Advances in the Theory of Nonlinear Analysis and its Applications **4** (2020) No. 3, 143–151.
https://doi.org/10.31197/atnaa.753458
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648 Peer-Reviewed Scientific Journal

On a final value problem for parabolic equation on the sphere with linear and nonlinear source

Nguyen Duc Phuong[a], Tran Thanh Binh[b], Nguyen Hoang Luc*[b]

[a] *Faculty of fundamental science, Industrial university of Ho Chi Minh City, Viet Nam.*
[b] *Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*

Abstract

Parabolic equation on the unit sphere arise naturally in geophysics and oceanography when we model a physical quantity on large scales. In this paper, we consider a problem of finding the initial state for backward parabolic problem on the sphere. This backward parabolic problem is ill-posed in the sense of Hadamard. The solutions may be not exists and if they exists then the solution does not continuous depends on the given observation. The backward problem for homogeneous parabolic problem was recently considered in the paper Q.T. L. Gia, N.H. Tuan, T. Tran. However, there are very few results on the backward problem of nonlinear parabolic equation on the sphere. In this paper, we do not consider the its existence, we only study the stability of the solution if it exists. By applying some regularized method and some techniques on the spherical harmonics, we approximate the problem and then obtain the convalescence rate between the regularized solution and the exact solution.

Keywords: Cauchy problem; parabolic on the sphere; Ill-posed problem; Convergence estimates.
2010 MSC: 35K05, 35K99, 47J06

1. Introduction

Boundary value problem and parabolic equations and also related models have many applications in various fields, for example, heat transfer, biology, physics. Let us refer some works concerning the existence of parabolic equations, for example [13, 14, 15, 16, 17] and the references therein.

Email addresses: nguyenducphuong@iuh.edu.vn (Nguyen Duc Phuong), tranthanhbinh@tdmu.edu.vn (Tran Thanh Binh), nguyenhoangluc@tdmu.edu.vn (Nguyen Hoang Luc*)
*Corresponding author.

Received June 16, 2020, Accepted: August 03, 2020, Online: August 05, 2020.

In this paper, we consider a final value problem (often called the backward problem) for the following parabolic equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \Delta^* u(x,t) = F(u(x,t)), \quad (x,t) \in S^n \times [0,T] \\ u(x,T) = g(x), \end{cases} \tag{1}$$

where S^n is the unit sphere on R^n. This backward parabolic problem is ill-posed in the sense of Hadamard. Indeed, our above problem is non well-posed in the sense of Hadamard, the solutions may be not exists and if they exists then the solution does not continuous depends on the given observation. If the given data is noise by the measured data with small error then the corresponding solutions may have big errors. This is disadvantage point for compute the numerical solution of the problem. Let us assume that if the given final observation data g is noisy by the data g^ϵ which satisfies that

$$\|g^\epsilon - g\|_{L^2(S^n)} \leq \epsilon. \tag{2}$$

Here, our main goal in this paper is to construct a regularization problem and prove that it is well-posed. Backward problem is also ill-posed and there are many publications about regularization, for example, N.H. Tuan *et al.* [10, 11, 12]. Partial Differential Equations (PDEs) on the sphere has many applications in various fields, for example, physical geodesy, geophysics, oceanography, and biology. Let us refer the reader to many papers of Q.T. Le Gia and his group [8, 9]. However, to the best of our knowledge, there are limited results on backward problem of PDEs on the sphere.

To the best of author's knowledge, there are very few papers on backward problem on the sphere. In order to study the models on the sphere, we can apply some techniques and knowledge on Spherical harmonics. In this paper, we apply two various methods for approximate the backward problem. In the linear case $F = F(x,t)$, we use the Fourier truncation method. In the nonlinear case $F = F(u)$, we use the method of quasi-reversibility.

The paper is organized as follows. In the section 3, we use truncation method to give approximate solution. In section 4, we present a quasi-reversibility regularization method and establish the convergence estimates between the regularized solution and the exact solution.

