Fundamentals of Contemporary Mathematical Sciences
(**2020**) **1**(2) 49 – 62

Two Types of Quotient Structure of Co-Quasiordered Residuated Systems

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Received: 30 March 2020 **Accepted:** 18 July 2020

Abstract: In our article we introduced and analysed the concept of residuated relational systems ordered under co-quasiorder. In this article, as a continuation of the mentioned paper, we introduce two types of quotient structures of residuated relational systems are constructed, one of which is a specificity of Bishop's constructive framework and has no counterpart in the classical theory. The paper finished by a theorem which can be viewed as the first isomorphism theorem for these algebraic structures.

Keywords: Bishop's constructive mathematics, set with apartness, co-quasiordered residuated system, se-homomorphism.

1. Introduction

In [4], Bonzio and Chajda introduced and analyzed the structure of the residual relational system. Previously, this system was studied in Bonzio's doctoral thesis [3]. In our article [13], we are developed this concept within the Bishop's constructive framework. (On this principled-philosophical orientation, see for example [1, 2, 5, 6, 15].) In that article, we observed and analyzed a residuated relational system $\langle A, \cdot, \rightarrow, 1, R\rangle$ on a set with apartness $(A, =, \neq)$ as the carrier of the algebraic construction, and additionally R was a co-quasiorder relation on the set $(A, =, \neq)$.

In this article we continue our analyze of co-quasiordered residuated relational systems [13, 14]. Important contribution in this paper are Theorem 3.6 and Theorem 3.7, in which two types of quotient structures are constructed, one of which is one of the specificities of Bishop's constructive framework and has no counterpart in the classical theory. The second quotient structure appears naturally in this logical environment. The strong link between the two structures is described in Theorem 3.8. The paper finished by a theorem (Theorem 3.10) which can be viewed as the first isomorphism theorem for these algebraic structures.

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2010 *AMS Mathematics Subject Classification*: 03F65, 08A02

2. Preliminaries

The setting of this research is the Bishop's constructive mathematics **Bish** in the sense of the following books [1, 2, 5, 6] - a mathematics based on the intuitionistic logic **IL** (see [15]) and principled-philosophical orientation on Bishop's constructive mathematics.

In this paper, we literally use the notions and notations from our three previous papers [12–14] dedicated to this topic.

Let $(S,=,\neq)$ be a (constructive) set in the sense of Bishop [1], Mines et all. [6], Troelstra and van Dalen [15]. On set $S=(S,=,\neq)$ in this mathematics we look as on a relational system with an one binary relation extensive with respect to the equality. The relation "$\neq$" is a binary relation on S with the following properties:

$$\neg(x\neq x),\; x\neq y\Longrightarrow y\neq x,\; x\neq z\Longrightarrow x\neq y\vee y\neq z,$$
$$x\neq y\wedge y=z\Longrightarrow x\neq z.$$

It is called *apartness*. Let S and T be two sets with apartness. Then the relation "$\neq$" on $S\times T$ is defined by

$$(x,y)\neq(u,v)\Longleftrightarrow(x\neq u\vee y\neq v)$$

for any $x,u\in S$ and any $y,v\in T$.

Let Y be a subset of S and $x\in S$. We put it the following notation "$\lhd$" as a relation between an element x and subset Y with (for more details on this relation, the readers can see the following texts [11, 12])

$$x\lhd Y \quad\Longleftrightarrow\quad (\forall y\in Y)(\,x\neq y).$$

Following the orientation in books [1], [2], [5] we define a subset

$$Y^{\lhd}=\{x\in S: x\lhd Y\}$$

of S called the *complement of* Y *in* S.

For subset Y of S we say that it is a *strongly extensional subset* if

$$(\forall x,y\in S)(y\in Y\Longrightarrow x\neq y\;\vee\; x\in Y).$$

For a relation R on S it is called a *strongly extensional* if holds

$$(\forall x,y,z,u\in S)((x,y)\in R\implies((x,y)\neq(z,u)\vee(z,u)\in R))$$

. For example, for a mapping $f:S\longrightarrow T$ it is called a *strongly extensional* (shortly: *se-mapping*) if holds

$$(\forall x,y\in S)(f(x)\neq f(y)\implies x\neq y).$$

Also, the following specific terms for this domain should be mentioned:

- A se-mapping $f : S \longrightarrow T$ is an *embedding* if holds

$$(\forall x, y \in S)(x \neq y \implies f(x) \neq f(y)),$$

- A se-mapping $f : S \longrightarrow T$ is a *se-isomorphism* if it is injective, embedding and onto.

