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Local Existence and Blow Up of Solutions for a Coupled Viscoelastic Kirchhoff-Type Equations with Degenerate Damping

Erhan Pişkin[1] *Fatma Ekinci*[2,*]

[1] *Dicle University, Department of Mathematics, Diyarbakır, Turkey, ORCID: 0000-0001-6587-4479*
[2] *Dicle University, Department of Mathematics, Diyarbakır, Turkey, ORCID: 0000-0002-9409-3054*
* *Corresponding Author E-mail: episkin@dicle.edu.tr, ekincifatma2017@gmail.com*

Abstract: In this paper, we consider the initial boundary value problem of a coupled viscoelastic Kirchhoff-type equations with degenerate damping. Firstly, we prove a local existence theorem by using the Faedo-Galerkin approximations. Then, we study blow up of solutions when initial energy is possitive.

Keywords: Blow up, Degenerate damping, Kirchhoff type, Local existence, Viscoelastic equation.

1 Introduction

This paper are concerned with the local existence and blow up of solutions for the following viscoelastic Kirchhoff-type equation with degenerate damping:

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t \mu_1(t-s)\Delta u(s)ds + \left(|u|^k + |v|^l\right)|u_t|^{p-1}u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} - M(\|\nabla v\|^2)\Delta v + \int_0^t \mu_2(t-s)\Delta v(s)ds + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q-1}v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, $p, q \geq 1$, $j, k, l, \theta, \varrho \geq 0$; $\mu_i(.) : R^+ \to R^+$, $f_i(.\,,.) : R^2 \longrightarrow R$ $(i = 1, 2)$ are given functions to be specified later. $M(s)$ is a nonnegative C^1 function satisfying $M(s) = b_1 + b_2 s^\gamma$, $\gamma, s \geq 0$ and $b_1 = b_2 = 1$.

There is an extensive literature on this kind of problems. For instance, one of them is our work [1] where we investigated problem (1) and obtained global existence and the general decay result for the global solution. Then we proved blow-up result of solutions with negative initial energy. We now state other related problem in the literature: Firstly, we mention the pioneer work of Wu [2] where he established a general decay result of the system (1) for $M = 1$. Then, Pişkin et al. [3] studied local existence and uniqueness results by using the Faedo-Galerkin method. Also, some author studied existence, blow up and decay of the solutions (1) for $k = l = \theta = \varrho = 0$ and $M = 1$ (see [4]-[5]-[6]-[7]- [8]-[9]). Furthermore, Rammaha and Sakuntasathien [12] and Zennir et al. [10]-[11] studied system (1) for $M = 1$ and $\mu_i = 0$ $(i = 1, 2)$ and considered well posedness of solutions, the blow up and growth properties.

The content of this paper is organized as follows: In Section 2, we give necessary assumptions and notation that will be used later. In Section 3, firstly, we give definition of weak solution then, under some conditions we obtain the local existence of weak solutions by Galerkin's approximation. In Section 4, we obtain finite time blow up of solutions with positive initial energy.

Throughout this paper, we denote the standart $L^2(\Omega)$-norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$-norm by $\|.\| = \|.\|_{L^p(\Omega)}$. We need the following assumptions to state and prove our results.

$(\mathbf{A1})$ The relaxation functions μ_i $(i = 1, 2)$ are nonincreasing and satisfy, for $s \geq 0$

$$\mu_i(s) \geq 0, \ \ \mu_i'(s) \leq 0, \quad 1 - \int_0^\infty \mu_i(s)ds = l_i > 0. \tag{2}$$

$(\mathbf{A2})$ For the nonlinearity in damping, we suppose that $1 \leq p, q$ if $n = 1, 2$; $1 \leq p, q \leq \frac{n+2}{n-2}$ if $n \geq 3$.

We pick up the functions $f_1(u, v)$ and $f_2(u, v)$ as follows

$$\begin{array}{c} f_1(u, v) = a|u + v|^{2(r+1)}(u + v) + b|u|^r u|v|^{r+2}, \\ f_2(u, v) = a|u + v|^{2(r+1)}(u + v) + b|v|^r v|u|^{r+2} \end{array} \tag{3}$$

where $a, b > 0$ are constants and r satisfies

$$\begin{cases} -1 < r \ \text{ if } n = 1, 2, \\ -1 < r \leq \frac{3-n}{n-2} \text{ if } n \geq 3. \end{cases} \tag{4}$$