2. Preliminaries

From [8], we know that the eigenvalues for $-\Delta$ are as follows

$$\lambda_l = l(l+n-1), \quad l = 0, 1, 2, ...$$

and the eigenfunctions $Y_l(x)$ such that

$$\Delta Y_l(x) = -\lambda_l Y_l(x).$$

Let us denote the space V_l which contain all spherical harmonics in the following

$$\{Y_{lk}(x) : k = 1, 2, 3, ...N(n,l)\},$$

where

$$N(n,0) = 1, \quad N(n,l) = \frac{(2l+n-1)\Gamma(l+n-1)}{\Gamma(l+1)\Gamma(n)}, \quad l \geq 1.$$

For any function $f \in L^2(S^n)$, we have the expansion of spherical harmonics as follows

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \widehat{f}_{lk} Y_{lk}, \quad \widehat{f}_{lk} = \int_{S^n} f \overline{Y}_{lk} dS.$$

Here dS is called by the surface measure of S^n. The Sobolev space $H^\sigma(S^n)$ for $\sigma > 0$ is the space which consists of all function f such that

$$\|f\|^2_{H^\sigma(S^n)} = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} (1+\lambda_l)^\sigma |\widehat{f}_{lk}|^2 < \infty.$$

3. Inhomogeneous linear backward parabolic on the sphere

3.1. The inverse problem

Let $u(x,T) = g(x)$ and the source term $F(x,t)$ be given. The linear backward parabolic problem is of finding $u(x,0)$ from the system

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = F(x,t), \quad 0 < t < T, \\ u(x,T) = g(x), \ g \in H^{\sigma}(S^n). \end{cases} \tag{3}$$

Theorem 3.1. *The Problem* (3) *has a unique solution if and only if the following holds*

$$\sum_{l=0}^{\infty}\sum_{k=1}^{N(k,l)} e^{2\lambda T}\Big[\widehat{g}_{lk} - \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]^2 < \infty. \tag{4}$$

Then, its solution has the form

$$u(x,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(k,l)} e^{\lambda(T-t)}\Big[\widehat{g}_{lk} - \int_t^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]Y_{lk}(x). \tag{5}$$

Proof. Suppose the Problem (3) has a unique solution u. Let $u(x,0) = h(x) \in L^2(S^n)$. Then u is given by

$$u(x,t) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(k,l)} e^{-\lambda t}\Big[\widehat{h}_{lk} + \int_0^t e^{\lambda(s-t)}\widehat{F}_{lk}(s)ds\Big]Y_{lk}(x), \tag{6}$$

where $\widehat{h}_{lk} = \int_{S^n} u(x,0)\overline{Y_{\ell k}}dS$. By letting $t = T$, we have

$$g(x) = u(x,T) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(k,l)} e^{-\lambda T}\Big[\widehat{h}_{lk} + \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]Y_{lk}(x).$$

This implies that

$$\widehat{g}_{lk} = e^{-\lambda T}\Big[\widehat{h}_{lk} + \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big].$$

Or

$$\widehat{h}_{lk} = e^{\lambda T}\Big[\widehat{g}_{lk} - \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]. \tag{7}$$

Hence

$$\|u(.,0)\|^2_{L^2(S^n)} = \sum_{n=1}^{\infty}|\widehat{h}_{lk}|^2 = \sum_{n=1}^{\infty} e^{2\lambda T}\Big[\widehat{g}_{lk} - \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]^2 < \infty. \tag{8}$$

Suppose that (4) holds. Define the function

$$v(x) = \sum_{l=0}^{\infty}\sum_{k=1}^{N(k,l)} e^{\lambda T}\Big[\widehat{g}_{lk} - \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]Y_{lk}(x),$$

then from (4) we have $v \in L^2(S^n)$.
We consider the problem of finding a solution u from the original value $u(x,0) = v(x)$

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = F(x,t), \quad 0 < t < T, \\ u(x,0) = v(x), \ f \in L^2(S^n). \end{cases} \tag{9}$$

By Theorem 1, Problem (23) has a unique solution $u \in L^2(S^n)$. It is given by

$$u(x,t) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(k,l)} e^{-\lambda t}\Big[\widehat{v}_{lk} + \int_0^t e^{\lambda(s-t)}\widehat{F}_{lk}(s)ds\Big]Y_{lk}(x). \tag{10}$$

Since

$$\begin{aligned}\widehat{v}_{lk} = \int_{S^n} v(x)\overline{Y_{\ell k}}dS &= \int_{S^n}\left(\sum_{l=0}^{\infty}\sum_{k=1}^{N(k,l)} e^{\lambda T}\Big[\widehat{g}_{lk} - \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big]Y_{lk}(x)\right)\bar{Y}_{lk}dS \\ &= e^{\lambda T}\Big[\widehat{g}_{lk} - \int_0^T e^{\lambda(s-T)}\widehat{F}_{lk}(s)ds\Big].\end{aligned}$$

