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Optimality Conditions in One Stochastic Control Problem

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Abstract: For one stochastic optimal control problem described by a linear Ito stochastic equation and linear quality functional a necessary and sufficient optimality condition form of the Pontryagin maximum principle is obtained. In the case of convexity of the nonlinear quality functional, a sufficient optimality condition is obtained.In the deterministic case, many authors have studied such problems using the increment method. The considered work using a stochastic analogue of the increment method necessary and sufficient conditions for optimality as well as sufficient conditions are established.

Keywords: Ito equations, Optimal control, Optimality conditions.

1 Introduction

As is known, in the theory of optimal stochastic control when describing a stochastic controlled model, a convenient mathematical apparatus is Ito's stochastic differential equations [1]-[2]. Until now, many authors have obtained various types of necessary optimality conditions for control problems for Ito stochastic systems [3]-[5]. In this paper, using a stochastic analogue of the method of increments in one linear stochastic control problem, a necessary and sufficient condition for optimality in the form of the Pontryagin maximum principle [6] is established. In the case of convexity of the nonlinear quality functional, a sufficient optimality condition is proved.

2 Statement problem

Suppose (Ω, F, P) is a full probabilistic space with an allocated non-decreasing σ -algebras stream F_t, where $F_t = \sigma(w(s), \quad t_0 \le s \le t)$ and $w(t)$ is n -dimensional standard Wiener process. $L_F^2(t_0, t_1; R^n)$ is the space of concerted processes measurable in (t, ω) and F_t, $x(t, \omega)$: $[t_0, t_1] : \Omega \to R^n$, for which $\int_{t_0}^{t_1} \|x(t)\|^2 dt < +\infty$. Here, E is the symbol of mathematical expectation.
Consider system Ito type linear stochastic differential equations in form

$$dx(t) = (A(t)x(t) + f(t, u(t)))\, dt + B(t)x(t)dw(t), \quad t \in T = [t_0, t_1], \tag{1}$$

$$x(t_0) = x_0.$$

Here $x(t)$- is the desired n-dimensional vector, $A(t)$, $B(t)$- are given continuous $(n \times n)$- coefficients matrices, $f(t, u)$ - is a given n-dimensional continuous in (t, u) vector-function.

$$u(t) \in U \subset R^r, \tag{2}$$

where U is a given nonempty, bounded set. We call such controls admissible controls.
Required to find an admissible control $u(t)$ such that the solution $x(t)$ of system (1) delivers the smallest possible value to the functional

$$S(u) = Ec'x(t_1), \tag{3}$$

Here, c- known n- dimensional constant vector.
In this paper, using a stochastic analogue of method, the increment method, we obtain first order optimality conditions.

3 The increment formula of the quality criterion and the main results

Let $u(t)$ and $\bar{u}(t) = u(t) + \Delta u(t)$- two valid controls, $x(t)$ and $\bar{x}(t) = x(t) + \Delta x(t)$ - the corresponding solutions to system (1). Then it is clear that $\Delta x(t)$ - is a solution the problem

$$d\Delta x(t) = (A(t)\Delta x(t) + (f(t, \bar{u}(t)) - f(t, u(t))))\, dt + B(t)\Delta x(t)dw(t), \quad t \in T = [t_0, t_1],$$

$$\Delta x(t_0) = 0. \tag{4}$$

Since(4) is a linear stochastic inhomogeneous differential equation relatively $\Delta x(t)$, the solution can be represented as [5].

$$\Delta x(t) = \int_{t_0}^{t} F(t, \tau)(f(\tau, \bar{u}(\tau)) - f(\tau, u(\tau)))d\tau, \tag{5}$$

where $F(t, \tau)$ – $(n \times n)$ - matrix is solution problem

$$dF(t, \tau) = F(t, \tau)A(\tau)d\tau + F(t, \tau)B(\tau)dw(\tau), \quad t \in T = [t_0, t_1],$$

$$F(t, t) = I - (n \times n) - unit\, matrix.$$

From (5) clear, that

$$\Delta x(t_1) = \int_{t_0}^{t_1} F(t_1, t)(f(t, \bar{u}(t)) - f(t, u(t)))dt.$$

Then increment formula quality criterion (3) can be imagining as:

$$\Delta S(u) = S(\bar{u}) - S(u) = Ec'\Delta x(t_1) = E\int_{t_0}^{t_1} c'F(t_1, t)(f(t, \bar{u}(t)) - f(t, u(t)))dt.$$

Denote by

$$\psi(t) = -c'F(t_1, t),$$

$$H(t, u(t), \psi(t)) = \psi'(t)f(t, u(t))$$

Then we have

$$\Delta S(u) = -E\int_{t_0}^{t_1} [H(t, \bar{u}(t), \psi(t)) - H(t, u(t), \psi(t))]\, dt. \tag{6}$$

