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A New Family of Odd Generalized Nakagami (Nak-G) Distributions

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Abstract. In this article, we proposed a new family of generalized Nak-G distributions and study some of its statistical properties, such as moments, moment generating function, quantile function, and probability Weighted Moments. The Renyi entropy, expression of distribution order statistic and parameters of the model are estimated by means of maximum likelihood technique. We prove, by providing three applications to real-life data, that Nakagami Exponential (Nak-E) distribution could give a better fit when compared to its competitors.

1. Introduction

There has been recent developments focus on generalized classes of continuous distributions by adding at least one shape parameters to the baseline distribution, studying the properties of these distributions and using these distributions to model data in many applied areas which include engineering, biological studies, environmental sciences and economics. Numerous methods for generating new families of distributions have been proposed [8] many researchers. The beta-generalized family of distribution was developed , Kumaraswamy generated family of distributions [5], Beta-Nakagami distribution [19], Weibull generalized family of distributions [4], Additive weibull generated distributions [12], Kummer beta generalized family of distributions [17], the Exponentiated-G family [6], the Gamma-G (type I) [21], the Gamma-G family (type II) [18], the McDonald-G [1], the Log-Gamma-G [3], A new beta generated Kumaraswamy Marshall-Olkin-G family of distributions with applications [11], Beta Marshall-Olkin-G family [2] and Logistic-G family [20].
The Nakagami distribution is a continuous probability distribution related to gammadistribution with applications in measuring alternation of wireless signal traversing multiple paths. The Nakagami distribution has two parameters; $\lambda \geq 0.5$ is the shape parameter and $\beta >$ is scale parameter. The cumulative distribution function (cdf) is given by

$$F(x;\lambda,\beta) = \int_0^x \frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} t^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta} t^2\right) dt \qquad (1)$$

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probability density function (pdf) is given by

$$f(x;\lambda,\beta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}t^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t^{2}\right);\ x>0 \tag{2}$$

It reduces to Rayleigh distribution when $\lambda=1$ and half normal distribution when $\lambda=0.5$ The main aim of this study is to develop a new family of generated distributions for the generalized Nakagami distribution and study some of the mathematical and statistical properties of the proposed family of distributions.
This paper is organized as follows: In section 2, the Nakagami (Nak-G) family of distributions was defined. In section 3, a useful linear representation for its probability density function (pdf) was obtained, some mathematical properties and parameter estimators using maximum likelihood estimation are derived. In section 4, the goodness of fit of the distribution using real data was illustrated while section 5, gives the conclusion.

2. Constructions of the Nak-G Distributions

In this section, the probability density function (pdf), cumulative distribution function (cdf), survival function, hazard rate function (hrf), mean remaining lifetime function, order statistic, moment, moment generating function, Renyi and q entropies of Nak-G distributions are derived. We obtain the Nak-G distribution by considering the Nakagami generator applied to the odd ratio $G(x;\eta)/\bar{G}(x;\eta)$ where $G(x;\eta)$ is the cdf of baseline distribution and $\bar{G}(x;\eta)=1-G(x;\eta)$.
Let denote the cdf and pdf of baseline model, η is the parameter vector of the baseline distribution. Based on the family of distributions we define the cdf of Nak-G by replacing x with in equation (1) it become Nak-G distribution.

$$F(x;\lambda,\beta,\eta)=\int_{0}^{\frac{G(x;\eta)}{\bar{G}(x;\eta)}}\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}t^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t^{2}\right)dt \tag{3}$$

$$F(x;\lambda,\beta,\eta)=\frac{1}{\Gamma\lambda}\gamma\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right)$$

$$F(x;\lambda,\beta,\eta)=\gamma_{*}\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right) \tag{4}$$

Using expansion of incomplete gamma ratio function $\gamma_{*}(a,x)$ in [7] the above equation (4) can be expressed as:

$$\gamma_{*}\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right)=\sum_{q=0}^{\infty}\frac{(-1)^{q}\left\{\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right\}^{\lambda+q}}{(\lambda-1)!q!(\lambda+q)} \tag{5}$$

The pdf of the Nak-G is given by

$$f(x)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}g(x;\eta)\frac{[G(x;\eta)]^{2\lambda-1}}{[1-G(x;\eta)]^{2\lambda+1}}\exp\left(-\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right);\ x\in\Re \tag{6}$$

A random variable X with pdf in equation (6) is denoted by $X\sim Nak-G(x;\eta)$ the survival function and hazard rate function (hrf) of X are given by:

$$S(x) = 1 - \gamma_* \left(\lambda, \frac{\lambda}{\beta} \left(\frac{G(x;\eta)}{\bar{G}(x;\eta)} \right)^2 \right) \tag{7}$$

and

$$h(x) = \frac{\frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} g(x;\eta) \frac{[G(x;\eta)]^{2\lambda-1}}{[1-G(x;\eta)]^{2\lambda+1}} \exp\left(-\frac{\lambda}{\beta} \left(\frac{G(x;\eta)}{\bar{G}(x;\eta)} \right)^2 \right)}{1 - \gamma_* \left(\lambda, \frac{\lambda}{\beta} \left(\frac{G(x;\eta)}{\bar{G}(x;\eta)} \right)^2 \right)} \tag{8}$$

2.1. Linear Representation

In this section, we derive some very useful linear representation for the Nak-G density function. Note that.