We get

$$u(x,T) = g(x).$$

So, we deduce that u is a solution of Problem (23). And we also have (5). □

3.2. Truncation regularization method

In this subsection, we give a regularized solution as follows

$$u_\epsilon(x,t) = \sum_{l=0}^{\lambda_l \le \mathcal{M}(\epsilon)} \sum_{k=1}^{N(k,l)} e^{\lambda(T-t)}\Big[\widehat{g^\epsilon}_{lk} - \int_t^T e^{\lambda(s-T)}\widehat{F^\epsilon}_{lk}(s)ds\Big]Y_{lk}(x), \tag{11}$$

where $\mathcal{M}(\epsilon)$ is chosen later.

Theorem 3.2. *Let g^ϵ, F^ϵ be as follows*

$$\|g^\epsilon - g\|_{L^2(S^n)} + \|F^\epsilon - F\|_{L^\infty(0,T;L^2(S^n))} \le \epsilon. \tag{12}$$

Let us choose $\mathcal{M}(\epsilon) = \frac{1}{m}\log(1/\epsilon)$ for any $0 < m < 1$, then we obtain

$$\|u_\epsilon(.,t) - u(.,t)\|_{L^2(S^n)} \le \left(1 + \frac{1}{m}\log(1/\epsilon)\right)^{-\sigma}\|u\|_{L^\infty(0,T;H^\sigma(S^n))} + 2\epsilon^{1-m}. \tag{13}$$

Proof. Set the following function

$$v_\epsilon(x,t) = \sum_{l=0}^{\lambda_l \le \mathcal{M}(\epsilon)} \sum_{k=1}^{N(k,l)} e^{\lambda_l(T-t)}\Big[\widehat{g^\epsilon}_{lk} - \int_t^T e^{\lambda(s-T)}\widehat{F^\epsilon}_{lk}(s)ds\Big]Y_{lk}(x). \tag{14}$$

Let us first obtain the following estimate

$$\|u(.,t) - v_\epsilon(.,t)\|_{L^2(S^n)} = \sqrt{\sum_{l=0}^{\lambda_l > \mathcal{M}(\epsilon)}\sum_{k=1}^{N(k,l)} |\widehat{u}_{lk}|^2} = \sqrt{\sum_{l=0}^{\lambda_l > \mathcal{M}(\epsilon)}\sum_{k=1}^{N(k,l)} (1+\lambda_l)^{-\sigma}(1+\lambda_l)^{\sigma}|\widehat{u}_{lk}|^2}. \tag{15}$$

Noting that if $\lambda_l > \mathcal{M}(\epsilon)$ then we get

$$(1+\lambda_l)^{-\sigma} \le (1+\mathcal{M}(\epsilon))^{-\sigma}. \tag{16}$$

This latter inequality implies that

$$\|u(.,t) - v_\epsilon(.,t)\|_{L^2(S^n)} \le (1+\mathcal{M}(\epsilon))^{-\sigma}\|u(.,t)\|_{H^\sigma(S^n)} \le (1+\mathcal{M}(\epsilon))^{-\sigma}\|u\|_{L^\infty(0,T;H^\sigma(S^n))}. \tag{17}$$

