Conference Proceeding Science and Technology, 3(1), 2020, 136–140

Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)

An Investigation on Fractional Maximal Operator in Time Scales

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Lutfi Akin[1] and Yusuf Zeren[2]

[2] Department of Mathematics,Yildiz Technical University, Istanbul, Turkey, ORCID:0000-0001-8346-2208
** Corresponding Author E-mail: yzeren@yildiz.edu.tr*

Abstract: Dynamic equations, operators and inequalities have recently increased their motivating role in time scales. Time scales have been the field of study of many mathematicians and scientists working in different sciences for the last 30 years. Dynamic equations and inequalities on time scales have many applications in quantum mechanics, neural networks, heat transfer, electrical engineering, optics, economics and population dynamics. It is possible to give an example from the economics, seasonal investments and incomes. In this research, we will prove that, for $1 < p(x) < \infty$, the variable exponent $L^{p(\cdot)}$ norm of the restricted centered fractional Maximal delta integral operator $M^c_{a,\delta}$ equals the norm of the centered fractional Maximal delta integral operator M^c_a for all $0 < \delta < \infty$.
Keywords: Time scales, variable exponent, fractional Maximal operator.

1 Introduction

The theory of time scales was initiated by mathematician Stefan Hilger [1]. Later, this theory was quickly developed by many mathematicians and scientists working in other disciplines. And they have demonstrated various aspects of integral inequalities and operators [2]-[13]. Recently, time scales have played an important role in differential calculus, difference calculus, dynamics equations, quantum calculus and integral inequalities. Operators and integral inequalities have many applications. For example; electrical engineering, fluid dynamics, quantum mechanics, phsical problems, wave equations, heat transfer and economic problems [14, 21]-[25]-[26].

In this study, we will demonstrated some properties inequalities of fractional Maximal type with norm of $L^{p(\cdot)}(R^m)$ in time scales.

2 Preliminaries

Let $L^{p(\cdot)}(\Omega)$ denote the Banach function space of measurable functions f on Ω such that for some $\lambda > 0$, $\int_\Omega |\frac{f(t)}{\lambda}|^{p(t)}dt < \infty$ with norm

$$\|f\|_{p(.),\Omega} = \inf\{\lambda > 0; \int_\Omega \left(\frac{|}{f}(t)|\lambda\right)^{p(t)} dt \leq 1\}.$$

These spaces are referred to as the variable $L^{(p(.)}$ spaces. We can define the centered fractional maximal operator by

$$M^c_a f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{(n-a)/n}} \int_{B(x,r)\cap\Omega} |f(y|dy \tag{1}$$

and the uncentered fractional maximal operator by

$$M_a f(x) = \sup_{r>0} \frac{1}{|B|^{(n-a)/n}} \int_{B\cap\Omega} |f(y|dy. \tag{2}$$

Let's define the restricted centered fractional maximal operator and the restricted uncentered fractional maximal operator [22], respectively.

$$M^c_{a,\delta} f(x) = \sup_{\delta>r>0} \frac{1}{|B(x,r)|^{(n-a)/n}} \int_{B(x,r)\cap\Omega} |f(y|dy \tag{3}$$

and

$$M_{a,\delta} f(x) = \sup_{\delta>r>0,|t-x|<r} \frac{1}{|B(t,r)|^{(n-a)/n}} \int_{B\cap\Omega} |f(y|dy \tag{4}$$

for $x \in R^m$ and $\delta \in R_+$.

Definition 2.1. [22] If f a measurable function on R^m, then distribution function d_f on $[0, +\infty]$ is defined by

$$d_f(\beta) = |\{x \in R^m : |f(x)| > \beta\}| \tag{5}$$

where $|\{x \in R^m : |f(x)| > \beta\}|$ is the Lebesgue measurable of the measurable of $\{x \in R^m : |f(x)| > \beta\}$.

Lemma 2.2. [23] If $f \in L^p(R^m)$ with $0 < p < \infty$, then we have

$$\|f\|^p_{L^p(R^m)} = p\int_0^\infty \beta^{p-1} d_f(\beta)d\beta. \tag{6}$$

Lemma 2.3. (Lemma 2.4, [22]) If operators M_a and $M_{a,\delta}$ are defined as in (2) and (4), then the equality

$$d_{M_a f}(\beta) = \lim_{\delta\to\infty} d_{M_{a,\delta} f}(\beta) \tag{7}$$

holds $\forall f \in L^p(R^m)$ and $\beta > 0$.