As is usual in Bishop's constructive orientation, a dual concept Y, determined by apartness relation and strongly extensional predicates, to a classical algebraic concept X should be associated with these classical concept. This correlation is shown by proving that the strong complement $Y^{\lhd}$ of concept Y has the properties of concept X determined in the classical way. In this regard, see Lemma 2.2.

2.1. Co-Quasiorder Relation

The constructive notion of a co-quasiorder relation is the dual notion to the classical notion of a quasi-order relation. Let $(S, =, \neq)$ be a set with apartness. A consistent and co-transitive relation σ defined on S is called a *co-quasiorder* ([11, 12]):

- $(\forall x, y \in S)((x, y) \in \sigma \implies x \neq y)$ (consistency),
- $(\forall x, y, z \in S)((x, z) \in \sigma \implies ((x, y) \in \sigma \vee (y, z) \in \sigma))$ (co-transitivity).

Example 2.1 *A subset K of S is a detachable subset of S if holds*

$$(\forall u \in S)(u \in K \vee \neg(u \in K)).$$

If we additionally assume that K is a strongly extensional non empty subset of S, then in the case $\neg(u \in K)$ we have

$$x \in K \implies (\forall u \in S)(u \neq x \vee u \in K).$$

Thus it follows $u \neq x \in K$, and hence $u \lhd K$. Let us define a relation σ_K on S by

$$(\forall x, y \in S)((x, y) \in \sigma_K \iff (x \in K \wedge y \lhd K)).$$

Then, the following hold:

- $(x, y) \in \sigma_K \implies x \neq y$, *i.e. the relation σ_K is consistent.*
- *Let $x, y, z \in S$ be such that $(x, z) \in \sigma_K$. Then $x \in K \wedge z \lhd K$. On the other hand, for the element $y \in S$ we have $y \in K \vee y \lhd K$. In the first option $y \in K$ we have $(y, z) \in \sigma_K$. In the second option $y \lhd K$, we have $(x, y) \in \sigma_K$. So, σ_K is a co-transitive relation. Therefore, σ_K is a co-quasiorder relation on S.*

We accept that the empty set $\emptyset$ is also a co-quasiorder relation on set S. The strong complement $\sigma^{\lhd}$ of a co-quasiorder σ has the well known property.

Lemma 2.2 ([11], Lemma 2.2) *If σ is a co-quasiorder on S, then the relation $\sigma^{\lhd} = \{(x,y) \in S \times S : (x,y) \lhd \sigma\}$ is a quasi-order on S.*

In what follows, we will also use the following notion. A co-quasiorder relation "$\nleqslant$" is a *co-order* if it satisfies the following condition

$$(\forall x, y \in S)(x \neq y \implies (x \nleqslant y \vee y \nleqslant x)) \qquad \text{(linearity)}.$$

2.2. Co-Ordered Residuated System

In our paper [13], following the ideas of Bonzio and Chajda [4], we introduced the notion of residuated relational systems ordered under a co-quasiorder - residuated relational systems $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ where R is a co-quasiorder relation on set $(A, =, \neq)$.

Our intention in paper [13] was to introduce the concept of residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ where R is a co-quasiorder relation on set $A = (A, =, \neq)$. If R is a co-quasiorder relation on set $(A, =, \neq)$, then the axiom (2) in Definition 1.1 gives $(1,1) \in R \subseteq \neq$ which is a contradiction. That is why we transformed this axiom into the next formula

(2′) $(\forall x \in A)(x \neq 1 \Longrightarrow (x, 1) \in R)$.

Let $(A, =, \neq)$ be a set with apartness. A co-quasiordered residuated system is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \sigma\rangle$, where the axiom (2′) is replaced by (2) and where σ is a co-quasiorder on A.

Definition 2.3 ([13], Definition 2.1) *A co-quasiordered residuated relational system is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \sigma\rangle$, where $A = (A, =, \neq)$ is a set with apartness and where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and σ is a co-quasiorder relation on A and satisfying the following properties:*

(1) $\langle A, \cdot, 1\rangle$ *is a commutative monoid,*

(2′) $(\forall x \in A)(x \neq 1 \Longrightarrow (x, 1) \in \sigma)$,

(3) $(\forall x, y, z \in A)((x \cdot y, z) \in \sigma \Longleftrightarrow (x, y \rightarrow z) \in \sigma)$.

We will refer to the operation "$\cdot$"as multiplication, to "$\rightarrow$"as its residuum and to condition (3) as residuation.