Then, we get the following estimate

$$\begin{aligned}
\|u_\epsilon(.,t)-v_\epsilon(.,t)\|_{L^2(S^n)} &\le \Big\| \sum_{l=0}^{\lambda_l\le\mathcal{M}(\epsilon)}\sum_{k=1}^{N(k,l)} e^{\lambda_l(T-t)}\left(\widehat{h^\epsilon}_{lk}-\widehat{h}_{lk}\right)\Big\|_{L^2(S^n)} \\
&+ \Big\| \int_t^T \sum_{l=0}^{\lambda_l\le\mathcal{M}(\epsilon)}\sum_{k=1}^{N(k,l)} e^{\lambda_l(s-t)}\left(\widehat{F^\epsilon}_{lk}-\widehat{F}_{lk}\right) ds\Big\|_{L^2(S^n)} \\
&\le e^{\mathcal{M}(\epsilon)(T-t)}\sqrt{\sum_{l=0}^{\lambda_l\le\mathcal{M}(\epsilon)}\sum_{k=1}^{N(k,l)}\left(\widehat{h^\epsilon}_{lk}-\widehat{h}_{lk}\right)^2} \\
&+ \int_t^T \sqrt{\sum_{l=0}^{\lambda_l\le\mathcal{M}(\epsilon)}\sum_{k=1}^{N(k,l)} e^{2\lambda_l(s-t)}\left(\widehat{F^\epsilon}_{lk}-\widehat{F}_{lk}\right)^2} ds \\
&\le e^{\mathcal{M}(\epsilon)(T-t)}\|h^\epsilon-h\|_{L^2(S^n)} + \int_t^T e^{\mathcal{M}(\epsilon)(s-t)}\|F^\epsilon(.,s)-F(.,s)\|_{L^2(S^n)} ds \\
&\le e^{\mathcal{M}(\epsilon)(T-t)}\left(\|h^\epsilon-h\|_{L^2(S^n)}+\|F^\epsilon-F\|_{L^\infty(0,T;L^2(S^n))}\right). \qquad (18)
\end{aligned}$$

Since the fact that

$$\|h^\epsilon-h\|_{L^2(S^n)}+\|F^\epsilon-F\|_{L^\infty(0,T;L^2(S^n))}\le 2\epsilon,$$

we deduce that

$$\|u_\epsilon(.,t)-v_\epsilon(.,t)\|_{L^2(S^n)}\le 2\epsilon e^{\mathcal{M}(\epsilon)T}. \qquad (19)$$

Combining (17) and (19), we conclude that

$$\|u(.,t)-v_\epsilon(.,t)\|_{L^2(S^n)}\le (1+\mathcal{M}(\epsilon))^{-\sigma}\|u\|_{L^\infty(0,T;H^\sigma(S^n))}+2\epsilon e^{\mathcal{M}(\epsilon)T}. \qquad (20)$$

By choose $\mathcal{M}(\epsilon)=\frac{1}{m}\log(1/\epsilon)$ and noting that $0<m<1$, we find that

$$\|u(.,t)-v_\epsilon(.,t)\|_{L^2(S^n)}\le\left(1+\frac{1}{m}\log(1/\epsilon)\right)^{-\sigma}\|u\|_{L^\infty(0,T;H^\sigma(S^n))}+2\epsilon^{1-m}, \qquad (21)$$

which allows us to get the desired result. □

4. Nonlinear backward parabolic on the sphere

To more clear, we discuss some details on the direct problem for nonlinear parabolic equation on the sphere.

4.1. The direct problem

The direct problem is of finding $u(x,t)$ from the known data $u(x,0)$

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t)-\Delta^* u(x,t)=F(u(x,t)), \quad 0<t<T, \\ u(x,0)=u_0(x),\ u_0\in H^{2\sigma}(S^n). \end{cases} \qquad (22)$$

Theorem 4.1. *Let $F\in C^\infty(R^{n+1})$ and $u_0\in H^{2\sigma}(S^n)$ with $\sigma\ge 1$. Then there is a time $T>0$ such that (1.1), (1.2) has a unique solution u satisfying*

$$u\in C([0,T];H^{2\sigma}(S^{n-1}))\cap C^1([0,T];H^{2\sigma-2}(S^{n-1})).$$

And u has the form

$$u(x,t) = S(t)u_0 + \int_0^t S(s-t)F(u(s))ds,$$

where $S(t)f = e^{-t\Delta^*}$ are defined as

$$S(t)f = \sum_{L=0}^{\infty} \sum_{m=-L}^{m=L} e^{-\lambda_L t} \widehat{f}_{L,m} Y_{L,m}.$$

4.2. The backward in time problem

In this section, we are looking for solution u of the following backward in time problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \Delta^* u(x,t) = F(u(x,t)), \quad (x,t) \in S^n \times [0,T], \\ u(x,T) = u_T(x). \end{cases} \tag{23}$$

Let P_N be the orthogonal projection of $L^2(S^n)$ on to H_N, where H_N is the linear space spanned by the set

$$H_N = \{Y_{lk} : |k| = 0, 1, ..., l; l = 0, 1, 2, ...N\}.$$

For $w \in L^2(S^n)$ we let $P_N w = \sum_{L=0}^{N} \sum_{m=-L}^{m=L} \widehat{w}_{L,m} Y_{L,m}$. The approximate problem is

$$\begin{cases} \dfrac{\partial}{\partial t}u^N(x,t) - \Delta u^N(x,t) = P_N F(u^N(x,t)), \quad (x,t) \in S^2 \times [0,T], \\ u^N(x,T) = P_N u_T(x). \end{cases} \tag{24}$$

4.3. The backward in time problem with a global Lipschitz continuous source term

Lemma 4.2. *There exists a unique solution of problem* (24) *in* $C([0,T]; H^\sigma(S^n))$.