One can easily verify that

$$uf_1(u, v) + vf_2(u, v) = 2(r + 2)F(u, v), \ \forall (u, v) \in R^2, \tag{5}$$

where

$$F(u, v) = \frac{1}{2(r + 2)}\left[a|u + v|^{2(r+2)} + 2b|uv|^{r+2}\right]. \tag{6}$$

We introduce the energy function $E(t)$ as follows

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\left[(\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) + \frac{1}{\gamma + 1}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)})\right] \\ &+ \frac{1}{2}\left[(1 - \int_0^t \mu_1(s)ds)\|\nabla u(t)\|^2 + (1 - \int_0^t \mu_2(s)ds)\|\nabla v(t)\|^2\right] - \int_\Omega F(u, v)\,dx. \end{aligned} \tag{7}$$

where $(\mu_i \diamond \nabla w)(t) = \int_0^t \mu_i(t - s)\|\nabla w(t) - \nabla w(s)\|_2^2\,ds$. By computation, we get

$$\begin{aligned} E'(t) &= \frac{1}{2}\left[(\mu_1' \diamond \nabla u)(t) + (\mu_2' \diamond \nabla v)(t)\right] \\ &- \frac{1}{2}\left(\mu_1(t)\|\nabla u\|^2 + \mu_2(t)\|\nabla v\|^2\right) \\ &- \int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}dx - \int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}dx. \end{aligned} \tag{8}$$

2 Local existence

In this section, we state and proved local existence of weak solution of the problem (1). Firstly, we give the definition of weak solutions for the problem (1).

Definition 1. *We say that* (u,v) *is a weak solution of* (1) *on* $[0,T)$ *under the assumptions* $(A1)$, $(A2)$ *if* $u,v\in L^{\infty}\left(0,T;W_0^{1,2(\gamma+1)}\left(\Omega\right)\right)$, $u_t\in L^{\infty}\left(0,T;L^2\left(\Omega\right)\right)$, $v_t\in L^{\infty}\left(0,T;L^2\left(\Omega\right)\right)$ *and satisfies*

$$
\begin{aligned}
&\left\langle u'\left(t\right),\theta\right\rangle-\left\langle u^1,\theta\right\rangle+\int_0^t\left\langle\int_\Omega M(\|\nabla u\|^2)\nabla u(\alpha)d\alpha,\nabla\theta\right\rangle d\xi-\int_0^t\left\langle\int_0^s\mu_1(\xi-\alpha)\nabla u(\alpha)d\alpha,\nabla\theta\right\rangle d\xi\\
&+\int_0^t\left\langle\left(|u|^k+|v|^l\right)\left|u'(\xi)\right|^{p-1}u'(\xi),\theta\right\rangle d\xi\\
=\;&\int_0^t\left\langle f_1\left(u(\xi),v(\xi)\right),\theta\right\rangle d\xi,
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle v'\left(t\right),\phi\right\rangle-\left\langle v^1,\phi\right\rangle+\int_0^t\left\langle\int_\Omega M(\|\nabla v\|^2)\nabla v(\alpha)d\alpha,\nabla\phi\right\rangle d\xi-\int_0^t\left\langle\int_0^s\mu_2(\xi-\alpha)\nabla v(\alpha)d\alpha,\nabla\phi\right\rangle d\xi\\
&+\int_0^t\left\langle\left(|v|^\theta+|u|^\varrho\right)\left|v'(\xi)\right|^{q-1}v'(\xi),\phi\right\rangle d\xi\\
=\;&\int_0^t\left\langle f_2\left(u(\xi),v(\xi)\right),\phi\right\rangle d\xi,
\end{aligned}
$$

for almost everywhere $t\in 0,T)$ *and any test functions* $\theta,\phi\in W_0^{1,2(\gamma+1)}\left(\Omega\right)$.

Theorem 2. *(Local existence). Assume assumptions* $(A1)$, $(A2)$*, (4) and* $n=1,2,3$ *hold. Then, for some* $T>0$ *problem* (1) *has at least a local weak solution* (u,v) *on* $[0,T)$.

Proof: For the convenience of the readers, we merely show the main steps and indicate the modifications. We follows the standard Fadeo–Galerkin approximation to establish to show the existence of solution (1). The combination of the Faedo-Galerkin method and the compactness argument gives us a efficent method that allows us to deal with some evolution equations with degenerate damping terms.

