https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 1, Pages 64–73 (2021)
DOI: 10.31801/cfsuasmas.774658
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: July 28, 2020; Accepted: November 12, 2020

BIPOLAR FUZZY SOFT (D-)METRIC SPACES

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Abstract. In this study, we have introduced the bipolar fuzzy soft (D-)metric space which is based on bipolar fuzzy soft point of bipolar fuzzy soft sets and give some of their properties. Also, the bipolar fuzzy soft sequences and bipolar fuzzy soft cauchy sequences concepts have been studied with the help of defined metric spaces and some of their properties have been investigated. In addition to all this, many examples are given in order to better understand the concepts and features studied and contribute to a better understanding of the paper.

1. Introduction

Uncertainty is involved in most of the fields like engineering, economic and social disciplines etc. All uncertainty problems that a person encounters in his life cannot be solved by using classic mathematical skills. Because; information may be incomplete, not fully reliable, vague, contradictory or deficient in some other way. These various information deficiencies may result in fuzziness or vagueness. One of the first studies on the solution of uncertainty problems was the fuzzy set theory given by Zadeh [15] in 1965. In the following years, in 1994, Zhang [16] initiated the concept of bipolar fuzzy sets. In addition, many set theory such as rough sets [7] (this theory is based on equivalence relations), soft sets [6], bipolar fuzzy soft sets [1] have been proposed to solve the uncertainty problem in the most ideal way. Bipolar fuzzy sets, one of these set theories, are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. Abdullah et al. [1] introduced the notion of bipolar fuzzy soft set which is a combination of bipolar fuzzy set and soft set. In recent years, we can easily say that the studies for the solution of uncertainty problems are increasing day by day [3, 8–11, 13, 14].

2020 *Mathematics Subject Classification.* 03E72, 54E99.

Keywords and phrases. Bipolar fuzzy soft sets, bipolar fuzzy soft metric space, bipolar fuzzy soft D-metric space.

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Metric spaces are one of the most useful and important concepts in mathematics and applied sciences. Many researchers have studied to generalizations the concept of metric spaces. For example, concepts of D-metric spaces and 2-metric spaces were introduced in [4] and [5], respectively. Then, Beaulaa and Gunaseeli [2] introduced the definition of the fuzzy soft metric space. Also in [12], Sayed and Alahmari introduced the notions of some mappings and proved some fixed point theorems in fuzzy soft metric spaces.

In this paper, we have defined bipolar fuzzy soft (D-)metric space in terms of bipolar fuzzy soft points. Moreover, we introduce the concepts of bipolar fuzzy soft sequence and bipolar fuzzy soft cauchy sequence and examine the connection between them. In addition, all the given properties related to bipolar fuzzy soft (D-)metric spaces are supported by examples.

2. Preliminaries

Here, we remind some basic information from the literature for subsequent use.

Definition 1. *[16, 17] Let U be any nonempty set. Then a bipolar fuzzy set, is an object of the form*

$$A = \{u, < \mu_A^+(u), \mu_A^-(u) >: u \in U\}$$

and $\mu_A^+ : U \to [0,1]$ and $\mu_A^- : U \to [-1,0]$, $\mu_A^+(u)$ is a positive material and $\mu_A^-(u)$ is a negative material of $u \in U$. For simplicity, we donate the bipolar fuzzy set as $A =< \mu_A^+, \mu_A^- >$ in its place of $A = \{u, < \mu_A^+(u), \mu_A^-(u) >: u \in U\}$.

Definition 2. *[16,17] Let $A_1 =< \mu_{A_1}^+, \mu_{A_1}^- >$ and $A_2 =< \mu_{A_2}^+, \mu_{A_2}^- >$ be two bipolar fuzzy sets, on U. Then,*
(i) $A_1' = \{< 1 - \mu_{A_1}^+(u), -1 - \mu_{A_1}^-(u) >\}$,
(ii) $A_1 \cup A_2 =< max(\mu_{A_1}^+(u), \mu_{A_2}^+(u)), min(\mu_{A_1}^-(u), \mu_{A_2}^-(u)) >$,
(iii) $A_1 \cap A_2 =< min(\mu_{A_1}^+(u), \mu_{A_2}^+(u)), max(\mu_{A_1}^-(u), \mu_{A_2}^-(u)) >$.