Proof. Let u^N and v^N be two solution of problem (24) such that $u^N, v^N \in C([0,T]; H^\sigma(S^n))$. Put

$$u_k^N(t) = e^{-k(t-T)}u^N(t),\ v_k^N(t) = e^{-k(t-T)}v^N(t),\ w_k^N(t) = u_k^N(t) - v_k^N(t) \quad k > 0.$$

By direct computation, we have w_k^N satisfying the equation

$$\frac{d}{dt}w_k^N - \Delta w_k^N(t) - k w_k^N(t) = e^{k(t-T)} \left(P_N F(t, u_k^N(t)) - P_N F(t, v_k^N(t))\right). \tag{25}$$

It follows that

$$\begin{aligned} < \frac{d}{dt}w_k^N(t) &- \Delta w_k^N(t) - k w_k^N(t), w_k^N(t) >_{H^\sigma(S^n)} \\ &= < e^{k(t-T)} \left(P_N F(t, u_k^N(t)) - P_N F(t, v_k^N(t))\right), w_k^N(t) >_{H^\sigma(S^n)}. \end{aligned} \tag{26}$$

Recall the the Lipchitz property of F given in Theorem 1, we have

$$\begin{aligned} \Big| < e^{k(t-T)} \left(P_N F(t, u_k^N(t)) - P_N F(t, v_k^N(t))\right), w_k^N(t) >_{H^\sigma(S^n)} \Big| \\ &\le e^{k(t-T)} \|P_N F(t, u_k^N(t)) - P_N F(t, v_k^N(t))\|_{H^\sigma} \|w_k^N(t)\|_{H^\sigma(S^n)} \\ &\le e^{k(t-T)} \|F(t, u_k^N(t)) - F(t, v_k^N(t))\|_{H^\sigma} \|w_k^N(t)\|_{H^\sigma(S^n)} \\ &\le K \|w_k^N(t)\|^2_{H^\sigma(S^n)}. \end{aligned}$$

Therefore, we get

$$< e^{k(t-T)} \left(P_N F(t, u_k^N(t)) - P_N F(t, v_k^N(t))\right), w_k^N(t) >_{H^\sigma(S^n)} \ge -K \|w_k^N(t)\|^2_{H^\sigma(S^n)}, \tag{27}$$

and we have

$$\begin{aligned} |<\Delta w_k^N(t), w_k^N(t)>_{H^\sigma(S^n)}| &= \sum_{L=1}^{N}\sum_{m=-L}^{m=L}\lambda_L^{2\sigma+1}|\widehat{(w_k^N)}_{L,m}|^2 \\ &\leq \lambda_N\sum_{L=1}^{N}\sum_{m=-L}^{m=L}\lambda_L^{2\sigma}(S^n)|\widehat{(w_k^N)}_{L,m}|^2 = \lambda_N\|w_k^N(t)\|^2_{H^\sigma(S^n)}. \end{aligned} \tag{28}$$

Combining (26), (27), (28), we get

$$\frac{1}{2}\frac{d}{dt}\|w_k^N(t)\|^2_{H^\sigma(S^n)} \geq k\|w_k^N(t)\|^2_{H^\sigma(S^n)} - K\|w_k^N(t)\|^2_{H^\sigma(S^n)} - \lambda_N\|w_k^N(t)\|^2_{H^\sigma(S^n)}.$$

By taking the integration with respect to s from t to T, we have

$$\|w_k^N(T)\|^2_{H^\sigma(S^n)} - \|w_k^N(t)\|^2_{H^\sigma(S^n)} \geq 2\int_t^T (k-K-\lambda_N)\,\|w_k^N(s)\|^2_{H^\sigma(S^n)}ds.$$

