[\left(\int_a^b|h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}+\left(\int_a^b|g(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p}\right]\leq 2L^{1/p}\left(\int_a^b|g(\gamma)+h(\gamma)|^p\Diamond_\alpha\gamma\right)^{1/p} \tag{18}$$

Thus, we have proved (13) inequality.

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