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \tag{9}$$

Therefore, applying equation (9) to (6)

$$f(x) = \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} g(x;\eta) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\lambda}{\beta} \right)^k \frac{[G(x;\eta)]^{2(\lambda+k)-1}}{[1-G(x;\eta)]^{2(\lambda+k)+1}} \tag{10}$$

Consider the binomial expansion theorem
$(1-z)^{-b} = \sum_{j=0}^{\infty} \binom{b+j-1}{j} z^j, |z| < 1, b > 0$ then

$$[1 - G(x;\eta)]^{-2(\lambda+k)+1} = \sum_{j=0}^{\infty} \binom{2(\lambda+k)+j}{j} [G(x;\eta)]^j, [2(\lambda+k)+1] > 0 \tag{11}$$

Therefore, applying equation (11) to (10)

$$f(x) = \frac{2\lambda^\lambda [2(\lambda+k) + 2j(\lambda+k) + j]}{\Gamma(\lambda)\beta^\lambda [2(\lambda+k)+j]} \sum_{k,j=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\lambda}{\beta} \right)^k \binom{2(\lambda+k)+j}{j}$$
$$g(x;\eta)[G(x;\eta)]^{2(\lambda+k)+j-1} \tag{12}$$

Also, the pdf equation (12) can be written as

$$f(x) = \sum_{k,j=0}^{\infty} \pi_{k,j} h_{2(\lambda+k)+j}(x) \tag{13}$$

where

$$\pi_{k,j} = \frac{2}{\Gamma(\lambda)} \frac{(-1)^k}{k!} \left(\frac{\lambda}{\beta} \right)^{\lambda+k} \binom{2(\lambda+k)+j}{j}$$

and

$$h_{2(\lambda+k)+j}(x) = (2(\lambda+k)+j) g(x;\eta)[G(x;\eta)]^{2(\lambda+k)+j-1}$$

Equation (13) can be well-defined as an infinite linear combination of exponentiated -G (exp $-G$) densities. Similarly, the cdf of the Nak-G family can also be expressed as a linear combination of exponentiated-G (exp $-G$) cdfs given by

$$F(x) = \sum_{k,j}^{\infty} \pi_{k,j} H_{2(\lambda+k)+j}(x) \tag{14}$$

where $H_{2(\lambda+k)+j}(x) = [G(x;\eta)]^{2(\lambda+k)+j}$ is the cdf of the exp $-G$ family with power parameter.

3. The Nakagami Exponential (NE) Distribution

Our baseline distribution, the Exponential distribution with parameter α has its cdf and pdf given by:

$$G(x;\alpha) = 1 - e^{-\alpha x} \tag{15}$$

$$g(x;\alpha) = \alpha e^{-\alpha x}; \alpha > 0, x > 0 \tag{16}$$

Substituting equation (15) and (16) in (4) and (6) then, the cdf and pdf of NE distribution can be written as

$$F_{NE}(x) = \gamma_* \left(\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}} \right)^2 \right) \tag{17}$$

$$f_{NE}(x) = \frac{2\lambda^{\lambda} \alpha e^{-\alpha x} \left(1 - e^{-\alpha x}\right)^{2\lambda - 1}}{\Gamma(\lambda) \beta^{\lambda} \left(e^{-\alpha x}\right)^{2\lambda + 1}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} \tag{18}$$

3.1. *Investigation of the Proposed (NE) Distribution for PDF*

To show that the proposed distribution is a proper pdf, we proceed to show as follows:

$$\int_0^{\infty} f(x)dx = 1 \tag{19}$$

$$\int_0^{\infty} \frac{2\lambda^{\lambda} \alpha e^{-\alpha x} \left(1 - e^{-\alpha x}\right)^{2\lambda - 1}}{\Gamma(\lambda) \beta^{\lambda} \left(e^{-\alpha x}\right)^{2\lambda + 1}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} dx = 1$$

$$y = \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}} \right)^2 \tag{20}$$

$$\frac{\partial y}{\partial x} = \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}} \right) \left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}} \right)$$

$$\partial x = \frac{\beta e^{-2\alpha x}}{2\alpha\lambda(1 - e^{-}\alpha x)} \partial y$$

$$\int_0^{x} f(x)dx \equiv \frac{\lambda^{\lambda - 1}}{\Gamma(\lambda)\beta^{\lambda - 1}} \int_0^{\infty} \frac{(1 - e^{-\alpha})^{2\lambda - 2}}{(e^{-\alpha})^{2\lambda - 2}} e^{-y} \partial y \tag{21}$$

from (20)

$$\left(\frac{y\beta}{\lambda}\right)^{\frac{1}{2}} \tag{22}$$

Therefore, from (21) and (22) we obtained

$$\frac{1}{\Gamma(\lambda)}\int_0^\infty y^{\lambda-1}e^{-y}\partial y = 1 \tag{23}$$

Hence Nakagami Exponential Distribution is pdf

3.2. *Expansion for Nakagami Exponential Distribution*

In this part a simple form for the probability density function of NE distribution is derived. Applying equation (9) into (18) we obtained

$$f_{NE}(x) = \frac{2\alpha}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\left(1-e^{-\alpha x}\right)^{2(\lambda+k)-1}\left(e^{-\alpha x}\right)^{-2(\lambda+k)} \tag{24}$$