Definition 3. *[6] Let U be an initial universe, E be the set of parameters, $A \subset E$ and $P(U)$ is the power set of U. Then (F, A) is called a soft set, where $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A), or as the set of ϵ-approximate elements of the soft set.*

Definition 4. *[1] Let U be a universe, E a set of parameters and $A \subseteq E$. Define $f : A \to BF^U$, where BF^U is the collection of all bipolar fuzzy subsets of U. Then (f, A), denoted by f_A, is said to be a bipolar fuzzy soft set over a universe U. It is defined by*

$$f_A = \{(u, \mu_e^+(u), \mu_e^-(u)) : \forall u \in U, e \in A\}$$

Definition 5. *[1] Let U be a universe and E a set of attributes. Then, (U, E) is the collection of all bipolar fuzzy soft sets on U with attributes from E and is said to be bipolar fuzzy soft class.*

Definition 6. *[1] Let f_A and g_B be two bipolar fuzzy soft sets over a common universe U. We say that f_A is a bipolar fuzzy soft subset of g_B, if*
(i) $A \subseteq B$ and
(ii) For all $e \in A$, $f(e)$ is a bipolar fuzzy subset of $g(e)$.

We write $f_A \widehat{\subseteq} g_B$.

Moreover, we say that f_A and g_B are bipolar fuzzy soft equal sets if f_A is a bipolar fuzzy soft subset of g_B and g_B is a bipolar fuzzy soft subset of f_A.

Definition 7. *[1] The complement of a bipolar fuzzy soft set f_A is denoted ${f_A}^c$ and is defined by ${f_A}^c = \{(u, 1 - \mu_e^+(u), -1 - \mu_e^-(u)) : \forall u \in U, e \in E\}$.*

It should be noted that $1 - f(e)$ denotes the fuzzy complement of $f(e)$ for $e \in A$.

Definition 8. *[1] Let f_A and g_B be two bipolar fuzzy soft sets over a common universe U. Then*
(i) The union of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cup} g_B$ over U, where $C = A \cup B$, $h : C \to BF^U$ and

$$h(e) = \begin{cases} f(e) & \text{if } e \in A \setminus B \\ g(e) & \text{if } e \in B \setminus A \\ f(e) \cup g(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$.
(ii) The restricted union of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cup}_{\mathcal{R}} g_B$ over U, where $C = A \cap B \neq \emptyset$, $h : C \to BF^U$ and $h(e) = f(e) \cup g(e)$ for all $e \in C$.
(iii) The extended intersection of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cap} g_B$ over U, where $C = A \cup B$, $h : C \to BF^U$ and

$$h(e) = \begin{cases} f(e) & \text{if } e \in A \setminus B \\ g(e) & \text{if } e \in B \setminus A \\ f(e) \cap g(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$.
(iv) The restricted intersection of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cap}_{\mathcal{R}} g_B$ over U, where $C = A \cap B \neq \emptyset$, $h : C \to BF^U$ and $h(e) = f(e) \cap g(e)$ for all $e \in C$.

3. Bipolar Fuzzy Soft Points

In this section, we discuss the notion of bipolar fuzzy soft (D-) metric spaces using bipolar fuzzy soft points.

Definition 9. *Let U be a universe, E a set of parameters and $A \subseteq E$. Define $f : A \to BF^U$, where BF^U is the collection of all bipolar fuzzy subsets of U. Then (f, A), denoted by f_A, is said to be a bipolar fuzzy soft set over a universe U. It is defined by*

$$f_A = \Big\{ \big(e, \{< u, \mu^+_{f_A}(u), \mu^-_{f_A}(u) >: u \in U\}\big) : e \in A \subseteq E \Big\}$$

The set of all bipolar fuzzy soft set over (U, E) is denoted by $BFS(U, E)$.