Let the sequence $\left\{e_j:j=1,2,...\right\}$ is an orthogonal basis for $L^2(\Omega)\cap W_0^{1,2(\gamma+1)}$. By virtue of the theory of ordinary differential equations guarantee that has a unique local solution. We construct approximate solutions $(u_M(t),v_M(t))$ $(M=1,2,3,...)$ in the form

$$u_M(t)=\sum_{j=1}^{M}u_{M,j}(t)e_j,\quad v_M(t)=\sum_{j=1}^{M}v_{M,j}(t)e_j.$$

Approximate system

$$
\begin{aligned}
&\left\langle u''_M(t),e_j\right\rangle+\left\langle\int_\Omega M\left(\|\nabla u_M\left(t\right)\|^2\right)\nabla u_M(t),\nabla e_j\right\rangle-\left\langle\int_0^t\mu_1(t-s)\nabla u_M(s)ds,\nabla e_j\right\rangle\\
&+\left\langle\left(\left|u_M\left(t\right)\right|^k+\left|v_M\left(t\right)\right|^l\right)\left|u'_M(t)\right|^{p-1}u'_M(t),e_j\right\rangle\\
=\;&\left\langle f_1(u_M(t),v_M(t)),e_j\right\rangle,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\left\langle v''_M(t), e_j\right\rangle + \left\langle \int_\Omega M\left(\|\nabla v_M(t)\|^2\right)\nabla v_M(t), \nabla e_j\right\rangle - \left\langle \int_0^t \mu_2(t-s)\nabla v_M(s)ds, \nabla e_j\right\rangle \\
&+\left\langle \left(|v_M(t)|^\theta + |u_M(t)|^\varrho\right)\left|v'_M(t)\right|^{q-1} v'_M(t), e_j\right\rangle \\
=\ & \left\langle f_2(u_M(t), v_M(t)), e_j\right\rangle, \qquad (10)
\end{aligned}
$$

with initial data

$$
\begin{aligned}
u_M(0) &= \sum_{j=1}^{M} u_{M,j}(0)e_j, \quad v_M(0) = \sum_{j=1}^{M} v_{M,j}(0)e_j, \\
u'_M(0) &= \sum_{j=1}^{N} u'_{M,j}(0)e_j, \quad v'_M(0) = \sum_{j=1}^{N} v_{M,j}(0)e_j,
\end{aligned} \qquad (11)
$$

where

$$
u_{M,j}(0) = \left\langle u^0, e_j\right\rangle, \ v_{M,j}(0) = \left\langle v^0, e_j\right\rangle, \ u'_{M,j}(0) = \left\langle u^1, e_j\right\rangle, \ v'_{M,j}(0) = \left\langle v^1, e_j\right\rangle. \qquad (12)
$$

A priori estimate

Multiply (9) by $u'_{M,j}(t)$, (10) by $v'_{M,j}(t)$, and summing with respect j from 1 to M, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|u'_M(t)\right\|^2 + \left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla u_M(t)\|^2 + (\mu_1 \diamond \nabla u_M)(t) + \frac{1}{\gamma+1}\|\nabla u_M(t)\|^{2(\gamma+1)}\right] \\
&+\frac{1}{2}\mu_1(t)\|\nabla u_M(t)\|^2 - \frac{1}{2}\left(\mu'_1 \diamond \nabla u_M\right)(t) + \int_\Omega \left(|u_M(t)|^k + |v_M(t)|^l\right)\left|u'_M(t)\right|^{p+1} dx \\
=\ & \int_\Omega f_1(u_M(t), v_M(t))u'_M(t)dx, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|v'_M(t)\right\|^2 + \left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla v_M(t)\|^2 + (\mu_2 \diamond \nabla v_M)(t) + \frac{1}{\gamma+1}\|\nabla v_M(t)\|^{2(\gamma+1)}\right] \\
&+\frac{1}{2}\mu_2(t)\|\nabla v_M(t)\|^2 - \frac{1}{2}\left(\mu'_2 \diamond \nabla v_M\right)(t) + \int_\Omega \left(|v_M(t)|^\theta + |u_M(t)|^\varrho\right)\left|v'_M(t)\right|^{q+1} dx \\
=\ & \int_\Omega f_2(u_M(t), v_M(t))v'_M(t)dx. \qquad (14)
\end{aligned}
$$