Using the expression for $\psi(t)$, we find a stochastic differential equation that this function satisfies.
From the definition of the function $\psi(t)$ it is clear that

$$d\psi(t) = -c'F(t_1, t)A(t)dt - c'F(t_1, t)B(t)dw(t) = A(t)\psi(t)dt + B(t)\psi(t)dw(t),$$

$$\psi(t_1) = -c'F(t_1, t_1) = -c.$$

Therefore, it was proved that $\psi(t)$ is the following system of stochastic equations:

$$d\psi(t) = A(t)\psi(t)dt + B(t)\psi(t)dw(t), \quad t \in T = [t_0, t_1],$$

$$\psi(t_1) = -c.$$

Using the relation (6), the following is proved

Theorem 1. *For the optimality of the admissible control $u(t)$ in the considered problem (1)-(3), it is necessary and sufficient that*

$$\max_{v \in U} EH(t, v, \psi(t)) = EH(t, u(t), \psi(t)), \tag{7}$$

satisfied for all $t \in T$.

Proof.
Necessity. Let us assume that $u(t)$ is optimal. We prove that the relationship is fulfilled (7). In the power of optimal control $u(t)$ from the formulas of attachment (6) follows that for any $\bar{u}(t) \in U, t \in T$

$$-E\int_{t_0}^{t_1}\left[H(t,\bar{u}(t),\psi(t))-H(t,u(t),\psi(t))\right]dt \geq 0.$$

Using the function $\bar{u}(t)$, we define it in the form

$$\bar{u}(t)=\begin{cases} v, & t\in[\theta,\theta+\varepsilon)\,, \\ u(t), & t\bar{\in}\,[\theta,\theta+\varepsilon)\,, \end{cases}$$

where θ - is the arbitrary point of control continuity $u(t)$, $t \in T$, $v \in U$ - is the arbitrary vector, and $\varepsilon > 0$ is the arbitrary sufficient number. Then from last inequality implies

$$E\int_{\theta}^{\theta+\varepsilon}\left[H(t,v,\psi(t))-H(t,u(t),\psi(t))\right]dt \leq 0.$$

From here, using the average value formula, we obtain

$$EH(\theta,v,\psi(\theta)) \leq EH(\theta,u(\theta),\psi(\theta)). \tag{8}$$

Hence, since $v \in U$, $\theta \in T$ - are arbitrary, the maximum condition (7) follows.
Sufficiency. Suppose that maximum condition (7) is satisfied for an admissible control $u(t)$ and prove that in this case the control $u(t)$ is optimal for the considered problem (1)-(3).
From condition (7) for any $v = v(\theta) \in U$ it follows that

$$E\left\{H(\theta,v(\theta),\psi(\theta))-H(\theta,u(\theta),\psi(\theta))\right\} \leq 0.$$

Hence, since $\theta \in T$ is arbitrary, it follows that

$$E\int_{t_0}^{t_1}\left[H(\theta,v(\theta),\psi(\theta))-H(\theta,u(\theta),\psi(\theta))\right]d\theta \leq 0. \tag{9}$$

Taking into account inequalities (9), we obtain that

$$\Delta S(u)=S(v(\theta))-S(u(\theta))=-E\int_{t_0}^{t_1}\left[H(\theta,v(\theta),\psi(\theta))-H(\theta,u(\theta),\psi(\theta))\right]d\theta \geq 0,$$

i.e. for all $v(\theta) \in U$,

$$S(v(\theta)) \geq S(u(\theta)).$$

In other words, $u(t)$ is the optimal control.
This completely proves the theorem.
Now consider the case of a nonlinear but convex quality functional. Let it be required to find a minimum of functionality

$$S(u)=\varphi(x(t_1)), \tag{10}$$

under restrictions (1)-(2).
Here $\varphi(x)$ - is given continuously differentiable convex scalar function.
Then, using Taylor's formula the increment of functional (10) is written in form

$$\Delta S(u)=E\{\varphi_x'(x(t_1))\Delta x(t_1)+o(\|\Delta x(t_1)\|)\}.$$

Put

$$\psi(t)=-\varphi_x'(x(t_1))F(t_1,t),$$

$$H(t,u(t),\psi(t))=\psi'(t)f(t,u(t)).$$

Then we have

$$\Delta S(u)=-E\int_{t_0}^{t_1}\left[H(t,\bar{u}(t),\psi(t))-H(t,u(t),\psi(t))\right]dt+Eo(\|\Delta x(t_1)\|). \tag{11}$$

Due to $\varphi(x)$ - convex differentiable function

$$Eo(\|\Delta x(t_1)\|) \geq 0.$$

Therefore, from inequality (11) follows

$$\Delta S(u) \geq -E\int_{t_0}^{t_1} [H(t, \bar{u}(t), \psi(t)) - H(t, u(t), \psi(t))]\, dt.$$

From here we obtain following

Theorem 2. *For optimality of the admissible control $u(t)$ in the considered problem (10), (1),(2) it is sufficient that for all $v(t) \in U$ satisfied equality*

$$E\int_{t_0}^{t_1} [H(t, v(t), \psi(t)) - H(t, u(t), \psi(t))]\, dt \leq 0.$$

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