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Common best proximity points theorems for H-contractive non-self mappings

Parvaneh Lo'lo'[a], Mehdi Shabibi[b]

[a] *Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Behbahan, Iran.*
[b] *Department of Mathematics, Islamic Azad University, Mehran Branch, Mehran, Iran.*

Abstract

Fixed point theory and contractive mappings are popular tools in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis.There are many works on different types of contractions to find a fixed point in metric spaces. Improving and extending some kind of those, in this paper, we introduce a new version of H-contradiction for four mappings in a metric space (X,d). Then, we prove the existence and uniqueness of a common best proximity point for four non-self mappings. An example is also given to support our main result. The related fixed point theorem are also proved.

Keywords: Common best proximity points, Metric space, H-contractive condition.
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1. Introduction and Preliminaries

Since the first results of Banach in 1922, various authors have been studying fixed points, and, in recent years, best proximity points of mappings in metric spaces. Their discoveries are still being generalized in many directions; see [1] to [10]. In a recent paper, Wardowski [11] presented a new contraction, which called F-contraction and proved a fixed point results in complete metric spaces. Then Omidvari et al.[12] proved existence of a unique best proximity point for F-contractive non-self mappings. In this paper, we extend their results by introduce a new version of Wardowski's contraction for four mappings in a complete metric space and estabilish a new common best proximity point theorem. Next, by an example and a fixed point result, we support our main results and show some applications of them.

Email addresses: `lolo@bkatu.ac.ir` (Parvaneh Lo'lo'), `mehdi_math1983@yahoo.com` (Mehdi Shabibi)

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Given two non-empty subsets A and B of a metric space (X, d) , the following notions and notations are used in the sequel .

$$d(A,B) = inf\{d(a,b) : a \in A, b \in B\}$$
$$A_0 = \{a \in A : d(a,b) = d(A,B) \text{for some } b \in B\}$$
$$B_0 = \{b \in B : d(a,b) = d(A,B) \text{for some } a \in A\}.$$

Definition 1.1. *An element $u \in A$ is said to be a common best proximity point of the non-self mappings $f_1, f_2, ..., f_n : A \to B$ if it satisfies the condition that*

$$d(u, f_1u) = d(u, f_2u) = ... = d(u, f_nu) = d(A,B).$$

Definition 1.2. *The mappings $f : A \to B$ and $g : A \to B$ are said to be commute proximally if they satisfy the condition that*

$$[d(u, fx) = d(v, gx) = d(A,B)] \Rightarrow fv = gu.$$

Definition 1.3. *If $A_0 \neq \varnothing$ then the pair (A, B) is said to have P-property if and only if for any $a_1, a_2 \in A_0$ and $b_2, b_2 \in B_0$*

$$\begin{cases} d(a_1, b_1) = d(A,B) \\ d(a_2, b_2) = d(A,B) \end{cases} \Longrightarrow d(a_1, a_2) = d(b_1, b_2)$$

2. Main Results

We begin our study with following definition

Definition 2.1. *Let $H : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:*

(H_1) *H is strictly increasing, i.e*
$\alpha < \beta \Longrightarrow H(\alpha) < H(\beta) \qquad \forall \alpha, \beta \in \mathbb{R}_+,$

(H_2) *For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers*
$\lim_{n\to\infty} \alpha_n = 0 \Longleftrightarrow \lim_{n\to\infty} H(\alpha_n) = -\infty,$

(H_3) *There exists $k \in (0, 1)$such that* $\lim_{\alpha\to 0^+} \alpha^k H(\alpha) = 0,$

then self mappings $f, g, S, T : X \to X$ are said to satisfy an H-contractive condition if there exists $C > 0$ such that

$$\forall x, y \in X \quad s.t \quad d(fx, gy) > 0 \quad \Longrightarrow \quad C + H(d(fx, gy)) \leq H(m)$$

and

$$m = max\{d(Sx, Ty), d(fx, Sx), d(Ty, gy), \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}.$$

Theorem 2.1. *Let A and B be nonempty subsets of a complete metric space (X,d) such that A_0 is closed and nonempty. Let the non-self mappings $f,g,S,T : A \to B$ satisfy the following conditions:*

i) $\{f,S\}$ *and* $\{g,T\}$ *commute proximally,*

ii) *pair* (A,B) *has the P-property,*

iii) f,g,S *and* T *are continuous,*

iv) f,g,S *and* T *satisfy the* H*-contractive condition.*

v) $f(A_0) \subseteq T(A_0), g(A_0) \subseteq S(A_0)$ *and* $g(A_0) \subseteq B_0, f(A_0) \subseteq B_0$.