Apart from the difference in the carrier of this constructed algebraic structure, the difference between the residuated relational system in our definition and the definition in article [4] is in the strong extensionality of the internal binary operations in A. Let us note that the internal operations "$\cdot$" and "$\rightarrow$" are total strongly extensional function from $A \times A$ into A:

$$(\forall a, b, a', b' \in A)(a \cdot b \neq a' \cdot b' \Longrightarrow (a, b) \neq (a', b')),$$

$$(\forall a, b, a', b' \in A)(a \rightarrow b \neq a' \rightarrow b' \Longrightarrow (a, b) \neq (a', b')).$$

Proposition 2.4 ([13], Proposition 2.3) *Let $\mathfrak{A}$ be a co-quasiordered residuated relational system. Then*

$$(\forall x, y \in A)((x, y) \in \sigma \Longleftrightarrow 1 \neq x \rightarrow y).$$

In the following theorem we show that the co-quasiorder σ is compatible with the internal operation "$\cdot$".

Theorem 2.5 ([13], Theorem 2.1) *Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then*

$$(\forall x, y, a, b \in A)(a \cdot x, a \cdot y) \in \sigma \ \vee \ (x \cdot b, y \cdot b) \in \sigma \Longrightarrow (x, y) \in \sigma).$$

In the following theorem we show that the co-quasiorder σ is left compatible and right anti-compatible with the internal operation "$\rightarrow$".

Theorem 2.6 ([13], Theorem 2.2) *Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then*

(a) $(\forall x, y, a \in A)((a \rightarrow x, a \rightarrow y) \in \sigma \Longrightarrow (x, y) \in \sigma)$.

(b) $(\forall x, y, b \in A)((y \rightarrow b, x \rightarrow b) \in \sigma \Longrightarrow (x, y) \in \sigma)$.

Speaking by the language of classical algebra, when we speak of the compatibility of the internal binary operations "$\cdot$"and "$\rightarrow$" with the relation σ, we mean on the cancellativity of these operations with respect to σ.

3. The Main Results

3.1. Concept of Co-Ideals

Definition 3.1 *A subset K of A is a* co-ideal *of a residuated system $\mathfrak{A}$ ordered under a co-quasiorder σ if the following conditions hold*

(K1) $(\forall x, y \in A)(x \cdot y \in K \Longrightarrow x \in K \ \vee \ y \in K)$,

(K2) $(\forall x, y \in A)(x \in K \Longrightarrow (x, y) \in \sigma \ \vee \ y \in K)$.

Condition (K1) states that a co-ideal K is a co-subgroupoid in $(A, \cdot)$.

Remark 3.2 *If we assume that $\sigma \cap \sigma^{-1} = \emptyset$, then we conclude that $\neg(1 \in K)$. Indeed, suppose that $1 \in K$ and $y \lhd K$. Then $1 \in K \Longrightarrow ((1, y) \in \sigma \vee y \in K)$ by (K2). We get a contradiction in both cases. So, $\neg(1 \in K)$ if there exists an element $y \in A$ such that $y \lhd K$.*

3.2. Mappings on Co-Quasiordered Residuated Systems

In this subsection, we introduce the concept of se-homomorphism between residuated relational systems ordered under co-quasiorders. Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1_A, \sigma\rangle$ and $\mathfrak{B} = \langle B, \cdot, \rightarrow, 1_B, \tau\rangle$ be two co-quasiordered residuated systems. We can not make a graphical distinction between operations in $\mathfrak{A}$ and $\mathfrak{B}$ systems since it is clear from context which carrier of the algebraic structure is involved.

Definition 3.3 *A se-mapping $f : A \longrightarrow B$ is a* se-homomorphism *between systems $\mathfrak{A}$ and $\mathfrak{B}$ if the following hold*

(1) $f(1_A) = 1_B$,

(2) $(\forall x, y \in A)(f(x \cdot y) = f(x) \cdot f(y))$,

(3) $(\forall x, y \in A)(f(x \rightarrow y) = f(x) \rightarrow f(y))$,

(4) *f is an isotone mapping if*

$$(\forall x, y \in A)((x, y) \in \sigma \implies (f(x), f(y)) \in \tau),$$

(5) *f is a reverse isotone mapping if*

$$(\forall x, y \in A)((f(x), f(y)) \in \tau \implies (x, y) \in \sigma).$$

Onto se-homomorphism is a se-epimorphism. Injective se-homomorphism is a se-monomorphism.

The following statements can be proven without difficulty.