This follows that

$$\|u^N(T)-v^N(T)\|^2_{H^\sigma(S^n)} - e^{-2k(t-T)}\|u^N(t)-v^N(t)\|^2_{H^\sigma(S^n)} \tag{29}$$

$$\geq 2\int_t^T (k-K-\lambda_N)\,e^{-2k(s-T)}\|u^N(s)-v^N(s)\|^2_{H^\sigma(S^n)}ds. \tag{30}$$

Choosing $k = K+\lambda_N$ and noting that $u^N(T)-v^N(T)=0$, we get $u^N(t)=v^N(t)\ \ \forall 0\leq t\leq T$. This ends the proof. □

Theorem 4.3. *Assume that the Problem with $g\in H^\sigma$ has a weak solution $u\in C([0,T];H^\sigma(S^n))$. For any $\epsilon>0$, let $g_\epsilon\in H^\sigma(S^n)$ such that*

$$\|g_\epsilon - g\|_{H^\sigma(S^n)}\leq\epsilon.$$

Suppose that F satisfies the Lipschitz condition on H^σ, i.e., there exists a constant K such that

$$\|F(u)-F(v)\|_{H^\sigma(S^n)}\leq K\|u-v\|_{H^\sigma(S^n)}, \tag{31}$$

for any $u,v\in H^\sigma$. Denote by u_ϵ the solution of Problem (24) with $g=g_\epsilon$.

(i) If u satisfies the following condition

$$\sum_{L=0}^{\infty}\sum_{m=-L}^{m=L}\lambda_L^{2\sigma}e^{2t\lambda_L}|\widehat{u(t)}_{L,m}|^2\leq A_1^2,$$

then

$$\|u_\epsilon(t)-u(t)\|^2_{H^\sigma(S^n)}\leq e^{-2\lambda_N t}\left(4e^{2\lambda_N T}\epsilon^2+2A_1^2\right)\exp\{4TK^2(T-t)\}.$$

(ii) If u satisfies the following condition

$$\sum_{L=0}^{\infty}\sum_{m=-L}^{m=L}\lambda_L^{2\sigma+2r}e^{2t\lambda_L}|\widehat{u(t)}_{L,m}|^2\leq A_2^2,$$

then

$$\|u_\epsilon(t)-u(t)\|^2_{H^\sigma(S^n)}\leq e^{-2\lambda_N t}\left(4e^{2\lambda_N T}\epsilon^2+2\lambda_N^{-2r}A_2^2\right)\exp\{4TK^2(T-t)\}.$$

Proof. First, we have the following estimate

$$\begin{aligned}
&\|u_\epsilon(t) - P_N u(t)\|^2_{H^\sigma(S^n)} \\
&= \sum_{L=1}^{N} \lambda_L^{2\sigma} \sum_{m=-L}^{m=L} \left|(\widehat{u}_\epsilon)_{L,m} - (\widehat{u})_{L,m}\right|^2 \\
&= \sum_{L=1}^{N} \lambda_L^{2\sigma} \sum_{m=-L}^{m=L} \left|e^{\lambda_L(T-t)}(\widehat{g_\epsilon - g})_{L,m} - \int_t^T e^{\lambda_L(s-t)} \left(F(\widehat{s,u^\epsilon(s)}) - F(\widehat{s,u(s)})\right)_{L,m} ds\right|^2 \\
&\le 2\sum_{L=1}^{N} \lambda_L^{2\sigma} \sum_{m=-L}^{m=L} e^{2\lambda_N(T-t)} \left|(\widehat{g_\epsilon - g})_{L,m}\right|^2 \\
&\qquad +2T \int_t^T \sum_{L=1}^{N} \lambda_L^{2\sigma} \sum_{m=-L}^{m=L} e^{2\lambda_N(s-t)} \left| \left(F(\widehat{s,u_\epsilon(s)}) - F(\widehat{s,u(s)})\right)_{L,m} \right|^2 ds \\
&\le 2e^{2\lambda_N(T-t)} \|g_\epsilon - g\|^2_{H^\sigma} + 2T \int_t^T e^{2\lambda_N(s-t)} \left\|F(s,u_\epsilon(s)) - F(s,u(s))\right\|^2_{H^\sigma(S^n)} ds \\
&\le 2e^{2\lambda_N(T-t)}\epsilon^2 + 2TK^2 \int_t^T e^{2\lambda_N(s-t)} \left\|u_\epsilon(s) - u(s)\right\|^2_{H^\sigma(S^n)} ds. \qquad (32)
\end{aligned}$$