Example 10. *Let $U = \{u_1, u_2, u_3\}$ be the set of three houses under consideration and $E = \{e_1, e_2, e_3, e_4, e_5\} = \{$ scenic,expensive, large and comfortable, garden, traditional $\}$ be the set of parameters and $A = \{e_2, e_3, e_5\} \subset E$. Then,*

$$f_A = \left\{ \begin{array}{c} \left(e_2, \left\{\left\langle u_1, 0.37, -0,54\right\rangle, \left\langle u_2, 0.43, -0,23\right\rangle, \left\langle u_3, 0.12, -0,87\right\rangle\right\}\right), \\ \left(e_3, \left\{\left\langle u_1, 0.11, -0,76\right\rangle, \left\langle u_2, 0.45, -0,98\right\rangle, \left\langle u_3, 0.61, -0,21\right\rangle\right\}\right), \\ \left(e_5, \left\{\left\langle u_1, 0.41, -0,17\right\rangle, \left\langle u_2, 0.91, -0,19\right\rangle, \left\langle u_3, 0.83, -0,47\right\rangle\right\}\right) \end{array} \right\}.$$

Definition 11. (i) *A bipolar fuzzy soft set f_A is said to be the absolute bipolar fuzzy soft set over U, if $f(e) = \{< u, 1, -1 >: u \in U\}$ for all $e \in A$. It is denoted by A^U.*
(ii) *A bipolar fuzzy soft set f_A is said to be the null bipolar fuzzy soft set over U, if $f(e) = \{< u, 0, 0 >: u \in U\}$ for all $e \in A$. It is denoted by $A^\emptyset$.*

Definition 12. *The complement of a bipolar fuzzy soft set f_A is denoted ${f_A}^c$ and is defined by $f^c_A = \Big\{ \big(e, \{< u, 1 - \mu^+_{f_A}(u), -1 - \mu^-_{f_A}(u) >: u \in U\}\big) : e \in A \subseteq E \Big\}$.*

Definition 13. *A bipolar fuzzy soft point in a bipolar fuzzy soft set f_A is defined as an element $(e, f(e))$ of f_A, for $e \in A$ and is denoted by e_{f_A}, if for the element $e \in A$, $f(e) \neq \{< u, 0, 0 >: u \in U\}$ and $f(e') = \{< u, 0, 0 >: u \in U\}$, for all $e' \in A - \{e\}$.*

Proposition 14. *The union of any collection of bipolar fuzzy soft points can be considered as a bipolar fuzzy soft set and every bipolar fuzzy soft set can be expressed as the union of all bipolar fuzzy soft points.*

$$f_A = \bigcup_{e_{f_A} \widehat{\in} f_A} e_{f_A}$$

Definition 15. *The complement of a bipolar fuzzy soft point e_{f_A} is a bipolar fuzzy soft point $e^c_{f_A}$ such that $f^c(e) = (f(e))^c$ for all $e \in A$.*

Definition 16. *A bipolar fuzzy soft point $e_{f_A} \in g_A$, g_A being a bipolar fuzzy soft set if for $e \in A$, $f(e) \leq g(e)$ i.e., $\mu^+_{f_A}(u) \leq \mu^+_{g_A}(u)$ and $\mu^-_{g_A}(u) \leq \mu^-_{f_A}(u)$, $u \in U$.*

Example 17. *Let $U = \{u_1, u_2\}$ and $A = \{e_1, e_2, e_3\} \subseteq E$. Then,*

$$(e_2)_{f_A} = \{< u_1, 0.75, -0.32 >, < u_2, 0.40, -0, 52 >\}$$

is a bipolar fuzzy soft point. Its complement is given by:

$$(e_2)^c_{f_A} = \{< u_1, 0.25, -0.68 >, < u_2, 0.60, -0, 48 >\}$$

For another bipolar fuzzy soft set g_A defined on same (U, E), let

$$g_A(e_2) = \{< u_1, 0.85, -0.72 >, < u_2, 0.45, -0, 65 >\}$$

Then $f(e_2) \leq g(e_2)$ for $e_2 \in A$ i.e., $(e_2)_{f_A} \in g_A$.