Lemma 3.4 *Let $f : A \longrightarrow B$ be a se-mapping between systems $\mathfrak{A}$ and $\mathfrak{B}$. Then*

(a) *The relation $f^{-1}(\tau) = \{(x, y) \in A \times A : (f(x), f(y)) \in \tau\}$ is a co-quasiorder relation on $\mathfrak{A}$,*

(b) *f is a reverse isotone se-mapping if and only if $f^{-1}(\tau) \subseteq \sigma$,*

(c) *f is an isotone se-mapping if and only if $\sigma \subseteq f^{-1}(\tau)$.*

Example 3.5 *Let f be a reverse isotone se-homomorphism between systems $\mathfrak{A}$ and $\mathfrak{B}$. The subset $f^{-1}(\{1_B\}^{\lhd})$ is a co-ideal of $\mathfrak{A}$. Indeed:*

$$\begin{aligned} x \cdot y \in f^{-1}(\{1_B\}^{\lhd}) &\iff f(x) \cdot f(y) = f(x \cdot y) \neq 1_B \\ &\implies f(x) \neq 1_B \wedge f(y) \neq 1_B \\ &\iff x \in f^{-1}(\{1_B\}^{\lhd}) \wedge y \in f^{-1}(\{1_B\}^{\lhd}), \end{aligned}$$

where $\wedge$ is the notation for logical conjunction.

Let $x, y \in A$ be elements such that $x \in f^{-1}(\{1_B\}^{\lhd})$. Then $f(x) \neq 1_B$. Thus, $(f(x), 1_B) \in \tau$ by (2′). It follows that $(f(x), f(y)) \in \tau \vee (f(y), 1_B) \in \tau$ by co-transitivity of τ. Hence, $(x, y) \in \sigma$ or $f(y) \neq 1_B$. So, the set $f^{-1}(\{1_B\}^{\lhd})$ satisfies the condition (K2).

3.3. Quotient Structures

In what follows, we use the following two specific terms that appear in this logical environment. The first of these is the notion of co-equivalence on a set with apartness S. A relation q on S is a *co-equivalence* on S if it is consistent, symmetric and co-transitive (see, for example [8]). In addition to the above, suppose S is supplied by an internal binary operation "$\cdot$". In that case, co-equality q is a *co-congruence* ([7, 12]) on structure $(S, \cdot)$ if the following holds

$$(\forall x, y, u, v \in S)((x \cdot u, y \cdot v) \in q \Longrightarrow ((x, y) \in q \vee (u, v) \in q)).$$

If the previous formula is valid, we say that co-equivalence q is *compatible* with the operation in S. Without much difficulty, it can be shown that this formula is equivalent to the following formula

$$(\forall x, y, u, v \in S)(((u \cdot x, u \cdot y) \in q \vee (x \cdot v, y \cdot v) \in q) \Longrightarrow (x, y) \in q).$$

Speaking in the classical algebra language, a co-equality q is a co-congruence in S if the operation in S is cancellative with respect to q.

Let us recall that any co-quasiorder relation σ on a set $(A, =, \neq)$ generates a co-equality relation $\theta = \sigma \cup \sigma^{-1}$ ([9], Lemma 1) (θ is a co-equality relation on set A if it is a consistent, symmetric and co-transitive relation on A). We only need to prove that θ is compatible with the multiplication and its residuum. According to Theorem 2.1 and Theorem 2.2 in [12], the co-quasiorder relation σ is compatible with the operation "$\cdot$". σ is left compatible and right anti-compatible with the operation "$\rightarrow$". Now it's easy to show that θ is compatible with these operations. So, θ is a co-congruence on A. The importance of relation θ is justified by the fact that the quotient-set $A/(\theta^{\triangleleft}, \theta) = \{[x] : x \in A\}$ with

$$(\forall x, y \in A)(([x] =_1 [y] \iff (x, y) \lhd \theta) \text{ and } ([x] \neq_1 [y] \iff (x, y) \in \theta))$$

is naturally constructed groupoid (for example, see [9, 10]) ordered under a co-order "$\nleqslant$" ([10], Lemma 1) defined by

$$(\forall x, y \in A)([x] \bullet [y] =_1 [x \cdot y])$$

and

$$(\forall x, y \in A)([x] \nleqslant_1 [y] \iff (x, y) \in \sigma).$$

Therefore, $(A/(\theta^{\triangleleft}, \theta), \bullet, \nleqslant_1)$ is a commutative monoid ordered under the co-order relation "$\nleqslant$". In addition, it is obvious that

$$(\forall x \in A)([x] \neq_1 [1] \Longrightarrow [x] \nleqslant_1 [1])$$

is valid under assumption

(S) $(\forall x \in A)\neg((1, x) \in \sigma)$.