We also have

a. The case $\sum_{L=0}^{\infty}\sum_{m=-L}^{m=L} \lambda_L^{2\sigma} e^{2t\lambda_L} |\widehat{u(t)}_{L,m}|^2 \le A_1^2$. Then

$$\begin{aligned}
\|u(t) - P_N u(t)\|^2_{H^\sigma} &= \sum_{L=N+1}^{\infty} \lambda_L^{2\sigma} \sum_{m=-L}^{m=L} |\widehat{u(t)}_{L,m}|^2 \\
&\le e^{-2\lambda_N t} \sum_{L=N+1}^{\infty} \sum_{m=-L}^{m=L} \lambda_L^{2\sigma} e^{2t\lambda_L} |\widehat{u(t)}_{L,m}|^2 \\
&\le e^{-2\lambda_N t} A_1^2. \qquad (33)
\end{aligned}$$

b. The case $\sum_{L=0}^{\infty}\sum_{m=-L}^{m=L} \lambda_L^{2\sigma+2r} e^{2t\lambda_L} |\widehat{u(t)}_{L,m}|^2 \le A_2^2$. Then

$$\begin{aligned}
\|u(t) - P_N u(t)\|^2_{H^\sigma(S^n)} &= \sum_{L=N+1}^{\infty} \lambda_L^{2\sigma} \sum_{m=-L}^{m=L} |\widehat{u(t)}_{L,m}|^2 \\
&\le e^{-2\lambda_N t} \lambda_N^{-2r} \sum_{L=N+1}^{\infty} \sum_{m=-L}^{m=L} \lambda_L^{2\sigma+2r} e^{2t\lambda_L} |\widehat{u(t)}_{L,m}|^2 \\
&\le e^{-2\lambda_N t} \lambda_N^{-2r} A_2^2. \qquad (34)
\end{aligned}$$

(i) With the case (a), combining (32) and (33), we obtain

$$\begin{aligned}
\|u_\epsilon(t) - u(t)\|^2_{H^\sigma(S^n)} &\le 2\|u_\epsilon(t) - P_N u(t)\|_{H^\sigma(S^n)} + 2\|u(t) - P_N u(t)\|_{H^\sigma(S^n)} \\
&\le 4e^{2\lambda_N(T-t)}\epsilon^2 + 4TK^2 \int_t^T e^{2\lambda_N(s-t)} \left\|u_\epsilon(s) - u(s)\right\|^2_{H^\sigma(S^n)} ds + 2e^{-2\lambda_N t} A_1^2.
\end{aligned}$$

Hence

$$e^{2\lambda_N t}\|u_\epsilon(t) - u(t)\|^2_{H^\sigma} \le \left(4e^{2\lambda_N T}\epsilon^2 + 2A_1^2\right) + 4TK^2 \int_t^T e^{2\lambda_N s} \left\|u_\epsilon(s) - u(s)\right\|^2_{H^\sigma} ds.$$

Applying the Gronwall's inequality, we obtain

$$e^{2\lambda_N t}\|u_\epsilon(t) - u(t)\|^2_{H^\sigma(S^n)} \le \left(4e^{2\lambda_N T}\epsilon^2 + 2A_1^2\right) \exp\{4TK^2(T-t)\}.$$

This implies that

$$\|u_\epsilon(t)-u(t)\|^2_{H^\sigma(S^n)} \le e^{-2\lambda_N t}\left(4e^{2\lambda_N T}\epsilon^2+2A_1^2\right)\exp\{4TK^2(T-t)\}.$$

(ii) With the case (b), combining (32) and (34), we obtain

$$\begin{aligned}\|u_\epsilon(t)-u(t)\|^2_{H^\sigma(S^n)} &\le 2\|u_\epsilon(t)-P_Nu(t)\|_{H^\sigma}+2\|u(t)-P_Nu(t)\|_{H^\sigma(S^n)}\\ &\le 4e^{2\lambda_N(T-t)}\epsilon^2+4TK^2\int_t^T e^{2\lambda_N(s-t)}\left\|u_\epsilon(s)-u(s)\right\|^2_{H^\sigma(S^n)}ds\\ &\quad+2e^{-2\lambda_N t}\lambda_N^{-2r}A_2^2. \end{aligned}\tag{35}$$