Then f,g,S and T have unique common best proximity point .

Proof. Fix a_0 in A_0, since $f(A_0) \subseteq T(A_0)$, then there exists an element a_1 in A_0 such that $f(a_0) = T(a_1)$. Similarly, a point $a_2 \in A_0$ can be chosen such that $g(a_1) = S(a_2)$. Continuing this process, we obtain a sequence $\{a_n\} \subseteq A_0$ such that

$$f(a_{2n}) = T(a_{2n+1}), \qquad g(a_{2n+1}) = S(a_{2n+2}).$$

Since $f(A_0) \subseteq B_0$ and $g(A_0) \subseteq B_0$, there exists $\{u_n\} \subseteq A_0$ such that

$$d(u_{2n}, f(a_{2n})) = d(A,B) \ \ and \ \ d(u_{2n+1}, g(a_{2n+1})) = d(A,B). \tag{1}$$

We wil prove that the sequence $\{u_n\}$ is convergent in A_0.
(A,B) satisfies the P-property therefore from (1) we obtain

$$d(u_{2n}, u_{2n+1}) = d(fa_{2n}, ga_{2n+1}). \tag{2}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(fa_{2n_0}, ga_{2n_0+1}) = 0$, them by (2) we have $d(u_{2n_0}, u_{2n_0+1}) = 0$ that implies $u_{2n_0} = u_{2n_0+1}$. If $u_{2n_0+1} \neq u_{2n_0+2}$ then by (2) , $d(fa_{2n_0+2}, ga_{2n_0+1}) > 0$ and therefor

$$\begin{aligned}
& H(d(u_{2n_0+1}, u_{2n_0+2})) = H(d(fa_{2n_0+2}, ga_{2n_0+1})) \\
\leq \ & -C + H(\max\{d(Sa_{2n_0+2}, Ta_{2n_0+1}), d(fa_{2n_0+2}, Sa_{2n_0+2}), \\
& \quad d(Ta_{2n_0+1}, ga_{2n_0+1}), \frac{1}{2}[d(Sa_{2n_0+2}, ga_{2n_0+1}) + d(fa_{2n_0+2}, Ta_{2n_0+1})]\}) \\
= \ & -C + H(\max\{d(u_{2n_0}, u_{2n_0+1}), d(u_{2n_0+1}, u_{2n_0+2}), d(u_{2n_0}, u_{2n_0+1}), \\
& \qquad \frac{1}{2}[d(u_{2n_0+1}, u_{2n_0+1}) + d(u_{2n_0}, u_{2n_0+2})]\}),
\end{aligned}$$

and consequently

$$H(d(u_{2n_0+1}, u_{2n_0+2})) \leq -C + H(\max\{0, d(u_{2n_0+1}, u_{2n_0+2}), \frac{1}{2}d(u_{2n_0}, u_{2n_0+2})\}).$$

Thus (note that $\frac{1}{2}d(u_{2n_0}, u_{2n_0+2}) \leq \frac{1}{2}[d(u_{2n_0}, u_{2n_0+1}) + d(u_{2n_0+1}, u_{2n_0+2})]$):

$$H(d(u_{2n_0+1}, u_{2n_0+2})) \leq \ -C + H(d(u_{2n_0+1}, u_{2n_0+2})).$$

Therefor $C \leq \ 0$, which is a contranction and $u_{2n_0} = u_{2n_0+1} = u_{2n_0+2}$. So $u_n = u_{2n_0}$ for all $n \geq \ 2n_0$, and u_n is convergent in A_0. Now let $d(fa_{2n}, ga_{2n+1}) \neq \ 0$ for all $n \in \mathbb{N}$. Since the pair (A,B) has p-property, by (2) we have