If we put $[y] \rightarrow_1 [z] :=_1 [y \rightarrow z]$, we get

$$[x] \bullet [y] \nleqslant_1 [z] \iff [x \cdot y] \nleqslant_1 [z]$$

$$\iff (x \cdot y, z) \in \sigma \iff (x, y \rightarrow z) \in \sigma$$

$$\iff [x] \nleqslant_1 [y \rightarrow z] \iff [x] \nleqslant_1 [y] \rightarrow_1 [z].$$

In order to justify this procedure, it is sufficient to verify that "$\rightarrow_1$" is a strongly extensional internal operation in $A/(\theta^{\lhd}, \theta)$ correctly determined on this way.

To illustrate the techniques used in the proof, we show that "$\rightarrow_1$" is strictly a well-defined extensive internal operation on $A/(\theta^{\lhd}, \theta)$. Let $y, y', z, z', u, v \in A$ be arbitrary elements such that $(u, v) \in \theta$.

(i) First suppose that $[y] =_1 [y']$ and $[z] =_1 [z']$. Then $(y, y') \lhd \theta$ and $(z, z') \lhd \theta$. Second, from $(u, v) \in \theta$ it follows that

$$(u, y \rightarrow z) \in \theta \vee (y \rightarrow z, y' \rightarrow z') \in \theta \vee (y' \rightarrow z', v) \in \theta$$

by co-transitivity of θ. Since from the second option it follows that $(y, y') \in \theta$ or $(z, z') \in \theta$ by compatibility of θ with the operation "$\rightarrow$", we have a contradiction with hypothesis. Therefore, $(u, y \rightarrow z) \in \theta$ or $(y' \rightarrow z', v) \in \theta$. Thus, $[y \rightarrow z] \neq_1 [u]$ or $[y' \rightarrow z'] \neq_1 [v]$ and $([y] \rightarrow_1 [y'], [z] \rightarrow_1 [z']) \lhd \theta$. So, we have $[y] \rightarrow_1 [y'] =_1 [z] \rightarrow_1 [z']$ which means that "$\rightarrow_1$" is a well-defined function on $A/(\theta^{\lhd}, \theta)$.

(ii) On the other hand, suppose that $[y] \rightarrow_1 [y'] \neq_1 [z] \rightarrow_1 [z']$. Then $[y \rightarrow y'] \neq_1 [z \rightarrow z']$. This means that $(y \rightarrow y', z \rightarrow z') \in \theta$. Thus, it follows immediately that $(y, y') \in \theta \vee (z, z') \in \theta$, since θ is compatible with the operation "$\rightarrow$". This proves that "$\rightarrow_1$" is a strictly extensional function on $A/(\theta^{\lhd}, \theta)$.

Therefore, demonstrations (i) and (ii) prove that "$\rightarrow_1$" is a correctly determined internal binary operation on $A/(\theta^{\lhd}, \theta)$.

We can state the theorem in which the previous analysis is summarized.

Theorem 3.6 *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \sigma\rangle$ be a co-quasiordered residuated relational system where the relation σ satisfies the additional condition* (S). *Then we can construct residual relational systems $\langle (A/(\theta^{\lhd}, \theta), =_1, \neq_1), \bullet, [1], \nleqslant_1\rangle$ where $\theta = \sigma \cup \sigma^{-1}$ ordered under the co-order "$\nleqslant_1$".*

Also, we can construct the family $[A : \theta] = \{x\theta : x \in A\}$ with

$$(\forall x, y \in A)((x\theta =_1 y\theta \iff (x, y) \lhd \theta) \text{ and } (x\theta \neq_1 y\theta \iff (x, y) \in \theta)).$$

Analogous to the previous case, a strongly extensional internal operation "$*$" can be naturally constructed on $[A:\theta]$ as follows (for example, see [9, 10])

$$(\forall x, y \in A)(x\theta * y\theta =_1 (x \cdot y)\theta).$$

In addition, a co-order "$\nleqslant_1$", defined by

$$(\forall x, y \in A)(x\theta \nleqslant_1 y\theta \iff (x, y) \in \sigma),$$

is compatible with the operation in $[A:\theta]$.