The binomial expansion of $(1-e^{-\alpha x})$ can be expressed as $\sum_{i=0}^{\infty}(-1)^i\binom{2(\lambda+k)-1}{i}e^{-\alpha x}$ Therefore, equation (24) will take the following form

$$f_{NE}(x) = \frac{2\alpha}{\Gamma(\lambda)}\sum_{k,i=0}^{\infty}\frac{(-1)^{k+i}}{k!}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\binom{2(\lambda+k)-1}{i}\left(e^{-\alpha x}\right)^{i-2(\lambda+k)} \tag{25}$$

Therefore, the NE pdf distribution is reduced to

$$f_{NE}(x) = \frac{2\alpha}{\Gamma(\lambda)}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\sum_{k,i=0}^{\infty}\omega_{k,i}\left(e^{-\alpha x}\right)^{i-2(\lambda+k)} \tag{26}$$

where $\omega_{k,i} = \frac{(-1)^{k+i}}{k!}\binom{2(\lambda+k)-1}{i}$.
While the cumulative distribution function (cdf), survival function and hazard functions are given respectively by equations (27), (28) and (29).

$$F_{NE}(x) = \gamma_*\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right] \tag{27}$$

$$S_{NE}(x) = 1-\gamma_*\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right] \tag{28}$$

$$H_{NE}(x) = \frac{\frac{2\alpha}{\Gamma(\lambda)}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\sum_{k,i=0}^{\infty}\omega_{k,i}\left(e^{-\alpha x}\right)^{i-2(\lambda+k)}}{1-\gamma_*\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right]} \tag{29}$$

3.3. *Some Mathematical and Statistical Properties*

In this section, some general mathematical and statistical properties of Nak-G distribution are derived.

3.4. Moment and Moment Generating Function

In this subsection, the r^{th} moment and moment generating function for Nak-G distribution will be derived. The rth moment of random variable can be obtained from pdf equation in (13) as follows;

$$\mu_r' = \int_0^\infty x^r f(x)\partial x = \pi_{k,j} \sum_{k,j=0}^{\infty} x^r h_{2(\lambda+k)+j}(x)\partial x$$

therefore,

$$\mu_r' = \pi_{k,j} I_{r,2(\lambda+k)+j}, r = 1, 2, 3, \ldots \tag{30}$$

where

$$I_{r,2(\lambda+k)+j} = \sum_{k,j=0}^{\infty} x^r h_{2(\lambda+k)+j}(x)\partial x$$

The mean and variance of Nak-G distribution are obtained, respectively as follows

$$E(x) = \pi_{k,j} I_{r,2(\lambda+k)+j} \tag{31}$$

where,

$$I_{r,2(\lambda+k)+j} = \sum_{k,j=0}^{\infty} x h_{2(\lambda+k)+j}(x)\partial x$$

and

$$Var(x) = \pi_{k,j} I_{2,2(\lambda+k)+j} - \left[\pi_{k,j} I_{1,2(\lambda+k)+j}\right]^2 \tag{32}$$

, where

$$I_{2,2(\lambda+k)+j} = \sum_{k,j=0}^{\infty} x^2 h_{2(\lambda+k)+j}(x)\partial x \tag{33}$$

From equation (30) the measures of skewness γ_1 and kurtosis γ_2 of Nak-G distribution can be expressed as follows

$$\gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1'\mu_1'^3}{(\mu_2' - \mu_1'^2)^{\frac{3}{2}}}, \tag{34}$$

$$\gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2} \tag{35}$$

Furthermore, the moment generating function can be obtained by using pdf equation (13) as follows

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r \mu_r'}{r!} = \sum_{r=0}^{\infty} \frac{t^r \pi_{k,j} I_{r,2(\lambda+k)+j}}{r!} \tag{36}$$

3.4.1. Moment for Nakagami Exponential Distribution

moment can be obtained by using pdf in equation (26) as follows

$$E(X^r) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}} \int_0^{\infty} x^r e^{-\alpha x[i-2(\lambda+k)]} \partial x \tag{37}$$

Let

$$u = \alpha x[i-2(\lambda+k)] \Rightarrow \frac{\partial x}{\partial y} = \alpha[i-2(\lambda+k)]$$

$$\frac{\partial u}{\alpha[i-2(\lambda+k)]} = \partial x$$

$$E(X^r) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}} \int_0^{\infty} \frac{u^r}{\alpha^r[i-2(\lambda+k)]^r} e^{-u} \frac{\partial u}{\alpha[i-2(\lambda+k)]}$$

$$= \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}} \int_0^{\infty} \frac{u^r}{\alpha^{r+1}[i-2(\lambda+k)]^{r+1}} e^{-u} \partial u$$

$$E(X^r) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha\Gamma(r+1)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha^{r+1}[i-2(\lambda+k)]^{r+1}}; \; r = 1,2,3,\ldots \tag{38}$$

The mean and variance of NE distribution are obtained, respectively as follows

$$E(X) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\Gamma(2)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha[i-2(\lambda+k)]^2} \tag{39}$$

$$Var(x) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{4\lambda^{\lambda+k}\Gamma(2)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha^2[i-2(\lambda+k)]^3} - \left[\sum_{r=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\Gamma(2)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha[i-2(\lambda+k)]^2}\right]^2 \tag{40}$$

Furthermore, the moment generating function can be obtained by using pdf in equation (26) as follows

$$M_X(t) = E(e^{tX}) = \frac{2\alpha}{\Gamma(\lambda)} \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}} \int_0^{\infty} e^{-x[\alpha[i-2(\lambda+k)]-t]} \partial x$$