$$\begin{aligned}
& H(d(u_{2n}, u_{2n+1})) = H(d(fa_{2n}, ga_{2n+1})) \\
\leq \ & -C + H(\max\{d(Sa_{2n}, Ta_{2n+1}), d(fa_{2n}, Sa_{2n}), \\
& \quad d(Ta_{2n+1}, ga_{2n+1}), \frac{1}{2}[d(Sa_{2n}, ga_{2n+1}) + d(fa_{2n}, Ta_{2n+1})]\}) \\
= \ & -C + H(\max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n-1}), d(u_{2n}, u_{2n+1}), \\
& \qquad \frac{1}{2}[d(u_{2n-1}, u_{2n+1}) + d(u_{2n}, u_{2n})]\}).
\end{aligned}$$

Then we have

$$H(d(u_{2n}, u_{2n+1})) \leq -C + H(\max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}),$$

using the preceding description

$$H(d(u_{2n}, u_{2n+1})) \leq \ -C + H(d(u_{2n-1}, u_{2n})). \tag{3}$$

Similarly

$$\begin{aligned} & H(d(u_{2n+1}, u_{2n+2})) = H(d(fa_{2n+2}, ga_{2n+1})) \\ \leq\ & -C + H(\max\{d(Sa_{2n+2}, Ta_{2n+1}), d(fa_{2n+2}, Sa_{2n+2}), \\ & d(Ta_{2n+1}, ga_{2n+1}), \frac{1}{2}[d(Sa_{2n+2}, ga_{2n+1}) + d(fa_{2n+2}, Ta_{2n+1})]\}) \\ =\ & -C + H(\max\{d(u_{2n+1}, u_{2n}), d(u_{2n+2}, u_{2n+1}), d(u_{2n}, u_{2n+1}), \\ & \qquad \frac{1}{2}[d(u_{2n+1}, u_{2n+1}) + d(u_{2n+2}, u_{2n})]\}). \end{aligned}$$

Thus (not that $\frac{1}{2}d(u_{2n+2}, u_{2n}) \leq \frac{1}{2}[d(u_{2n+2}, u_{2n+1}) + d(u_{2n+1}, u_{2n})]$

$$H(d(u_{2n+1}, u_{2n+2})) \leq -C + H(d(u_{2n}, u_{2n+1})). \tag{4}$$

Therefor, by (3) and (4) we have

$$H(d(u_n, u_{n+1})) \leq -C + H(d(u_{n-1}, u_n)),$$

and then

$$H(d(u_n, u_{n+1})) \leq -nC + H(d(u_0, u_1)). \tag{5}$$

Put $\alpha_n =: d(u_n, u_{n+1})$.
By (5), we obtain $\lim_{n\to\infty} H(\alpha_n) = -\infty$ that together with (H_2) gives

$$\lim_{n\to\infty} \alpha_n = 0. \tag{6}$$

Also from (H_3) we have

$$\exists k \in (0,1) \qquad \text{such that} \qquad \lim_{n\to\infty} \alpha_n^k H(\alpha_n) = 0 \tag{7}$$

On the Other hand, by (5)

$$H(\alpha_n) - H(\alpha_0) \leq \ -nC$$

Therefor

$$\alpha_n^k H(\alpha_n) - \alpha_n^k H(\alpha_0) \leq \ -n\alpha_n^k C \leq \ 0$$

Letting $n \longrightarrow \infty$ in the above inequality and using (6) and (7) , we obtain

$$\lim_{n\to\infty} n\alpha_n^k = 0$$

Hence there exists $N_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq N_1$. Therefor for any $n \geq N_1$

$$\alpha_n \leq \frac{1}{n^{\frac{1}{k}}}$$

This means that series $\sum_{i=1}^{\infty} \alpha_i$ is convergent, then

$$\forall \epsilon > 0 \quad \exists N \geq 0 \quad \text{such that} \quad m \geq n \geq N, \quad \sum_{i=n}^{m} \alpha_i \leq \epsilon. \tag{8}$$