4. Bipolar Fuzzy Soft Metric Spaces

Definition 18. *Let $BFSP(U, A)$ be the collection of all bipolar fuzzy soft points over (U, A). Then the bipolar fuzzy soft metric in terms of bipolar fuzzy soft points is defined by a mapping $d : BFSP(U, A) \times BFSP(U, A) \to [0, 2]$ satisfying the following conditions:*

$(BFS\ M_d1)$ $d(e_{f_A}, e_{g_A}) \geq 0$, $\forall e_{f_A}, e_{g_A} \in BFSP(U, A)$.
$(BFS\ M_d2)$ $d(e_{f_A}, e_{g_A}) = 0 \Leftrightarrow e_{f_A} = e_{g_A}$.
$(BFS\ M_d3)$ $d(e_{f_A}, e_{g_A}) = d(e_{g_A}, e_{f_A})$.
$(BFS\ M_d4)$ $d(e_{f_A}, e_{g_A}) \leq d(e_{f_A}, e_{h_A}) + d(e_{h_A}, e_{g_A})$, $\forall e_{f_A}, e_{g_A}, e_{h_A} \in BFSP(U, A)$.

Then $BFSP(U, A)$ is said to form a bipolar fuzzy soft metric space $(BFSM_dS)$ with respect to the $BFSM_dS$ 'd' over (U, A) and is denoted by $(BFSP(U, A), d)$. Here $e_{f_A} = e_{g_A}$ means that, $\mu^+_{f_A}(u) = \mu^+_{g_A}(u)$ and $\mu^-_{f_A}(u) = \mu^-_{g_A}(u)$ for all $u \in U$.

Example 19. *Define*

$$d(e_{f_A}, e_{g_A}) = min_{u_k}\{|\mu^+_{f_A}(u_k) - \mu^+_{g_A}(u_k)| + |\mu^-_{f_A}(u_k) - \mu^-_{g_A}(u_k)|\}$$

Clearly, $d(e_{f_A}, e_{g_A}) \geq 0$ and $d(e_{f_A}, e_{g_A}) = 0 \Leftrightarrow e_{f_A} = e_{g_A}$. Also $d(e_{f_A}, e_{g_A}) = d(e_{g_A}, e_{f_A})$. To verify the final condition, we use the triangle inequality

$$\begin{aligned} d(e_{f_A}, e_{g_A}) &= min_{u_k}\{|\mu^+_{f_A}(u_k) - \mu^+_{g_A}(u_k)| + |\mu^-_{f_A}(u_k) - \mu^-_{g_A}(u_k)|\} \\ &= min_{u_k}\{|\mu^+_{f_A}(u_k) - \mu^+_{h_A}(u_k) + \mu^+_{h_A}(u_k) - \mu^+_{g_A}(u_k)| \\ &\quad + |\mu^-_{f_A}(u_k) - \mu^-_{h_A}(u_k) + \mu^-_{h_A}(u_k) - \mu^-_{g_A}(u_k)|\} \\ &\leq min_{u_k}\{|\mu^+_{f_A}(u_k) - \mu^+_{h_A}(u_k)| + |\mu^+_{h_A}(u_k) - \mu^+_{g_A}(u_k)| + |\mu^-_{f_A}(u_k) \\ &\quad - \mu^-_{h_A}(u_k)| + |\mu^-_{h_A}(u_k) - \mu^-_{g_A}(u_k)|\} \end{aligned}$$

$$= \quad d(e_{f_A}, e_{h_A}) + d(e_{h_A}, e_{g_A})$$

Thus d defined above is called a bipolar fuzzy soft metric over (U, A).