Let's give information about the time scales that will help us in our work. T is a non-empty closed subset of R. $[a, b]$ is an arbitrary interval on time scale T. And $[a, b]_T$ is denoted by $[a, b] \cap T$.

Definition 2.4. [19] The mappings $\sigma, \rho : T \to T$ defined by $\sigma(t) = \inf s \in T : s > t, \rho(t) = \sup s \in T : s > t$, for $t \in T$. Respectively, $\sigma(t)$ is forward jump operator and $\rho(t)$ is backward jump operator.
If $\sigma(t) > t$, then t is right-scattered and if $\sigma(t) = t$, then t is called right-dense. If $\rho(t) < t$, then t is left-scattered and if $\rho(t) = t$, then t is called left-dense.

Definition 2.5. [19] Let two mappings $\mu, \vartheta : T \to R^+$ such that $\mu(t) = \sigma(t) - t, \vartheta(t) = t - \rho(t)$ are called graininess mappings.

Definition 2.6. [20] If $H : T \to R$ is defined a $\Delta-$ antiderivative of $h : T \to R$, then $H = h(t)$ holds for $\forall t \in T$. And we define the $\Delta-$ integral of h by

$$\int_s^t h(\tau)\Delta\tau = H(t) - H(s)$$

for $s, t \in T$.

In [24], we can define the restricted centered fractional maximal delta integral operator and the restricted uncentered fractional maximal delta integral operator, respectively

$$M^c_{a,\delta} f(x) = \sup_{\delta>r>0} \frac{1}{|B(x,r)|^{(m-a)/m}} \int_{B(x,r)\cap\Omega} |f(y|\Delta y$$

and

$$M_{a,\delta} f(x) = \sup_{\delta>r>0,|t-x|<r} \frac{1}{|B|^{(n-a)/n}} \int_{B(t,r)\cap\Omega} |f(y|\Delta y$$

for $x \in R^m$ and $\delta \in R_+$.

3 Main Result

Theorem 3.1. Let $M_{a,\delta} f$ be defined by (4) and $\delta > 0$. If function f is $\Delta-$ integrable, then

$$\|M_{a,\delta} f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_a f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

holds for $1 < p(.) \le \infty$.

Proof. We conclude from the definition of the operator $M_{a,\delta} f$ in (4) that

$$M_{a,\delta} f(\delta x) = \sup_{\delta>r>0,|t-\delta x|<r} \frac{1}{|B|^{(n-a)/n}} \int_{B(t,r)} |f(y|\Delta y = \sup_{\delta>r>0,|t-x|<r/\delta} \frac{1}{|v^m r^{(m-a)/m}|^{(n-a)/n}} \int_{|t|<r} |f(\delta t - y|\Delta y$$

$$\sup_{\delta>r>0,|t-x|<r/\delta} \frac{\delta^{(m-a)/m}}{|v^m r^{(m-a)/m}|^{(n-a)/n}} \int_{|t|<r/\delta} |f(\delta(t-y)|\Delta y = \sup_{1>\frac{r}{\delta}>0,|t-x|<r/\delta} \frac{1}{|v^m(\frac{r}{m})^{(m-a)/m}|^{(n-a)/n}} \int_{|t|<r/\delta} |(\tau_\delta f)(t-y|\Delta y$$

$$\sup_{1>r>0,|t-x|<r} \frac{1}{v^m r^{(m-a)/m}} \int_{|t|<r} |(\tau_\delta f)(x-y|\Delta y = M_{a,1}(\tau_\delta f)(x) \tag{8}$$

Thus we have

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_{a,1}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \tag{9}$$

for all $\delta > 0$ and $1 < p(.) < \infty$. Next we will prove that

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_af\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

for $1 < p(.) \leq \infty$.
If $f \in L^{p(.)}(R^m)$, then we have $Mf \in L^{p(.)}(R^m$. It follows from Lemma 2.2, Lemma 2.3, and equation (9) that