Summing (13) and (14) and integrating from 0 to $t \leq T_M$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left[\left\|u'_M(t)\right\|^2 + \left\|v'_M(t)\right\|^2 + \left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla u_M(t)\|^2 + \left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla v_M(t)\|^2\right] \\
&+\frac{1}{2}\left[(\mu_1 \diamond \nabla u_M)(t) + (\mu_2 \diamond \nabla v_M)(t)\right] + \frac{1}{2}\int_0^t \left[\mu_1(s)\|\nabla u_M(s)\|^2 + \mu_2(s)\|\nabla v_M(s)\|^2\right] ds \\
&-\frac{1}{2}\int_0^t \left[\left(\mu'_1 \diamond \nabla u_M\right)(s) + \left(\mu'_2 \diamond \nabla v_M\right)(s)\right] ds + \frac{1}{2(\gamma+1)}\left[\|\nabla u_M(t)\|^{2(\gamma+1)} + \|\nabla v_M(t)\|^{2(\gamma+1)}\right] \\
&+\int_0^t\int_\Omega \left(|u_M(s)|^k + |v_M(s)|^l\right)\left|u'_M(s)\right|^{p+1} dxds + \int_0^t\int_\Omega \left(|v_M(s)|^\theta + |u_M(s)|^\varrho\right)\left|v'_M(s)\right|^{q+1} dxds \\
=\ & \frac{1}{2}\left(\left\|u'_M(0)\right\|^2 + \left\|v'_M(0)\right\|^2 + \|\nabla u_M(0)\|^2 + \|\nabla v_M(0)\|^2\right) \\
&+\frac{1}{2(\gamma+1)}\left(\|\nabla u_M(0)\|^{2(\gamma+1)} + \|\nabla v_M(0)\|^{2(\gamma+1)}\right) \\
&+\int_0^t\int_\Omega \left[f_1(u_M(s), v_M(s))u'_M(s) + f_2(u_M(s), v_M(s))v'_M(s)\right] dxds \\
\leq\ & C_0 + \int_0^t\int_\Omega \left[f_1(u_M(s), v_M(s))u'_M(s) + f_2(u_M(s), v_M(s))v'_M(s)\right] dxds, \qquad (15)
\end{aligned}
$$

where positive constant $C_0 = C\left(\left|u^0\right|_{H^1(\Omega)}, \left|v^0\right|_{H^1(\Omega)}, \left|u^1\right|_{L^2(\Omega)}, \left|v^1\right|_{L^2(\Omega)}, \left|u^0\right|_{W^{1,2(\gamma+1)}(\Omega)}, \left|v^0\right|_{W^{1,2(\gamma+1)}(\Omega)}\right)$.

To estimate the last term in (15) applying (3) and using Young inequalities, Hölder inequalities and Sobolev embedding theorem, we have

$$\begin{aligned}\left|\int_\Omega f_1(u_M, v_M)u'_M dx\right| &\leq C\int_\Omega \left(|u_M+v_M|^{2r+3}\left|u'_M\right| + |v_M|^{r+2}|u_M|^{r+1}\left|u'_M\right|\right)dx \\ &\leq C\left[\left(\|u_M\|_{2(2r+3)}^{2r+3} + \|v_M\|_{2(2r+3)}^{2r+3}\right)\left\|u'_M\right\| + \|u_M\|_{4(r+1)}^{r+1}\|v_M\|_{4(r+2)}^{r+2}\left\|u'_M\right\|\right] \\ &\leq C\left[\|\nabla u_M\|^{2(2r+3)} + \|\nabla v_M\|^{2(2r+3)} + \|\nabla u_M\|^{2(r+1)}\|\nabla v_M\|^{2(r+2)} + \left\|u'_M\right\|^2\right] \qquad (16)\end{aligned}$$

In the same way, we obtain

$$\left|\int_\Omega f_2(u_M, v_M)v'_N dx\right| \leq C\left[\|\nabla u_M\|^{2(2r+3)} + \|\nabla v_M\|^{2(2r+3)} + \|\nabla u_M\|^{2(r+2)}\|\nabla v_M\|^{2(r+1)} + \left\|v'_M\right\|^2\right]. \tag{17}$$