Definition 20. *A sequence of bipolar fuzzy soft points $\{(e_n)_{f_A}\}$ in a $(BFSP(U,A), d)$ is said to converge in $(BFSP(U,A), d)$ if there exists a bipolar fuzzy soft point $e_{f_A} \in BFSP(U,A)$ such that $d((e_n)_{f_A}, e_{f_A}) \to 0$ as $n \to \infty$ or $(e_n)_{f_A} \to e_{f_A}$ as $n \to \infty$. Analytically for every $\epsilon > 0$ there exists a natural number n_0 such that $d((e_n)_{f_A}, e_{f_A}) < \epsilon$, for all $n \geq n_0$.*

Definition 21. *A sequence $\{(e_n)_{f_A}\}$ of bipolar fuzzy soft point in a $(BFSP(U,A), d)$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d((e_m)_{f_A}, (e_n)_{f_A}) < \epsilon$, for all $m, n \geq n_0$ i.e., $d((e_m)_{f_A}, (e_n)_{f_A}) \to 0$ as $m, n \to \infty$.*

Example 22. *Let $E = \mathbb{N}$ be the parameter set and $U = \mathbb{Z}$ be the universal set. Define a mapping $f : A \to BF^{\mathbb{Z}}$ where, for any $n \in A \subseteq \mathbb{N}$ and $u \in \mathbb{Z}$,*

$$\mu^+_{f_A}(u) = \begin{cases} 0, & \text{if } u \text{ is odd} \\ \frac{4}{3n+1}, & \text{if } u \text{ is even} \end{cases} \quad \text{and} \quad \mu^-_{f_A}(u) = \begin{cases} \frac{2-n}{2n}, & \text{if } u \text{ is odd} \\ 0, & \text{if } u \text{ is even} \end{cases}$$

Here, $\{(e_n)_{f_A}\} \to (0, -\frac{1}{2})$ for odd integers and $\{(e_n)_{f_A}\} \to (0,0)$ for even integers. Hence, $\{(e_n)_{f_A}\}$ is a divergent bipolar fuzzy soft sequence over $(\mathbb{Z}, A)$.

Let's give an example of a Cauchy sequence for mapping $g : A \to BF^{\mathbb{Z}}$ where, for any $n \in A \subseteq \mathbb{N}$ and $u \in \mathbb{Z}$,

$$\mu^+_{g_A}(u) = \frac{2-n}{2n}, \quad \mu^-_{g_A}(u) = \frac{3n+4}{6n}$$

Define the distance function as

$$\begin{aligned} d((e_m)_{g_A}, (e_n)_{g_A}) &= min_{u_k}\{|\mu^+_{(e_m)_{g_A}}(u_k) - \mu^+_{(e_n)_{g_A}}(u_k)| + |\mu^-_{(e_m)_{g_A}}(u_k) - \mu^-_{(e_n)_{g_A}}(u_k)|\} \\ &= |\mu^+_{(e_m)_{g_A}}(u_k) - \mu^+_{(e_n)_{g_A}}(u_k)| + |\mu^-_{(e_m)_{g_A}}(u_k) - \mu^-_{(e_n)_{g_A}}(u_k)| \\ &= |\frac{2-m}{2m} - \frac{2-n}{2n}| + |\frac{3m+4}{6m} - \frac{3n+4}{6n}| \to 0, \quad (m, n \to \infty) \end{aligned}$$

Hence $\{(e_n)_{g_A}\}$ is a Cauchy sequence.

Definition 23. *A bipolar fuzzy soft metric space $(BFSP(U,A), d)$ is said to be complete if every cauchy sequence in $BFSP(U,A)$ converges to some bipolar fuzzy soft point of $BFSP(U,A)$.*

Lemma 24. *Let $(BFSP(U,A), d)$ be a bipolar fuzzy soft metric space. Then every bipolar fuzzy soft convergent sequence is a bipolar fuzzy soft cauchy sequence.*

Proof. Let $\{(e_n)_{f_A}\}$ be a bipolar fuzzy soft convergent sequence in $BFSP(U,A)$ converging to e_{f_A}. Then

$$(e_n)_{f_A}(u) = \begin{cases} \mu_{(e_n)_{f_A}}(u), & \text{if } s = e_n \\ 0, & \text{if } s \neq e_n \end{cases}, \quad e_{f_A}(u) = \begin{cases} \mu_{e_{f_A}}(u), & \text{if } s = e \\ 0, & \text{if } s \neq e \end{cases}$$