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)} = p(.)\int_0^\infty \mu^{p(.)-1}d_{M_af}(\mu)\Delta\mu = p(.)\int_0^\infty \mu^{p(.)-1}\lim_{\delta\to\infty} d_{M_{a,\delta}f}(\mu)\Delta\mu = \lim_{\delta\to\infty} p(.)\int_0^\infty \mu^{p(.)-1}d_{M_{a,\delta}f}(\mu)\Delta\mu = \lim_{\delta\to\infty}\|M_{a,\delta}\|^{p(.)}_{L^{p(.)}}$$

$$\lim_{\delta\to\infty}\|M_{a,\delta}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}\|f\|^{p(.)}_{L^{p(.)}(R^m)} = \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}\|f\|^{p(.)}_{L^{p(.)}(R^m)} \tag{10}$$

Since we have the obvious inequality

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \geq \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \tag{11}$$

we derive from (10) that

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

Thus, we get the desired result.

This completes the proof of Theorem 3.1.

Theorem 3.2. Let $M_{a,\delta}f$ be defined by (4) and $\delta > 0$. If function f is $\nabla-$ integrable, then

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_af\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

holds for $1 < p(.) \leq \infty$.

Proof. The proof of the theorem can be made analogous to the proof of Theorem 3.1 by using the properties of the $\nabla-$derivative.

We conclude from the definition of the operator $M_{a,\delta}f$ in (4) that

$$M_{a,\delta}f(\delta x) = \sup_{\delta>r>0,|t-\delta x|<r}\frac{1}{|B|^{(n-a)/n}}\int_{B(t,r)}|f(y|\nabla y = \sup_{\delta>r>0,|t-x|<r/\delta}\frac{1}{|v^m r^{(m-a)/m}|^{(n-a)/n}}\int_{|t|<r}|f(\delta t-y|\nabla y$$

$$\sup_{\delta>r>0,|t-x|<r/\delta}\frac{\delta^{(m-a)/m}}{|v^m r^{(m-a)/m}|^{(n-a)/n}}\int_{|t|<r/\delta}|f(\delta(t-y)|\nabla y = \sup_{1>\frac{r}{\delta}>0,|t-x|<r/\delta}\frac{1}{|v^m(\frac{r}{m})^{(m-a)/m}|^{(n-a)/n}}\int_{|t|<r/\delta}|(\tau_\delta f)(t-y|\nabla y$$

$$\sup_{1>r>0,|t-x|<r}\frac{1}{v^m r^{(m-a)/m}}\int_{|t|<r}|(\tau_\delta f)(x-y|\nabla y = M_{a,1}(\tau_\delta f)(x) \tag{12}$$

Thus we have

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_{a,1}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \tag{13}$$

for all $\delta > 0$ and $1 < p(.) < \infty$. Next we will prove that

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_af\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

for $1 < p(.) \leq \infty$.
If $f \in L^{p(.)}(R^m)$, then we have $Mf \in L^{p(.)}(R^m$. It follows from Lemma 2.2, Lemma 2.3, and equation (13) that

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)} = p(.)\int_0^\infty \mu^{p(.)-1}d_{M_af}(\mu)\nabla\mu = p(.)\int_0^\infty \mu^{p(.)-1}\lim_{\delta\to\infty} d_{M_{a,\delta}f}(\mu)\nabla\mu = \lim_{\delta\to\infty} p(.)\int_0^\infty \mu^{p(.)-1}d_{M_{a,\delta}f}(\mu)\nabla\mu = \lim_{\delta\to\infty}\|M_{a,\delta}\|^{p(.)}_{L^{p(.)}}$$

$$\lim_{\delta\to\infty}\|M_{a,\delta}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}\|f\|^{p(.)}_{L^{p(.)}(R^m)} = \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}\|f\|^{p(.)}_{L^{p(.)}(R^m)} \tag{14}$$

Since we have the obvious inequality

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \geq \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \tag{15}$$

we derive from (14) that

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

Thus, we get the desired result.

This completes the proof of Theorem 3.2.

Theorem 3.3. Let $M_{a,\delta}f$ be defined by (4) and $\delta > 0$. If function f is $\Diamond_\alpha-$ integrable, then

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_a f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

holds for $1 < p(.) \leq \infty$.