Now, by putting

$$y_M(t) := \left\|u'_M(t)\right\|^2 + \left\|v'_M(t)\right\|^2 + \|\nabla u_M(t)\|^2 + \|\nabla v_M(t)\|^2 + \frac{1}{l(\gamma+1)}\left[\|\nabla u_M(t)\|^{2(\gamma+1)} + \|\nabla v_M(t)\|^{2(\gamma+1)}\right],$$

where $l = \min\{l_1, l_2\} < 1$. Then, we infer from (15)-(17)

$$\begin{aligned}& y_M(t) + \frac{1}{l}\left[(\mu_1 \diamond \nabla u_M)(t) + (\mu_2 \diamond \nabla v_M)(t)\right] + \frac{1}{l}\int_0^t\left[\mu_1(s)\|\nabla u_M(s)\|^2 + \mu_2(s)\|\nabla v_M(s)\|^2\right]ds \\ & -\frac{1}{l}\int_0^t\left[\left(\mu'_1 \diamond \nabla u_M\right)(s) + \left(\mu'_2 \diamond \nabla v_M\right)(s)\right]ds + \frac{2}{l}\int_0^t\int_\Omega\left(|u_M(s)|^k + |v_M(s)|^l\right)\left|u'_M(s)\right|^{p+1}dxds \\ & +\frac{2}{l}\int_0^t\int_\Omega\left(|v_M(s)|^\theta + |u_M(s)|^\varrho\right)\left|v'_M(s)\right|^{q+1}dxds \\ \leq\ & C_0 + C\int_0^t {y_M}^{(2r+3)}(s)\,ds, \end{aligned} \tag{18}$$

Particularly, $y_M(t)$ satisfies the inequality $y_M(t) \leq C_0 + C\int_0^t y_M^{(2r+3)}(s)\,ds$. Then, by applying Grönwall inequality, we obtain

$$y_M(t) \leq C_1 \text{ for all } t \in [0, T]. \tag{19}$$

The estimates follow from (18) and (19):

u_M, v_M are uniformly bounded in $L^\infty\left(0, T; W_0^{2(\gamma+1)}(\Omega)\right)$;

u'_M, v'_M are uniformly bounded in $L^\infty\left(0, T; L^2(\Omega)\right)$;

The sequences $\left\{\int_0^t\int_\Omega\left(|u_M(s)|^k + |v_M(s)|^l\right)\left|u'_M(s)\right|^{p+1}dxds\right\}$ and $\left\{\int_0^t\int_\Omega\left(|v_M(s)|^\theta + |u_M(s)|^\varrho\right)\left|v'_M(s)\right|^{q+1}dxds\right\}$ are uniformly bounded in $L^\infty(0, T)$.

Then

$$\begin{aligned} u_M \to u, \quad v_M \to v \quad \text{weakly } * \text{ in } L^\infty\left(0, T, W_0^{2(\gamma+1)}(\Omega)\right), \\ u'_M \to u', \quad v'_M \to v' \quad \text{weakly } * \text{ in } L^\infty\left(0, T, L^2(\Omega)\right), \end{aligned}$$

By applying the techniques in [3], we obtain the sequence of approximate solutions (u_M, v_M) satisfying

$$\begin{cases} \{u_M\},\ \{v_M\} \text{ are cauchy sequences in } L^\infty\left(0, T, W_0^{2(\gamma+1)}(\Omega)\right), \\ \{u'_M\},\ \{v'_M\} \text{ are cauchy sequences in } L^\infty\left(0, T, L^2(\Omega)\right). \end{cases}$$

Limiting process

Integrating (9) and (10) over $[0,T]$, we get

$$\begin{aligned} & \left\langle u_M'(t), e_j\right\rangle - \left\langle u_M'(0), e_j\right\rangle + \int_0^T \left\langle M\left(\|\nabla u_M(s)\|^2\right)\nabla u_M(s), \nabla e_j\right\rangle ds \\ & -\int_0^T \left\langle \int_0^s \mu_1(s-\tau)\nabla u_M(\tau)d\tau, \nabla e_j\right\rangle ds + \int_0^T \left\langle \left(|u_M|^k + |v_M|^l\right)|u_M'|^{p-1} u_M', e_j\right\rangle ds \\ = & \int_0^T \left\langle f_1(u_M(s), v_M(s)), e_j\right\rangle ds, \end{aligned} \tag{20}$$

$$\begin{aligned} & \left\langle v_M'(t), e_j\right\rangle - \left\langle v_M'(0), e_j\right\rangle + \int_0^T \left\langle M\left(\|\nabla v_M(s)\|^2\right)\nabla v_M(s), \nabla e_j\right\rangle ds \\ & -\int_0^T \left\langle \int_0^s \mu_2(s-\tau)\nabla v_M(\tau)d\tau, \nabla e_j\right\rangle ds + \int_0^T \left\langle \left(|v_M|^\theta + |u_M|^\varrho\right)|v_M'|^{q-1} v_M', e_j\right\rangle ds \\ = & \int_0^T \left\langle f_2(u_M(s), v_M(s)), e_j\right\rangle ds. \end{aligned} \tag{21}$$

Now, we can pass to the limit in (20) and (21) as $M \to \infty$. Therefore, this completes our proof of local existence of weak solution. □

3 Blow up of solutions

Our main result in this section is to show the blow up result of the solution of problem (1). For this purpose, we need the following lemmas.