where $\mu_{(e_n)_{f_A}}(u) = (\mu^+_{(e_n)_{f_A}}(u), \mu^-_{(e_n)_{f_A}}(u))$ and $\mu_{e_{f_A}}(u) = (\mu^+_{e_{f_A}}(u), \mu^-_{e_{f_A}}(u))$. Since $(e_n)_{f_A} \to e_{f_A}$ as $n \to \infty$, given $\frac{\epsilon}{2} > 0$, there exists $\frac{\delta}{2} > 0$ choose N such that $d((e_n)_{f_A}, e_{f_A}) < \frac{\delta}{2}$ implies $|\mu_{(e_n)_{f_A}}(u) - \mu_{e_{f_A}}(u)| < \frac{\epsilon}{2}$ for all $n \geq N \in \mathbb{N}$. Then for all $n, m \geq N \in \mathbb{N}$.

$$d((e_n)_{f_A}, (e_m)_{f_A}) \leq d((e_n)_{f_A}, e_{f_A}) + d(e_{f_A}, (e_m)_{f_A}) \leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$$

and

$$|\mu_{(e_n)_{f_A}}(u) - \mu_{(e_m)_{f_A}}(u)| \leq |\mu_{(e_n)_{f_A}}(u) - \mu_{e_{f_A}}(u)| + |\mu_{e_{f_A}}(u) - \mu_{(e_m)_{f_A}}(u)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.$$

□

5. Bipolar Fuzzy Soft D-Metric Spaces

Definition 25. *A mapping $D : BFSP(U, A) \times BFSP(U, A) \times BFSP(U, A) \to [0, 3]$ is called a bipolar fuzzy soft D-metric on the bipolar fuzzy soft set (U, A) that D satisfies the following conditions, for each bipolar fuzzy soft points $e_{f_A}, e_{g_A}, e_{h_A}, e_{t_A} \in BFSP(U, A)$,*

$(BFS\ M_D1)$ $D(e_{f_A}, e_{g_A}, e_{h_A}) \geq 0$ *and equality holds if and only if* $e_{f_A} = e_{g_A} = e_{h_A}$.
$(BFS\ M_D2)$ $D(e_{f_A}, e_{g_A}, e_{h_A}) = D(e_{g_A}, e_{f_A}, e_{h_A}) = D(e_{f_A}, e_{h_A}, e_{g_A}) = ...$
$(BFS\ M_D3)$ $D(e_{f_A}, e_{g_A}, e_{h_A}) \leq D(e_{f_A}, e_{g_A}, e_{t_A}) + D(e_{f_A}, e_{t_A}, e_{h_A}) + D(e_{t_A}, e_{g_A}, e_{h_A})$.

Then $BFSP(U, A)$ is said to form a bipolar fuzzy soft D-metric space $(BFSM_DS)$ with respect to the $BFSM_DS$ 'D' over (U, A) and is denoted by $(BFSP(U, A), D)$. Here $e_{f_A} = e_{g_A} = e_{h_A}$ means that, $\mu^+_{f_A}(u) = \mu^+_{g_A}(u) = \mu^+_{h_A}(u)$ and $\mu^-_{f_A}(u) = \mu^-_{g_A}(u) = \mu^-_{h_A}(u)$ for all $u \in U$.

Example 26. *Let U be a non-emty set and $A \subseteq E$ be the non-emty set of parameters. If we define mappings*

$$D_1, D_2 : BFSP(U, A) \times BFSP(U, A) \times BFSP(U, A) \to [0, 3]$$

by,

$$D_1(e_{f_A}, e_{g_A}, e_{h_A}) = \begin{cases} 0, & \text{if } e_{f_A} = e_{g_A} = e_{h_A} \\ 1, & \text{otherwise} \end{cases}$$

and

$$D_2(e_{f_A}, e_{g_A}, e_{h_A}) = \begin{cases} 0, & \text{if } e_{f_A} = e_{g_A} = e_{h_A} \\ 1, & \text{if } e_{f_A} = e_{g_A} \neq e_{h_A} \text{ or } e_{f_A} \neq e_{g_A} = e_{h_A} \text{ or } e_{f_A} \neq e_{g_A} = e_{h_A} \\ 2, & \text{if } e_{f_A} \neq e_{g_A} \neq e_{h_A} \end{cases}$$

for all of $e_{f_A}, e_{g_A}, e_{h_A} \in BFS(U,A)$. Then D_1 (D_2) is a bipolar fuzzy soft $D_1(D_2)$-metric on $BFS(U,A)$.