Proof. The proof of the theorem can be made analogous to the proof of Theorem 3.2 by using the properties of the $\Diamond_\alpha-$derivative.

Let $f(t)$ be differentiable on T for $\forall\alpha, t \in T$. Then, we define $f^{\Diamond_\alpha}(t)$ by

$$f^{\Diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t)$$

for $0 \leq \alpha \leq 1$ (for details see [20]).

If we get $\alpha, b, t \in T$ and $f : T \longrightarrow R$, then we have

$$\int_b^t f(\gamma)\Diamond_\alpha\gamma = \alpha\int_b^t f(\gamma)\Delta\gamma + (1-\alpha)\int_b^t f(\gamma)\nabla\gamma$$

for $0 \leq \alpha \leq 1$ (for details see [20]).

We conclude from the definition of the operator $M_{a,\delta}f$ in (4) that

$$M_{a,\delta}f(\delta x) = \sup_{\delta>r>0,|t-\delta x|<r}\frac{1}{|B|^{(n-a)/n}}\int_{B(t,r)}|f(y|\Diamond_\alpha y = \sup_{\delta>r>0,|t-x|<r/\delta}\frac{1}{|v^m r^{(m-a)/m}|^{(n-a)/n}}\int_{|t|<r}|f(\delta t-y|\Diamond_\alpha y$$

$$\sup_{\delta>r>0,|t-x|<r/\delta}\frac{\delta^{(m-a)/m}}{|v^m r^{(m-a)/m}|^{(n-a)/n}}\int_{|t|<r/\delta}|f(\delta(t-y)|\Diamond_\alpha y = \sup_{1>\frac{r}{\delta}>0,|t-x|<r/\delta}\frac{1}{|v^m(\frac{r}{m})^{(m-a)/m}|^{(n-a)/n}}\int_{|t|<r/\delta}|(\tau_\delta f)(t-y|\Diamond_\alpha y$$

$$\sup_{1>r>0,|t-x|<r}\frac{1}{v^m r^{(m-a)/m}}\int_{|t|<r}|(\tau_\delta f)(x-y|\Diamond_\alpha y = M_{a,1}(\tau_\delta f)(x) \tag{16}$$

Thus we have

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_{a,1}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \tag{17}$$

for all $\delta > 0$ and $1 < p(.) < \infty$. Next we will prove that

$$\|M_{a,\delta}f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_a f\|_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

for $1 < p(.) \leq \infty$.

If $f \in L^{p(.)}(R^m)$, then we have $Mf \in L^{p(.)}(R^m$. It follows from Lemma 2.2, Lemma 2.3, and equation (17) that

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)} = p(.)\int_0^\infty \mu^{p(.)-1}d_{M_a f}(\mu)\Diamond_\alpha\mu = p(.)\int_0^\infty \mu^{p(.)-1}\lim_{\delta\to\infty}d_{M_{a,\delta}f}(\mu)\Diamond_\alpha\mu = \lim_{\delta\to\infty}p(.)\int_0^\infty \mu^{p(.)-1}d_{M_{a,\delta}f}(\mu)\Diamond_\alpha\mu = \lim_{\delta\to\infty}\|M_{a,\delta}\|$$

$$\lim_{\delta\to\infty}\|M_{a,\delta}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}\|f\|^{p(.)}_{L^{p(.)}(R^m)} = \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}\|f\|^{p(.)}_{L^{p(.)}(R^m)} \tag{18}$$

Since we have the obvious inequality

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \geq \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} \tag{19}$$

we derive from (18) that

$$\|M_a\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)} = \|M_{a,1}\|^{p(.)}_{L^{p(.)}(R^m)\to L^{p(.)}(R^m)}$$

Thus, we get the desired result.

This completes the proof of Theorem 3.3.

Acknowledgments

The author would like to thank the referee for careful reading of the paper and valuable suggestions.