By the triangular inequality and(8)

$$d(u_m, u_n) \leq \alpha_{m-1} + \alpha_{m-2} + ... + \alpha_n \leq \sum_{i=n}^{m} \alpha_i \leq \epsilon$$

Therefor $\{u_n\}$ is a cauchy seqvence in A_0.
Since$\{u_n\} \subseteq A_0$ and A_0 is a closed subset of the complete metrice space (X, d), we can find $u \in A_0$ such that $\lim_{n\to\infty} u_n = u$.
By (1) and because of the fact $\{f, S\}$ and $\{g, T\}$ commute proximally, $fu_{2n-1} = Su_{2n}$ and $gu_{2n} = Tu_{2n+1}$. Therefore, the continuity of f, g, S and T and $n \to \infty$ ascertains that $fu = gu = Tu = Su$.
Since $f(A_0) \subseteq B_0$, there exists $a \in A_0$ such that

d(A,B)=d(a, fu)=d(a,gu)=d(a,Su)=d(a,Tu).

As $\{f, S\}$ and $\{g, T\}$ commute proximally, $fa = ga = Sa = Ta$. Then, since $f(A_0) \subseteq B_0$, there exists $x \in A_0$ such that

d(A,B)=d(x,fa)=d(x,ga)=d(x,Sa)=d(x,Ta).

Let $d(a, x) > 0$, because pair (A, B) has the P-property and
$d(a, fu) = d(x, ga) = d(A, B)$, we have $d(fu, ga) > 0$ and therefore

$$\begin{aligned} H(d(a,x)) &= H(d(fu,ga)) \\ &\leq -C + H(\max\{d(Su,Ta), d(fu,Su), d(Ta,ga), \\ &\qquad \frac{1}{2}[d(Su,ga) + d(fu,Ta)]\}) \\ &= -C + H(\max\{d(a,x), d(a,a), d(x,x), \\ &\qquad \frac{1}{2}[d(a,x) + d(a,x)]\}) \end{aligned}$$

then

$$d(a, x) \leq -C + d(a, x),$$

this results $C \leq 0$. wich is a contradiction and $d(a, x) = 0$ or $x = a$. Thus, it follows that

$$d(A, B) = d(a, fa) = d(a, ga) = d(a, Sa) = d(a, Ta), \tag{9}$$

then a is a common best proximity point of the mapping f, g, S and T.
Suppose that $a' \neq a$ is another common best proximity point of the mapping f, g, S and T, so that

$$d(A, B) = d(a', fa') = d(a', ga') = d(a', Ta') = d(a', Sa'). \tag{10}$$

As pair (A, B) has the P-property then from (9) and (10), we have

$$\begin{aligned} H(d(a,a')) &= H(d(fa,ga')) \\ &\leq -C + H(\max\{d(Sa,Ta'), d(fa,Sa), d(Ta',ga'), \\ &\qquad \frac{1}{2}[d(Sa,ga') + d(fa,Ta')]\}) \\ &= -C + H(\max\{d(a,a'), d(a,a), d(a',a'), \\ &\qquad \frac{1}{2}[d(a,a') + d(a,a')]\}), \end{aligned}$$

then

$$H(d(a, a')) \leq -C + H(d(a, a'))$$

which implies that $a = a'$. □

Now we illustrate our common best proximity point theorem by the following example.