Example 27. *Define*

$$D(e_{f_A^1}, e_{f_A^2}, e_{f_A^3}) = min_{u_k}\left\{ \sum_{\substack{i=1,\ j=2,3\\ i=2,\ j=3}} \left(|\mu^+_{f_A^i}(u_k) - \mu^+_{f_A^j}(u_k)| + |\mu^-_{f_A^i}(u_k) - \mu^-_{f_A^j}(u_k)|\right)\right\}$$

Clearly, $D(e_{f_A^1}, e_{f_A^2}, e_{f_A^3}) \geq 0$ and

$$D(e_{f_A^1}, e_{f_A^2}, e_{f_A^3}) = D(e_{f_A^2}, e_{f_A^1}, e_{f_A^3}) = D(e_{f_A^1}, e_{f_A^3}, e_{f_A^2}) = \dots .$$

To verify the final condition, we use the triangle inequality

$$\begin{aligned}
D(e_{f_A^1}, e_{f_A^2}, e_{f_A^3}) &= min_{u_k}\left\{ \sum_{\substack{i=1,\ j=2,3\\ i=2,\ j=3}} \left(|\mu^+_{f_A^i}(u_k) - \mu^+_{f_A^j}(u_k)| + |\mu^-_{f_A^i}(u_k) - \mu^-_{f_A^j}(u_k)|\right)\right\} \\
&= min_{u_k}\left\{ \sum_{\substack{i=1,\ j=2,3\\ i=2,\ j=3}} \left(|\mu^+_{f_A^i}(u_k) - \mu^+_{f_A^4}(u_k) + \mu^+_{f_A^4}(u_k) - \mu^+_{f_A^j}(u_k)|\right.\right. \\
&\quad \left.\left. + |\mu^-_{f_A^i}(u_k) - \mu^-_{f_A^4}(u_k) + \mu^-_{f_A^4}(u_k) - \mu^-_{f_A^j}(u_k)|\right)\right\} \\
&\leq min_{u_k}\left\{ \sum_{\substack{i=1,\ j=2,3\\ i=2,\ j=3}} \left(|\mu^+_{f_A^i}(u_k) - \mu^+_{f_A^4}(u_k)| + |\mu^+_{f_A^4}(u_k) - \mu^+_{f_A^j}(u_k)|\right.\right. \\
&\quad \left.\left. + |\mu^-_{f_A^i}(u_k) - \mu^-_{f_A^4}(u_k)| + |\mu^-_{f_A^4}(u_k) - \mu^-_{f_A^j}(u_k)|\right)\right\} \\
&\leq D(e_{f_A^1}, e_{f_A^2}, e_{f_A^4}) + D(e_{f_A^1}, e_{f_A^4}, e_{f_A^3}) + D(e_{f_A^4}, e_{f_A^2}, e_{f_A^3})
\end{aligned}$$

Thus D defined above is called a bipolar fuzzy soft D-metric over (U,A).

Definition 28. *Let $(BFSP(U,A), D)$ be a bipolar fuzzy soft D- metric space.*
(i) A bipolar fuzzy soft sequence $\{(e_n)_{f_A}\}$ in $(BFSP(U,A), D)$ converges to a bipolar fuzzy soft point $e_{f_A} \in BFSP(U,A)$ if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $D((e_n)_{f_A}, (e_m)_{f_A}, e_{f_A}) < \epsilon$.
(ii) A bipolar fuzzy soft sequence $\{(e_n)_{f_A}\}$ in $(BFSP(U,A), D)$ is called a Cauchy sequence if for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m > n$, $p \geq n_0$, $D((e_n)_{f_A}, (e_m)_{f_A}, (e_p)_{f_A}) < \epsilon$.
(iii) The bipolar fuzzy soft D-metric space $(BFSP(U,A), D)$ is said to be complete if every Cauchy sequence is convergent.