4 References

1 S.Hilger, *Ein Mabkettenkalkül mit Anwendung auf Zentrmsmannigfaltingkeiten,* Ph.D. Thesis, Univarsi. Wurzburg, 1988.
2 R.P. Agarwal, M. Bohner, A. Peterson, *Inequalities on time scales: A survey,* Math. Inequal. Appl. 4 (2001), 535-555. https://doi.org/10.7153/mia-04-48
3 E. Akin-Bohner, M. Bohner, F. Akin, *Pachpatte inequalities on time scales,* Journal of Inequalities in Pure and Applied Mathematics 6(1) (2005), 1-23.
4 W.N.Li, *Nonlinear Integral Inequalities in Two Independent Variables on Time Scales,* Adv Differ Equ. (2011), Article ID:283926. doi:10.1155/2011/283926.
5 G.A. Anastassiou, *Principles of delta fractional calculus on time scales and inequalities,* Mathematical and Computer Modelling 52 (2010), 556-566. https://doi.org/10.1016/j.mcm.2010.03.055.
6 F.-H. Wong, C.-C. Yeh, S.-L.Yu, C.-H. Hong, *Young's inequality and related results on time scales,* Appl. Math. Lett. 18 (2005), 983–988.
7 F.-H. Wong, C.-C. Yeh, W.-C. Lian, *An extension of Jensen's inequality on time scales,* Adv. Dynam. Syst. Appl. 1(1) (2006), 113-120.
8 J. Kuang, Applied inequalities. Shandong Science Press, Jinan, 2003.
9 D. Ucar, V.F. Hatipoglu, A. Akincali, *Fractional Integral Inequalıties On Time Scales,* Open J. Math. Sci. 2(1) (2018), 361-370.
10 U.M. Ozkan, M.Z. Sarikaya, H. Yildirim, *Extensions of certain integral inequalities on time scales,* Appl. Math. Lett. 21 (2008), 993-1000.
11 J.-F. Tian, M.-H. Ha, *Extensions of Holder-type inequalities on time scales and their applications,* J. Nonlinear Sci. Appl. 10 (2017), 937-953.
12 V. Kac, P. Cheung, Quantum Calculus. Universitext Springer, New York, 2002.
13 W.-G. Yang,*A functional generalization of diamond- integral Holder's inequality on time scales,* Appl. Math. Lett. 23 (2010), 1208-1212.
14 V. Spedding, *Taming nature's numbers,*New Scientist, July 19 (2003), 28-31.
15 C. C. Tisdell, A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling.* Nonlinear Anal. 68 (2008), 3504-3524.
16 M. Bohner, J. Heim, A. Liu, *Qualitative analysis of Solow model on time scales.* J. Concrete Appl. Math. 13 (2015), 183-197.
17 D. Brigo, F. Mercurio, *Discrete time vs continuous time stock-price dynamics and implications for option pricing.* Finance Stochast., 4 (2000), 147-159.
18 A. R. Seadawy, M. Iqbal and D. Lu, *Nonlinear wave solutions of the Kudryashov-Sinelshchikov dynamical equation in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity,*Journal of Taibah University for Science 13(1) (2019), 1060-1072, DOI: 10.1080/16583655.2019.1680170.
19 M. Bohner, R.P. Agarwal, *Basic calculus on time scales and some of its applications.* Resultate der Mathematic, 35 (1999), 3-22. https://doi.org/10.1007/BF03322019.
20 M. Bohner, A. Peterson, Dynamic equations on time scales, An introduction with applications Birkhauser, Boston, 2001. https://doi.org/10.1007/978-1-4612-0201-1
21 A. Tuna, S. Kutukcu, *Some integral inequalities on time scales.* Applied Mathematics and Mechanics (English Edition), 29(1) (2008), 23-28.
22 W. Mingquan, N. Xudong, W. Di and Y. Dunyan, *A note on Hardy-Littlewood maximal operators,* Journal of Inequalities and Applications, 2016:21 (2016), DOI 10.1186/s13660-016-0963-x.
23 L. Grafakos, Classical and Modern Fourier Analysis. China Machine Press, China (2005).
24 L. Akin, *On the Fractional Maximal Delta Integral Type Inequalities on Time Scales*. Fractal Fract. 4(2) (2020), 1-10.
25 L. Akin, *Compactness of Fractional Maximal Operator in Weighted and Variable Exponent Spaces*, Erzincan University, Journal of Science and Technology,12(1) (2019),185-190.
26 L. Akin,*On Two Weight Criterions for The Hardy Littlewood Maximal Operator in BFS*, Asian Journal of Science and Technology, 9(5) (2018), 8085-8089.