Therefore, the moment generating function of NE distribution takes the following form

$$M_X(t) = \frac{2\alpha}{\Gamma(\lambda)} \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}[\alpha[i-2(\lambda+k)]-t]}. \tag{41}$$

3.5. Probability Weighted Moments

[10] stated that for a random variable X, the Probability Weighted Moments (pwm) is given by:

$$\varphi_{s,r} = E\left[X^s F(x)^r\right] = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\partial x \tag{42}$$

we formally define PWM of Nak-G by means of equation (4) and (13)

$$\varphi_{s,r} = \int_0^{\infty} x^s \gamma_* \left[\left(\lambda, \frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right)\right]^r \sum_{k,j=0}^{\infty} \pi_{k,j,i,b} h_{2(\lambda+k)+j}(x)\partial x \tag{43}$$

$$\varphi_{s,r} = \int_0^{\infty} x^s \rho_{k,j,i,b} h_{2[\lambda(r+1)+i+k]+b+j}(x)\partial x \tag{44}$$

where,

$$\rho_{k,j,i,b} = \frac{\sum_{k,j,i,b=0}^{\infty} c_{r,i}\binom{2(\lambda r+i)+b-1}{b}\left(\frac{\lambda}{\beta}\right)^{\lambda r+1}\pi_{k,j}}{[\Gamma(\lambda)]^r}$$

3.6. Measures of Uncertainty

In this subsection, Renyi entropy will be mentioned as an important measure of uncertainty. The Rényi entropy of a random variable X is defined mathematically as follows:

$$I_R(\sigma) = \frac{1}{1-\sigma} log\left(\int_0^{\infty} f^{\sigma}(x)\partial x\right)$$

Where $\sigma > 0$ and $\sigma \neq 1$. Based on $f(x)$ of any distribution. From equation (18)

$$f_{NE}^{\sigma}(x) = \frac{2^{\sigma}\left(\lambda^{\lambda}\right)^{\sigma}\alpha^{\sigma}e^{-\sigma\alpha x}\left(1-e^{-\alpha x}\right)^{\sigma(2\lambda-1)}}{\left(\Gamma(\lambda)\right)^{\sigma}\beta^{\sigma\lambda}\left(e^{-\alpha x}\right)^{\sigma(2\lambda+1)}} e^{-\sigma\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} \tag{45}$$

Since the power series for the following exponential function can be expressed as

$$e^{-\sigma\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\sigma\frac{\lambda}{\beta}\right)^i\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^{2i}$$

Therefore equation (45) can be expressed as

$$f_{NE}^{\sigma}(x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{2\alpha}{\Gamma(\lambda)}\right)^{\sigma}\left(\frac{\lambda}{\beta}\right)^{\lambda\sigma+i}\sigma^i\frac{\left(1-e^{-\alpha x}\right)^{\sigma(2\lambda-1)+2i}}{\left(e^{-\alpha x}\right)^{2\sigma\left(\lambda+\frac{i}{\sigma}\right)}} \tag{46}$$

therefore, (46) is reduced to

$$f_{NE}^{\sigma}(x) = \sum_{i.j=0}^{\infty}\tau_{i,j}e^{\alpha x[2(\sigma\lambda+i)-j]} \tag{47}$$

where

$$\tau_{i,j} = \frac{(-1)^{i+j}}{i!}\binom{\sigma(2\lambda-1)2i}{j}\left(\frac{2\sigma}{\Gamma(\lambda)}\right)^{\sigma}\left(\frac{\lambda}{\beta}\right)^{\lambda\sigma+i}\sigma^{i}$$

since

$$\int_0^\infty f_{NE}^{\sigma}(x)\partial x = \int_0^\infty \sum_{i.j=0}^{\infty}\tau_{i,j}e^{\alpha x[2(\sigma\lambda+i)-j]}\partial x;$$

$$\int_0^\infty f_{NE}^{\sigma}(x)\partial x = \sum_{i.j=0}^{\infty}\frac{\tau_{i,j}}{\alpha[j-2(\sigma\lambda+i)]} \tag{48}$$

therefore, $I_R(\sigma)$ reduces to

$$I_R(\sigma) = \frac{1}{1-\sigma}log\left[\sum_{i.j=0}^{\infty}\frac{\tau_{i,j}}{\alpha[j-2(\sigma\lambda+i)]}\right] \tag{49}$$

3.7. Distribution of Order Statistic

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. denote the order statistics of a random sample, $X_1, X_2, \ldots, X_n$ from a Nak-G distribution with cdf equation (6) and pdf equation (5). Then the pdf of $X_{(j)}$ is given by

$$f_{x_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\sum_{z=0}^{n-j}(-1)^z\binom{n-j}{z}f_X(x)\,[F_X(x)]^{z+j-1} \tag{50}$$

$$f_{x_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\sum_{z=0}^{n-j}(-1)^z\binom{n-j}{z}\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}g(x;\eta)\frac{[G(x;\eta)]^{2\lambda-1}}{[1-G(x;\eta)]^{2\lambda+1}}\exp\left(-\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right)$$
$$\left[\gamma_*\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right)\right]^{z+j-1}$$

3.8. The Asymptotic Properties

We study the asymptotic behavior of NE distribution with a view to influential its performance limit as $x \to \infty$ is 0 and the limit as $x \to 0$ is 0.