Example 29. *Consider the sequences $\{(e_n)_{f_A}\}$ and $\{(e_n)_{g_A}\}$ given in Example 22. Here it is clear that $\{(e_n)_{f_A}\}$ in $(BFSP(U,A), D)$ is still a divergent bipolar fuzzy* *soft sequence over $(\mathbb{Z}, A)$. Moreover, let's show that $\{(e_n)_{g_A}\}$ in $(BFSP(U,A), D)$ is a Cauchy sequence with the help of the distance function as,*

$$\begin{aligned} D((e_n)_{g_A},(e_m)_{g_A},(e_p)_{g_A}) &= min_{u_k}\{|\mu^+_{(e_n)_{g_A}}(u_k)-\mu^+_{(e_m)_{g_A}}(u_k)|+|\mu^-_{(e_n)_{g_A}}(u_k) \\ &\quad -\mu^-_{(e_m)_{g_A}}(u_k)|+|\mu^+_{(e_n)_{g_A}}(u_k)-\mu^+_{(e_p)_{g_A}}(u_k)| \\ &\quad +|\mu^-_{(e_n)_{g_A}}(u_k)-\mu^-_{(e_p)_{g_A}}(u_k)| \\ &\quad +|\mu^+_{(e_m)_{g_A}}(u_k)-\mu^+_{(e_p)_{g_A}}(u_k)|+|\mu^-_{(e_m)_{g_A}}(u_k)-\mu^-_{(e_p)_{g_A}}(u_k)|\} \\ &= |\mu^+_{(e_n)_{g_A}}(u_k)-\mu^+_{(e_m)_{g_A}}(u_k)|+|\mu^-_{(e_n)_{g_A}}(u_k)-\mu^-_{(e_m)_{g_A}}(u_k)| \\ &\quad +|\mu^+_{(e_n)_{g_A}}(u_k)-\mu^+_{(e_p)_{g_A}}(u_k)|+|\mu^-_{(e_n)_{g_A}}(u_k)-\mu^-_{(e_p)_{g_A}}(u_k)| \\ &\quad +|\mu^+_{(e_m)_{g_A}}(u_k)-\mu^+_{(e_p)_{g_A}}(u_k)|+|\mu^-_{(e_m)_{g_A}}(u_k)-\mu^-_{(e_p)_{g_A}}(u_k)| \\ &= |\frac{2-n}{2n}-\frac{2-m}{2m}|+|\frac{3n+4}{6n}-\frac{3m+4}{6m}| \\ &\quad +|\frac{2-n}{2n}-\frac{2-p}{2p}|+|\frac{3n+4}{6n}-\frac{3p+4}{6p}| \\ &\quad +|\frac{2-m}{2m}-\frac{2-p}{2p}|+|\frac{3m+4}{6m}-\frac{3p+4}{6p}| \\ &\to 0, \quad (n,m,p\to\infty) \end{aligned}$$

Hence $\{(e_n)_{g_A}\}$ in $(BFSP(U,A), D)$ is a Cauchy sequence.

Lemma 30. *Let $(BFSP(U,A), D)$ be a bipolar fuzzy soft D-metric space. Then every bipolar fuzzy soft convergent sequence is a bipolar fuzzy soft cauchy sequence.*

Proof. The proof is similar to Lemma 24. □

6. Results and Discussion

The first aim to this paper is to introduce the notions of bipolar fuzzy soft metric space and bipolar fuzzy soft D-metric space. In order to define these concepts, the concept of bipolar fuzzy soft points has been brought to the literature and bipolar fuzzy soft points have been examined in detail. Moreover, the bipolar fuzzy soft sequences and bipolar fuzzy soft cauchy sequences were defined and some of their properties were examined. In addition, many examples have been given to better understand these new concepts and properties that have been brought to the literature. It would be an interesting problem to examine the limits to which bipolar fuzzy soft (D-)metric space can be extended. Moreover, we think that this study can be a pioneering study for similar metric studies on continuously developing set modeling.

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