Therefore, $([A:\theta], *, \nleqslant_1)$ is a commutative monoid ordered under the co-order relation "$\nleqslant_1$". In addition, it is obvious that

$$(\forall x \in A)(x\theta \neq_1 1\theta \implies x\theta \nleqslant_1 1\theta)$$

is valid since (S) holds. If we put $y\theta \rightarrow_1 z\theta = (y \rightarrow z)\theta$, we get

$$\begin{aligned} x\theta * y\theta \nleqslant_1 z\theta &\iff (x \cdot y)\theta \nleqslant_1 z\theta \\ &\iff (x \cdot y, z) \in \sigma \iff (x, y \rightarrow z) \in \sigma \\ &\iff x\theta \nleqslant_1 (y \rightarrow z)\theta \iff x\theta \nleqslant_1 y\theta \rightarrow_1 z\theta. \end{aligned}$$

Similarly, it is necessary to check that a strongly extensional internal binary operation "$\rightarrow_1$" in the set $[A:\theta]$ is correctly determined in this way.

To illustrate the techniques used in the proof, we show that "$\rightarrow_1$" is strictly a well-defined extensive internal operation on $[A:\theta]$. Let $y, y', z, z', u, v \in A$ be arbitrary elements such that $(u, v) \in \theta$.

(iii) First suppose that $y\theta =_1 y'\theta$ and $z\theta =_1 z'\theta$. Then $(y, y') \lhd \theta$ and $(z, z') \lhd \theta$. Second, from $(u, v) \in \theta$ it follows that

$$(u, y \rightarrow z) \in \theta \vee (y \rightarrow z, y' \rightarrow z') \in \theta \vee (y' \rightarrow z', v) \in \theta$$

by co-transitivity of θ. Since from the second option it follows that $(y, y') \in \theta$ or $(z, z') \in \theta$ by compatibility of θ with the operation "$\rightarrow$", we have a contradiction with hypothesis. Therefore, $(u, y \rightarrow z) \in \theta$ or $(y' \rightarrow z', v) \in \theta$. Thus, $(y \rightarrow z)\theta \neq_1 u\theta$ or $(y' \rightarrow z')\theta \neq_1 v\theta$ and $(y\theta \rightarrow_1 y'\theta, z\theta \rightarrow_1 z'\theta) \lhd \theta$. So, we have $y\theta \rightarrow_1 y'\theta =_1 z\theta \rightarrow_1 z'\theta$ which means that "$\rightarrow_1$" is a well-defined function on $[A:\theta]$.

(iv) On the other hand, suppose that $y\theta \rightarrow_1 y'\theta \neq_1 z\theta \rightarrow_1 z'\theta$. Then $(y \rightarrow y')\theta \neq_1 (z \rightarrow z')\theta$. This means that $(y \rightarrow y', z \rightarrow z') \in \theta$. Thus, it follows immediately that $(y, y') \in \theta \vee (z, z') \in \theta$, since θ is compatible with the operation "$\rightarrow$". This proves that "$\rightarrow_1$" is a strictly extensional function on $[A:\theta]$.

Therefore, demonstrations (iii) and (iv) prove that "$\rightarrow_1$" is a correctly determined internal binary operation on $[A:\theta]$.

We can state the theorem in which the previous analysis is summarized.

Theorem 3.7 *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \sigma\rangle$ be a co-quasiordered residuated relational system where the relation σ satisfies the additional condition* (S). *Then we can construct residual relational systems $\langle([A:\theta], =_1, \neq_1), *, 1\theta, \nleqslant_1\rangle$ where $\theta = \sigma \cup \sigma^{-1}$ ordered under co-order "$\nleqslant_1$".*

Although it is clear that this algebraic structure has no counterpart in the classical theory of residuated systems, it does appear naturally in this logical environment. There is a strong link between this residuated system and the residuated system constructed in Theorem 3.6. This connection is described in more detail in the following theorem.

Theorem 3.8 *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \sigma\rangle$ be a co-quasiordered residuated relational system where the relation σ satisfies the additional condition* (S). *Then there are a unique reverse isotone se-epimorphism $\pi : A \longrightarrow A/(\theta^{\lhd}, \theta)$, defined by $\pi(x) = [x]$, a unique se-epimorphism $\vartheta : A \longrightarrow [A:\theta]$, defined by $\vartheta(x) = x\theta$ and a unique onto, injective and embedding se-homomorphism $g : A/(\theta^{\lhd}, \theta) \longrightarrow [A:\theta]$, defined by $g([x]) = x\theta$, such that $\vartheta = g \circ \pi$.*

Proof (i) Let $x, y, u, v \in A$ be elements such that $x = y$ and $(u, v) \in \theta$. Then $(u, x) \in \theta \vee (x, y) \in \theta \vee (y, v) \in \theta$ by co-transitivity of θ. Since $(x, y) \in \theta$ is impossible by the consistency of θ, we have $(x, y) \neq (u, v) \in \theta$. Thus, $[x] =_1 [y]$ and hence $\pi(x) =_1 \pi(y)$. On the other hand, if we assume that $[x] =_1 \pi(x) \neq_1 \pi(y) =_1 [y]$, then we have $(x, y) \in \theta$. Thus, $x \neq y$. This shows that π is a well-defined se-mapping.