Proof:

These can be achieved as follows by taking the limiting behavior of the NE density function in equation (18).

$$\lim_{x\to\infty} f_{NE}(x) = \lim_{x\to\infty}\left[\frac{2\lambda^{\lambda}\alpha e^{-\alpha x}\,(1-e^{-\alpha x})^{2\lambda-1}}{\Gamma(\lambda)\beta^{\lambda}\,(e^{-\alpha x})^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2}\right] = 0$$
$$\lim_{x\to 0} f_{NE}(x) = \lim_{x\to 0}\left[\frac{2\lambda^{\lambda}\alpha e^{-\alpha x}\,(1-e^{-\alpha x})^{2\lambda-1}}{\Gamma(\lambda)\beta^{\lambda}\,(e^{-\alpha x})^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2}\right] = 0$$

Then $f_{NE}(x)$ has at least one mode.

3.9. Quantile Function

Quantile functions are normally used to describe a probability distribution, simulations and statistical application. Simulation techniques utilize quantile function to create simulated random variables for standard and new continuous distributions. In general, it is given as: $Q(u) = F^{-1}(u)$.
$U\ Uniform(0, 1)$. That is U follows a uniform distribution.
By considering equation (4) quantile function (qf) X is obtained as follows:

$$u = F(x;\lambda\beta\eta) = \frac{1}{\Gamma\lambda}\gamma\left(\lambda, \frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right) \tag{51}$$

$$x = \frac{-1}{\alpha}ln\left\langle 1 - \left\{\frac{\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, u\Gamma(\lambda))\right]}{1 + \left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, u\Gamma(\lambda))\right]}\right\}^{\frac{1}{2}}\right\rangle \tag{52}$$

3.10. Shape of the Crucial Functions

The shapes of the density and hazard function of the Nak-G family can be defined analytically. The critical points of the Nak-G density function equation (6) are the roots of the resulting equation:

$$\frac{g'(x;\eta)}{g(x;\eta)} + \frac{(2\lambda-1)g(x;\eta)}{G(x;\eta)} + \frac{(2\lambda+1)g(x;\eta)}{\bar{G}(x;\eta)} - \frac{2\lambda g(x;\eta)}{\beta\left[\bar{G}(x;\eta)\right]^3} = 0 \tag{53}$$

The critical points of Nak-G hazard function obtained in equation (8) are obtained from the following equation:

$$\frac{g'(x;\eta)}{g(x;\eta)} + \frac{(2\lambda-1)g(x;\eta)}{G(x;\eta)} + \frac{(2\lambda+1)g(x;\eta)}{\bar{G}(x;\eta)} - \frac{2\lambda g(x;\eta)}{\beta\left[\bar{G}(x;\eta)\right]^3} + e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2}\frac{2\lambda^\lambda\alpha e^{-\alpha x}\left(1-e^{-\alpha x}\right)^{2\lambda-1}}{\Gamma(\lambda)\beta^\lambda\left(e^{-\alpha x}\right)^{2\lambda+1}}$$
$$\left\langle\Gamma(\lambda) - \gamma\left(\lambda, \tfrac{\lambda}{\beta}\left(\tfrac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right)\right\rangle = 0$$

By using R software, we can examine equations (53) and (54) to determine the local maximums and minimums and inflexion points.

3.11. Maximum Likelihood Estimation

This subsection, deals with the ML estimators of the unknown parameters for the Nak-G family of distributions based on complete samples of size n. Let $X_1, X_2, \ldots, X_n$ be observed values from the Nak-G family with set of parameter $\Theta = (\lambda, \beta, \eta)$. The log-likelihood function for parameter vector $\Theta = (\lambda, \beta, \eta)$ is obtained from equation (6) as follows

$$\ell(\Theta) = nln2 + n\lambda ln\lambda - nln\Gamma(\lambda) - n\lambda ln\beta + \sum_{i=0}^{\infty} ln\left[g(x;\eta)\right] + (2\lambda-1)\cdot\sum_{i=0}^{\infty} ln\left[G(x;\eta)\right] - (2\lambda+1)\sum_{i=0}^{\infty} ln\left[1 - G(x;\eta)\right] -$$
$$\frac{\lambda}{\beta}\sum_{i=0}^{\infty} ln\left[W(x;\eta)\right]^2 \tag{54}$$

where $W(x;\eta) = \frac{G(x;\eta)}{1-G(x;\eta)}$

The components of the score function $U(\Theta) = \left(U_\lambda, U_\beta, U_\eta\right)$ are given by

$$U_\lambda = nln\lambda - n - n\Psi(\lambda) - nln\beta + 2\sum_{i=0}^{\infty} ln\left[G(x;\eta)\right] - 2\sum_{i=0}^{\infty} ln\left[1 - G(x;\eta)\right] - \frac{1}{\beta}\sum_{i=0}^{\infty} ln\left[W(x;\eta)\right] \quad (55)$$

$$U_\beta = n\frac{\lambda}{\beta} + \frac{\lambda \sum_{i=0}^{\infty} ln\left[W(x;\eta)\right]^2}{\beta^2} \quad (56)$$

$$U_\eta = \sum_{i=0}^{\infty} \frac{\partial g(x;\eta)/\partial\eta}{g(x;\eta)} + (2\lambda - 1)\sum_{i=0}^{\infty} \frac{\partial g(x;\eta)/\partial\eta}{G(x;\eta)} + (2\lambda + 1)\sum_{i=0}^{\infty} \frac{\partial g(x;\eta)/\partial\eta}{1 - G(x;\eta)} - \frac{2\lambda}{\beta} \cdot \sum_{i=0}^{\infty} W(x;\eta)w(x;\eta) \quad (57)$$

Setting $U_\lambda, U_\beta, U_\eta$ equate to zero and solving the equations simultaneously result to the ML estimates $\hat{\Theta} = (\hat{\lambda}, \hat{\beta}, \hat{\eta})$ of $\Theta = (\lambda, \beta, \eta)^\tau$.