Lemma 3. *[7]. There exist two positive constants* c_0 *and* c_1 *such that*

$$c_0\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \le 2(r+2)F(u,v) \le c_1\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \tag{22}$$

is satisfied.

Lemma 4. *Suppose that* (4) *holds. Then there exists* $\eta > 0$ *such that for the solution* (u,v)

$$\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \le \eta\left(\frac{1}{\gamma+1}\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) + l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right)^{r+2}. \tag{23}$$

Proof: Direct computation using Minkowski, Hölder's and Young's inequality and the embedding theorem yields the proof of this lemma. □

In order to state and prove our result and for sake of simplicity, we take $a = b = 1$. We introduce the following:

$$B = \eta^{\frac{1}{2(r+2)}},\ \alpha_1 = B^{-\frac{r+2}{r+1}},\ E_1 = \left(\frac{1}{2} - \frac{1}{2(r+2)}\right)\alpha_1^2,\ \ E_2 = \left(\frac{1}{2(\gamma+1)} - \frac{1}{2(r+2)}\right)\alpha_1^2, \tag{24}$$

where η is the optimal constant in (23). We define the functional

$$\Gamma(t) := \left(1 - \int_0^t \mu_1(s)\,ds\right)\|\nabla u\|^2 + \left(1 - \int_0^t \mu_2(s)\,ds\right)\|\nabla v\|^2 + (\mu_1 \diamond \nabla u) + (\mu_2 \diamond \nabla v). \tag{25}$$

The following lemma is very useful to prove our result for positive initial energy $E(0) > 0$, and it is similar to a lemma used firstly by Vitillaro [13].

Lemma 5. *[7]. Suppose that assumptions (A1) and (4) hold. Let (u, v) be a solution of (1). Moreover, assume that $E(0) < E_1$ and*

$$\left(\frac{1}{\gamma+1}(\|\nabla u(0)\|^{2(\gamma+1)} + \|\nabla v(0)\|^{2(\gamma+1)}) + \Gamma(0)\right)^{\frac{1}{2}} > \alpha_1. \tag{26}$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$\left(\frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) + \Gamma(t)\right)^{\frac{1}{2}} \geq \alpha_2, \text{ for } t > 0, \tag{27}$$

$$\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right)^{\frac{1}{2(r+2)}} \geq B\alpha_2, \text{ for } t > 0. \tag{28}$$

for all $t \in [0, T)$.

Theorem 6. *Assume that (A1), (A2) and (4) hold. Assume further that*

$$2(r+2) > \max\{2(\gamma+1), \ k+p+1, \ l+p+1, \ \theta+q+1, \ \varrho+q+1\}.$$

Then any the solution of the problem (1) with initial data satisfying

$$\left[\frac{1}{\gamma+1}(\|\nabla u(0)\|^{2(\gamma+1)} + \|\nabla v(0)\|^{2(\gamma+1)}) + \Gamma(0)\right]^{\frac{1}{2}} > \alpha_1, \ E(0) < E_2,$$

cannot exist for all time, where α_1 and E_2 are defined in (24).

Proof: We suppose that the solution exists for all time and we reach to a contradiction. Set

$$H(t) = E_2 - E(t). \tag{29}$$

By applying (7) and (29), we have

$$\begin{aligned} 0 \ < \ & H(0) \leq H(t) = E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{1}{2}\Gamma(t) \\ & - \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) + \int_\Omega F(u,v)\,dx. \end{aligned} \tag{30}$$

From (28) and (22), we have

$$\begin{aligned} & E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{1}{2}\Gamma(t) \\ & - \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) + \int_\Omega F(u,v)\,dx \\ \leq \ & \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right). \end{aligned} \tag{31}$$

By combining (30) and (31), we have

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right). \tag{32}$$

We then define with $\varepsilon > 0$

$$\Psi(t) = H^{1-\sigma}(t) + \varepsilon\left(\int_\Omega u_t u\,dx + \int_\Omega v_t v\,dx\right) \tag{33}$$

The reminder of the proof is similar to the proof of Theorem 12 combined with the proof in [1], and then we get the result. □

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