Let $x, y, u, v \in A$ be elements such that $x = y$ and $(u, v) \in \theta$. Then $(u, x) \in \theta \vee (x, y) \in \theta \vee (y, v) \in \theta$ by co-transitivity of θ. Since $(x, y) \in \theta$ is impossible by the consistency of θ, we have $(x, y) \neq (u, v) \in \theta$. Thus, $x\theta =_1 y\theta$ and hence $\vartheta(x) =_1 \vartheta(y)$. On the other hand, if we assume that $x\theta =_1 \vartheta(x) \neq_1 \vartheta(y) =_1 y\theta$, then we have $(x, y) \in \theta$. Thus, $x \neq y$. This shows that ϑ is a well-defined se-mapping.

In addition to the previous, the following equalities are valid

$$(\forall x, y \in A)(\pi(x \cdot y) =_1 [x \cdot y] =_1 [x] \bullet [y] =_1 \pi(x) \bullet \pi(y))$$

and

$$(\forall x, y \in A)(\pi(x \rightarrow y) =_1 [x \rightarrow y] =_1 [x] \rightarrow_1 [y] =_1 \pi(x) \rightarrow_1 \pi(y)).$$

So, π is a se-homomorphism. Since it is obvious that π is onto, we conclude that π is se-epimorphism. It's easy to check that π is unique.

(ii) Similar to the previous one, one can check that $\vartheta : A \longrightarrow [A : \theta]$, defined by $\vartheta(x) =_1 x\theta$, is a unique well-defined onto se-homomorphism.

(iii) Suppose that $x, y \in A$ are arbitrary elements. Let $g : A/(\theta^{\lhd}, \theta) \longrightarrow [A : \theta]$ be defined by $g([x]) :=_1 x\theta$. We conclude from two valid sequences of equalities

$$[x] =_1 [y] \iff (x, y) \lhd \theta \iff x\theta =_1 y\theta \iff g([x]) =_1 g([y])$$

and

$$g([x]) \neq_1 g([y]) \iff x\theta \neq_1 y\theta \iff (x, y) \in \theta \iff [x] \neq_1 [y],$$

that g is a well-defined injective and embedding se-mapping. It is obvious that g is onto.

Let us show that g is a homomorphism. Since we have

$$g([x] \bullet [y]) =_1 g([x \cdot y]) =_1 (x \cdot y)\theta =_1 x\theta * y\theta =_1 g([x]) * g([y])$$

and

$$g([x] \rightarrow_1 [y]) =_1 g([x \rightarrow y]) =_1 (x \rightarrow y)\theta =_1 x\theta \rightarrow_1 y\theta =_1 g([x]) \rightarrow_1 g([y]),$$

we have shown that g is a se-homomorphism.

(iv) Finally, the equality $(g \circ \pi)(x) =_1 g([x]) =_1 x\theta =_1 \vartheta(x)$ justifies the equality $\vartheta = g \circ \pi$ for any $x \in A$. $\square$

We end this section with the following theorem which can be viewed as the first isomorphism theorem for these algebraic systems. Before that, we need the following lemma.

Lemma 3.9 *Let f be a reverse isotone se-homomorphism between systems $\mathfrak{A}$ and $\mathfrak{B}$. Then relation*

$$q_f = \{(x, y) \in A \times A : f(x) \neq f(y)\}$$

is a co-congruence on $\mathfrak{A}$.

Theorem 3.10 *Let f be a reverse isotone se-homomorphism between systems $\mathfrak{A}$ and $\mathfrak{B}$ such that τ is a co-order relation in $\mathfrak{B}$. Then we can construct systems*

$$\langle (A/(q_f^{\lhd}, q_f), =_1, \neq_1), \bullet, [1], \nleqslant_1 \rangle \text{ and } \langle ([A : q_f], =_1, \neq_1), *, 1q, \nleqslant_1 \rangle$$

and there exist unique embedding se-monomorphisms $h_a : A/(q_f^{\lhd}, q_f) \longrightarrow B$ and $h_b : [A : q_f] \longrightarrow B$ such that

$$f = h_a \circ \pi = h_b \circ \vartheta = (h_b \circ g) \circ \pi.$$

Proof As is usual in such theorems, se-mappings h_a and h_b are determined as follows

$$(\forall x \in A)(h_a([x]_f) = f(x) = h_b(xq_f)).$$

Clearly, h_a and h_b are well-defined functions. Let $x, y \in A$ be elements such that

$$h_a([x]_f) \neq h_a([y]_f) \text{ and } h_b(xq_f) \neq h_b(yq_f).$$

Then $f(x) \neq f(y)$. Thus, $(x, y) \in q_f$ and hence $[x]_f \neq_1 [y]_f$ and $xq_f \neq_1 yq_f$. This means that h_a and h_b are se-mappings.