These estimates can not be solved algebraically and statistical software can be used to solve them numerically via iterative technique.

4. Result and Discussion

The first real life data set was obtained on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been formerly used by [15] and [16]. The data is as follows: 0.39, 0.85, 1.08 ,1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87 ,1.89, 2.03, 2.03 ,2.05, 2.12 ,2.35 ,2.41, 2.43, 2.48, 2.50, 2.53 ,2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87 ,2.88, 2.93, 2.95 ,2.96 ,2.97, 3.09, 3.11 ,3.11, 3.15, 3.15 ,3.19 ,3.22, 3.22, 3.27, 3.28, 3.31 ,3.31, 3.33 ,3.39, 3.39, 3.56 ,3.60 ,3.65, 3.68 ,3.70 ,3.75 ,4.20 ,4.38, 4.42, 4.70, 4.90

Table 1: MLEs and Goodness-of-fit measures for First Data Set

Model	MLE	ℓ	AIC	BIC	CAIC
Nak-Exp	λ = 1.2778 β=2.1709 α=0.2964	-85.88033	177.7607	187.3296	184.3296
GOG-Exp	λ = 0.6170 β = 4.0054 α = 0.4087	-85.92746	177.8549	187.4239	184.4239
Wei-Exp	λ = 0.7704 β = 2.4675 α = 0.2389	-85.97049	177.941	187.5099	184.5099
Kum-Exp	λ = 5.13720 β = 10.23005 α = 0.33140	-88.10031	182.2006	191.7696	188.7696
Beta-Exp	λ = 8.19864 β = 4.98148 α = 0.37362	-91.78444	189.5689	199.1378	196.1378
Exp.	α = 0.36235	-132.9944	267.9887	271.1785	270.1785
Gamma-Exp	λ = 0.337 β = 1.141 α = 11.458	-127.4033	260.8066	270.3756	267.3756

The second data set represents the times of failures and running times for sample of devices from an eld-tracking study of a larger system. The data set has been previously studied by [13] and [14]. The data set has thirty (30) observations and they are as follows: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00,3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66 The third real life data set [9] corresponds to fifty two ordered annual maximum antecedent rainfall

Table 2: MLEs and Goodness-of-fit measures for Second Data Set

Model	MLE	ℓ	AIC	BIC	CAIC
Nak-Exp	λ=0.35257 β=6.96172 α=0.54568	-38.92824	83.85647	88.06007	91.06007
Wei-Exp	λ=0.13324 β = 0.56404 α = 1.60267	-39.07062	84.14124	88.34483	91.34483
Exp.	α = 0.5648	-47.13504	96.27007	97.67128	98.67128

measurements in mm from Maple 264.9, 314.1, 364.6, 379.8, 419.3, 457.4, 459.4, 460, 490.3, 490.6, 502.2, 525.2, 526.8, 528.6, 528.6, 537.7, 539.6, 540.8, 551.0, 573.5, 579.2, 588.2, 588.7, 589.7, 592.1, 592.8, 600.8, 604.4, 608.4, 609.8, 619.2,626.4, 629.4, 636.4, 645.2, 657.6, 663.5, 664.9, 671.7, 673.0, 682.6, 689.8, 698, 698.6, 698.8, 703.2, 755.9, 786, 787.2, 798.6, 850.4, 895.1.

Table 3: MLEs and Goodness-of-fit measures for Third Data Set

Model	MLE	ℓ	AIC	BIC	CAIC
Nak-Exp	λ= 2.5182774 β=0.2482101 α=0.0006204	-329.275	664.5501	670.4037	673.4037
Wei-Exp	λ=1.6478687 β=1.5943460 α=0.0008296	-351.8995	709.799	715.6527	718.6527
Ext.-Burr III	a=12.0863 b=15.3622 α=0.5868 λ=15.4776 s=11.8405	-339.5244	689.0488	698.805	703.805

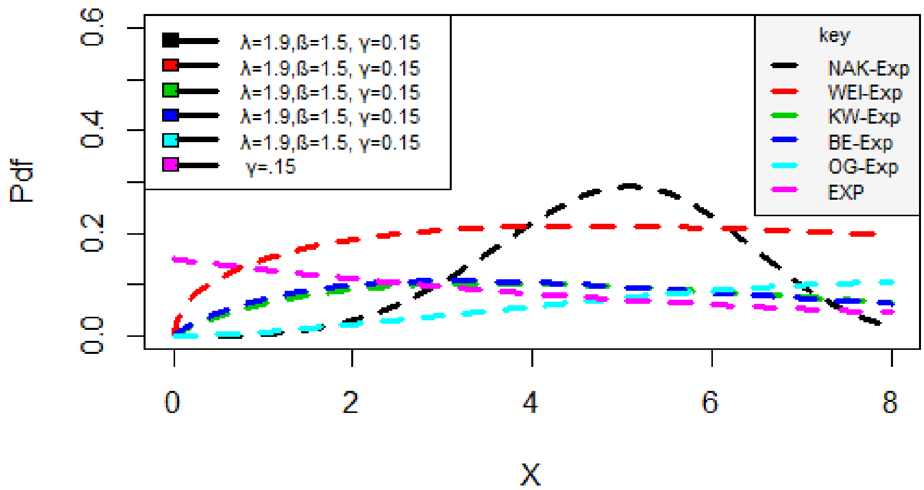

Figure 1: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp and Exp (λ= 1.9 (shape parameter) and β, γ= 1.5, 0.15 (scale parameters))