Example 2.1. *Let $X = [0,1] \times [0,1]$ and d be the Euclidean metric. Then (X,d) is a complete metric space. Let*

$$A := \{(0,a) : 0 \leq a \leq 1\}, \qquad B := \{(1,b) : 0 \leq b \leq 1\}.$$

Then $d(A,B) = 1$, $A_0 = A$ and $B_0 = B$. Let f, g, S and T defined as $f(0,x) = (1, \frac{x}{8})$, $g(0,x) = (1, \frac{x}{32})$, $S(0,x) = (1,x)$ and $T(0,x) = (1, \frac{x}{4})$. Then for all x and $y \in X$ we have

$$d(fx, gy) = |\frac{x}{8} - \frac{y}{32}| = \frac{1}{8} d(Sx, Ty).$$

Now if we define $H : \mathbb{R}^+ \to \mathbb{R}$ by $H(\alpha) = ln(\alpha)$ and $C = ln8$. Then clearly non-self mappings $f, g, S, T : A \to B$ are H-contraction. Now, all the required hypotheses of theorem 2.1 are satisfied. Clearly $(0,0)$ is unique common best proximity point of f, g, S and T.

By theorem 2.1 we also obtain the following common fixed point theorem.

Theorem 2.2. *Let (X,d) be a complete metric space . Let $f, g, S, T : X \to X$ be given continuous mappings and satisfy the H-contractive condition such that S and T commute f and g respectively. Further let $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$. Then f, g, S and T have unique common fixed point.*

Proof. We take the same sequence $\{u_n\}$ and u as in the proof of theorem 2.1. Due to the fact that S and T commute f and g respectively we have

$$fu_{2n-1} = Su_{2n}, \qquad gu_{2n} = Tu_{2n+1}.$$

By continuity of f, g, S, T and $n \to \infty$ we have

$$fu = Su, \qquad gu = Tu. \tag{11}$$

If $fu \neq gu$, since $f, g, S, T : X \to X$ satisfy the H-contractive condition and by (11)

$$\begin{aligned} H(d(fu, gu)) &\leq -C + H(max\{d(Su, Tu), d(fu, Su), d(Tu, gu), \\ &\qquad \frac{1}{2}[d(Su, gu) + d(fu, Tu)]\}) \\ &\leq -C + H(max\{d(fu, gu), d(fu, fu), d(gu, gu), \\ &\qquad \frac{1}{2}[(fu, gu) + (fu, gu)]\}), \end{aligned}$$

then $H(d(fu, gu)) \leq -C + H(d(fu, gu))$. Therefore $C \leq 0$, which is a contraction. Then $fu = gu$ and by (11), $fu = gu = Su = Tu$.
We set $a = fu = gu = Su = Tu$. Because of the fact T commute g we obtain

$$ga = gTu = Tgu = Ta.$$

If $a \neq ga$ therefor

$$\begin{aligned} H(d(a, ga)) &= H(d(fu, ga)) \\ &\leq -C + H(max\{d(Su, Ta), d(fu, Su), d(Ta, ga), \\ &\qquad \frac{1}{2}[d(Su, ga) + d(fu, Ta)]\}) \\ &\leq -C + H(max\{d(a, ga), a(a, a), d(ga, ga), \\ &\qquad \frac{1}{2}[(a, ga) + (a, ga)]\}). \end{aligned}$$

Therefore, $H(d(a, ga)) \leq -C + H(d(a, ga))$ and consequently $C \leq 0$, that is a contraction by $C > 0$. Therefore

$$a = ga = Ta. \tag{12}$$

Similarly, we can show that

$$a = fa = Sa. \tag{13}$$

Hence, by (12) and (13) we deduce that $a = fa = ga = Sa = Ta$. Therefore, a is a common fixed point of f, g, S and T.
Assume one contrary that, $p = fp = gp = Sp = Tp$ and $q = fq = gq = Sq = Tq$ but $p \neq q$.
We have

$$\begin{aligned} H(d(p,q)) &= H(d(fp, gq)) \\ &\leq -C + H(\max\{d(Sp, Tq), d(fp, Sp), d(Tq, gq), \\ &\qquad \frac{1}{2}[d(Sp, gq) + d(fp, Tq)]\}) \\ &\leq -C + H(\max\{d(p, q,), d(p, p), d(q, q), \frac{1}{2}[(p, q) + (p, q)]\}). \end{aligned}$$

Consequence $H(d(p, q)) \leq -C + H(d(p, q))$, then $C \leq 0$, a contradiction. Therefore, f, g, S and T have a unique common fixed point. □

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