On the other hand, if $[x]_f \neq_1 [y]_f$ or $xq_f \neq_1 yq_f$, then we have $(x, y) \in q_f$, that is, $f(x) \neq f(y)$ in both cases. Thus, the equality $h_a([x]_f) \neq h_a([y]_f)$ and the equality $h_b(xq_f) \neq h_b(yq_f)$ are valid equations. This shows that h_a and h_b are embedding.

Let us prove that h_a and h_b are injective mappings. Let $x, y, u, v \in A$ be such that $h_a([x]_f) = h_a([y]_f)$ or $h_b(xq_f) = h_b(yq_f)$ and $(u, v) \in q_f$. In both cases, we have $f(x) = f(y)$. It follows from $f(u) \neq f(v)$ that $f(u) \neq f(x) \vee f(x) \neq f(y) \vee f(y) \neq f(v)$ by co-transitivity of the apartness in B. Then $[x]_f \neq_1 [u]_f$ or $[v]_f \neq_1 [y]_f$ and $xq_f \neq_1 uq_f$ or $vq_f \neq_1 yq_f$. Thus, $([x]_f, [y]_f) \neq ([u]_f, [v]_f) \in q_f$ and $(xq_f, yq_f) \neq (uq_f, vq_f) \in q_f$. Since $u, v \in A$ are such that $(u, v) \in q_f$ are taken at will, we conclude that $([x]_f, [y]_f) \lhd q_f$ and $(xq_f, yq_f) \lhd q_f$ are valid. This means that $[x]_f =_1 [y]_f$ and $xq_f =_1 yq_f$. Therefore, h_a and h_b are injective mappings.

Since its rest of evidence consists of direct verification, we omit it. $\square$

4. Final Reflection

Bishop's constructive mathematics includes the following two aspects:

(1) The intuitionistic logic and

(2) The principled-philosophical orientations of constructivism.

Intuitionistic logic does not accept the TND principle (Tercium non datur principle (Lat.) - the principle of exclusion of the third) as an axiom. In addition, intuitionistic logic does not accept the validity of the "double negation"principle. In addition, the next deduction

$$F \vee G, \neg F \vDash G$$

is valid in intuitionistic logic. We have referred to this intuitionistically acceptable demonstration in more of our evidences. This makes it possible to have a difference relation in sets which is not a negation of the equality relation. Therefore, we accept that we consider set S as one relational system $(S, =, \neq)$ in Bishop's constructive mathematics. In Bishop's constructive algebra, we always encounter the following two problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined.

(b) Since every predicate has at least one of its duals, how to construct a dual of the algebraic concept defined by a given predicate(s).

In this case, we are faced with the problem of describing a residuated relational system based on a set with apartness as the carrier for constructing an algebraic structure. By our orientation that a groupoid $(A, \cdot)$ is ordered under a co-quasiorder relation instead of a quasi-order relation in this construction, a significantly different logical-sets framework is formed. In addition to the above, we have described some of the important features of a class of substructures (in this case - the class of co-ideals) in residuated relational systems constructed on sets with apartness in which both internal binary operations are strongly extensional functions in this report.

In the process of introducing new concepts in given algebraic structures, we have sought to almost always respect the following orientation. As is usual in Bishop's constructive orientation, a dual concept Y, determined by apartness relation and strongly extensional predicates, to a classical algebraic concept X should be associated with these classical concept. This correlation is shown by proving that the strong complement $Y^{\triangleleft}$ of concept Y has the properties of concept X determined in the classical way.

Many aspects of constructive mathematics are not just logical hygiene: avoid indirect proofs in favor of explicit constructions, detect and eliminate needless uses of the axiom of choice and so on. Of course, constructivism goes deeper than that. By accepting the non-existence of the TND principle, it is possible to have the multi-layered properties of algebraic objects and processes with them.

5. Acknowledgements

The author thanks the reviewers for suggestions that have improved the consistency of this paper. The author also thanks the editor of this journal for assistance during the technical design of the article.

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