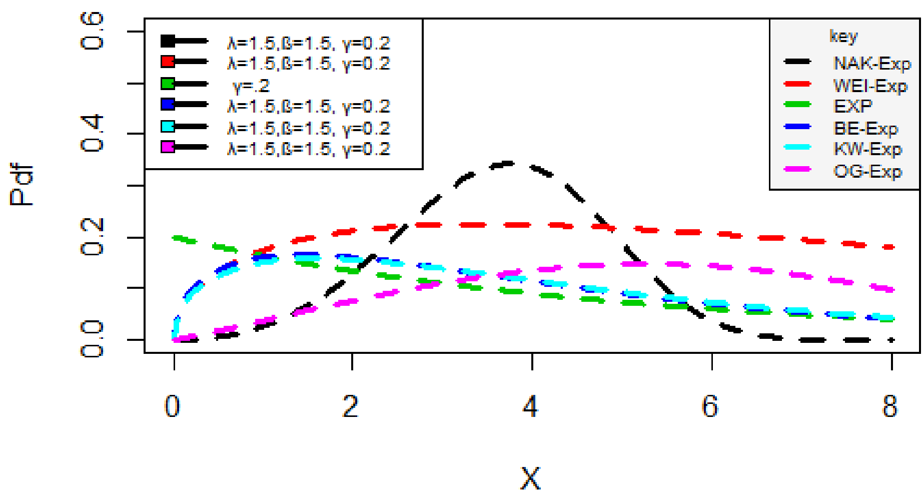

Figure 2: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp and Exp (λ= 1.5 (shape parameter) and β, γ= 1.5, 0.2 (scale parameters))

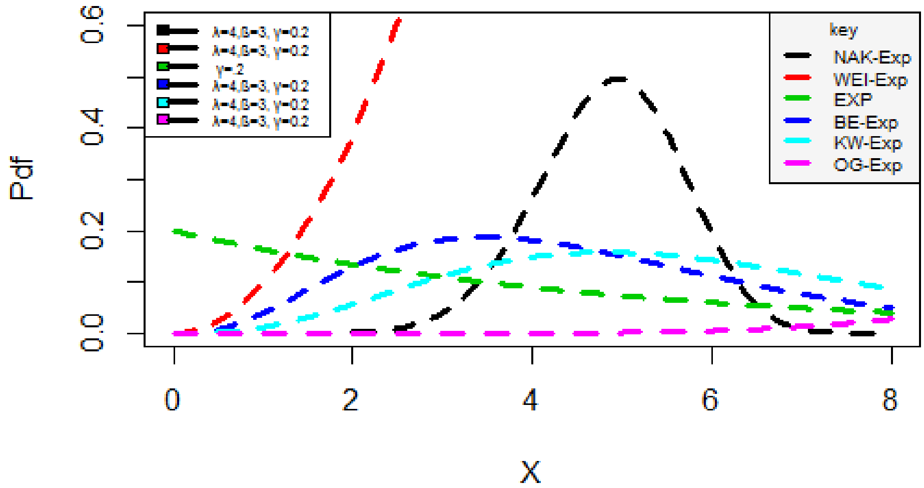

Figure 3: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp andExp (λ= 4 (shape parameter) and β, γ= 3, 0.2 (scale parameters))

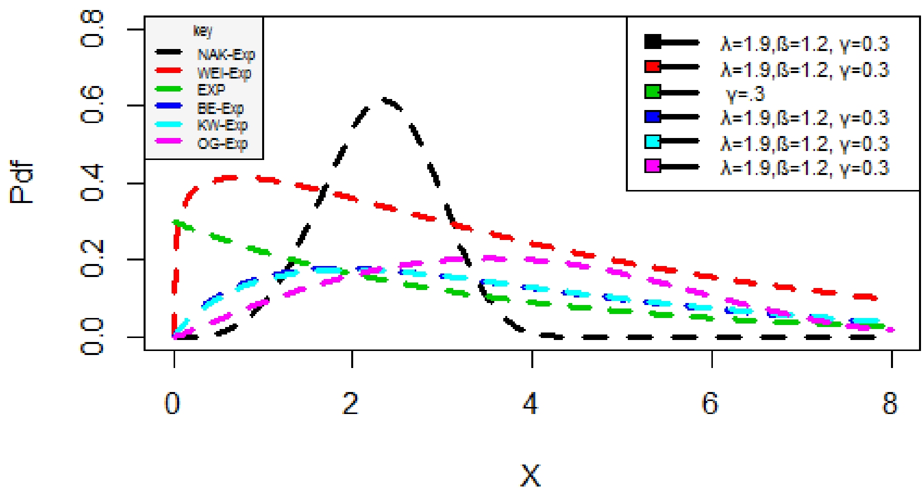

Figure 4: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp and Exp (λ= 1.9 (shape parameter) and β, γ =1.2, 0.3 (scale parameters))

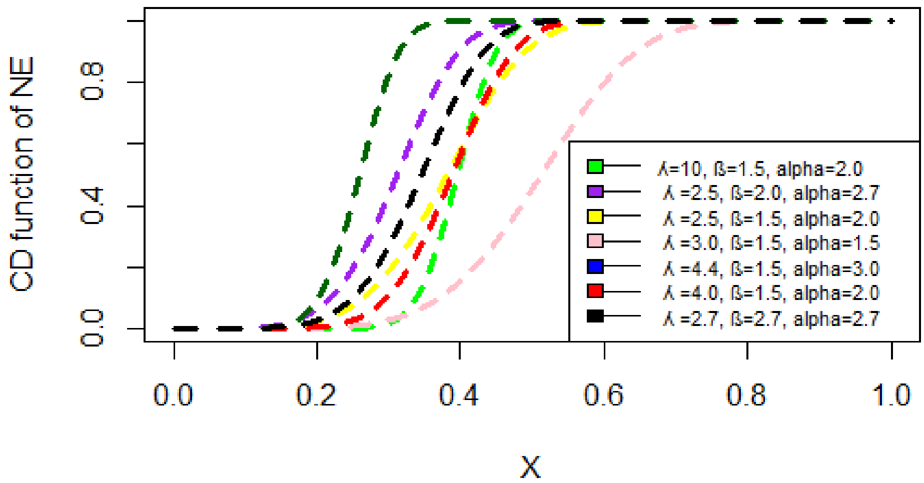

Figure 5: Graph of the Cummulative Distribution of Nak-Exp (λ= shape parameter and β, γ = scale parameters)

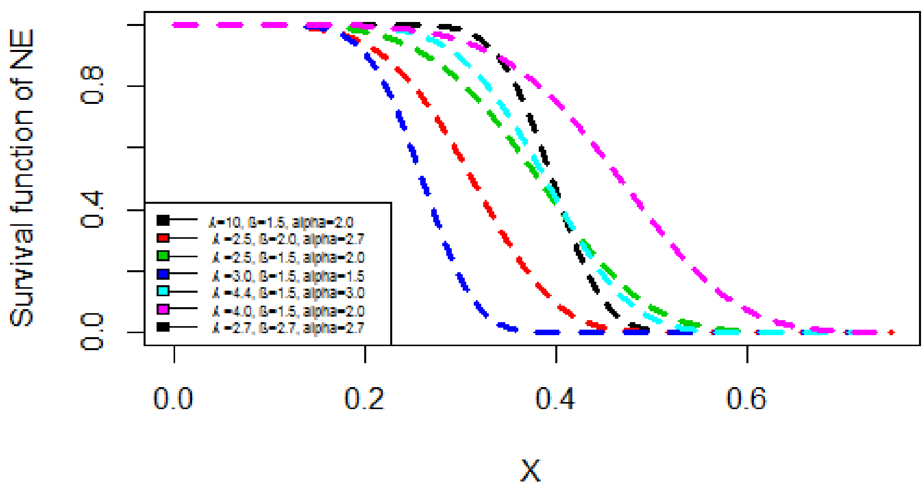

Figure 6: Graph of the Survival Function of Nak-Exp (λ= shape parameter and β, γ = scale parameters)

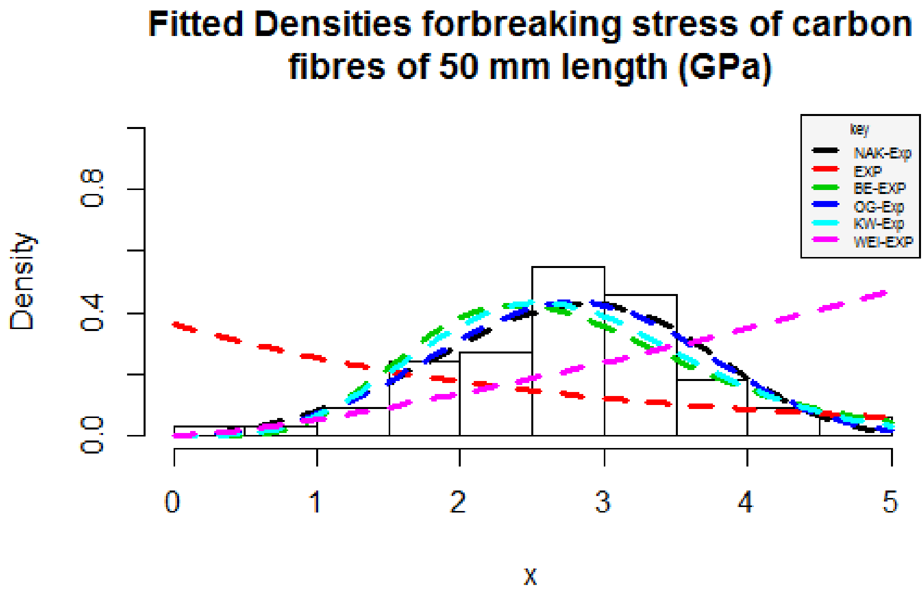

Figure 7: fitted Models on histogram of the first data set

4.1. CONCLUSION

For the first time, we propose a new family of Nakagami-G distributions by add two parameter to Exponential distribution called Nakagami Exponential distribution and some of its statistical properties of the new family were studied. The model parameters were estimated by using the maximum likelihood estimation technique. We finally fit the proposed model among others to real life data show that Nakagami Exponential distribution was found to provide a better fit than its competitors

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