Hence

$$e^{2\lambda_N t}\|u_\epsilon(t)-u(t)\|^2_{H^\sigma} \le \left(4e^{2\lambda_N T}\epsilon^2+2\lambda_N^{-2r}A_2^2\right)+4TK^2\int_t^T e^{2\lambda_N s}\left\|u_\epsilon(s)-u(s)\right\|^2_{H^\sigma}ds.$$

Applying the Gronwall's inequality, we obtain

$$e^{2\lambda_N t}\|u_\epsilon(t)-u(t)\|^2_{H^\sigma(S^n)} \le \left(4e^{2\lambda_N T}\epsilon^2+2\lambda_N^{-2r}A_2^2\right)\exp\{4TK^2(T-t)\}.$$

This implies that

$$\|u_\epsilon(t)-u(t)\|^2_{H^\sigma(S^n)} \le e^{-2\lambda_N t}\left(4e^{2\lambda_N T}\epsilon^2+2\lambda_N^{-2r}A_2^2\right)\exp\{4TK^2(T-t)\}.$$

□

References

[1] N.H. Tuan, L.D. Long, S. Tatar, *Tikhonov regularization method for a backward problem for the inhomogeneous time-fractional diffusion equation* Appl. Anal. 97 (2018), no. 5, 842–863.

[2] Q.T. Le Gia *Approximation of parabolic PDEs on spheres using collocation method,* Adv. Comput. Math., 22 (2005), 377–397.

[3] D.D. Trong, N.H. Tuan, *A nonhomogeneous backward heat problem: Regularization and error estimates,* Vol. 2008(2008), No. 33, pp. 1–14.

[4] D.D. Trong, N.H. Tuan, P.H. Quan, *A quasi-boundary value method for regularizing nonlinear ill-posed problems,* Vol. 2009(2009), No. 109, pp. 1–16.

[5] K. Sakamoto, M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems* J. Math. Anal. Appl. 382 (2011), no. 1, 426–447

[6] W. McLean, *Regularity of solutions to a time-fractional diffusion equation* ANZIAM J. 52 (2010), no. 2, 123–138.

[7] Q.T. L. Gia, N.H. Tuan, T. Tran, *Solving the backward heat equation on the unit sphere* ANZIAM J. (E) 56 (2016), pp. C262–C278.

[8] Q.T. L. Gia, *Approximation of parabolic PDEs on spheres using collocation method,* Adv. Comput. Math., 22 (2005), 377-397.

[9] Q.T. L. Gia, *Galerkin approximation of elliptic PDEs on spheres* J. Approx. Theory , 130 (2004), 125-149

[10] N.H. Tuan, D.D. Trong, *On a backward parabolic problem with local Lipschitz source* J. Math. Anal. Appl. 414 (2014), no. 2, 678–692.

[11] N.H. Tuan, V.V. Au, V.A. Khoa, D. Lesnic, *Identification of the population density of a species model with nonlocal diffusion and nonlinear reaction* Inverse Problems 33 (2017), no. 5, 055019, 40 pp.

[12] N.H. Tuan, V.V. Au, V.A. Khoa, *Analysis of a quasi-reversibility method for a terminal value quasi-linear parabolic problem with measurements* SIAM J. Math. Anal. 51 (2019), no. 1, 60–85.

[13] S. G. Georgiev, K. Zenir, *New results on IBVP for Class of Nonlinear Parabolic Equations,* Advances in the Theory of Nonlinear Analysis and its Application, Volume 2, Issue 4 , Jan 2018 , 202 – 216

[14] J. M. Jonnalagadda, *Existence Results for Solutions of Nabla Fractional Boundary Value Problems with General Boundary Conditions,* Advances in the Theory of Nonlinear Analysis and its Application, 4 (2020) No. 1, 29?42.

[15] K. Hamdache, D. Hamroun, *Asymptotic Behaviours for the Landau-Lifshitz-Bloch Equation,* Adv. Theory Nonlinear Anal. Appl. 3 (2019), 174 – 191.

[16] C.Zhang, J. Chen, *Convergence analysis of variational inequality and fixed point problems for pseudo-contractive mapping with Lipschitz assumption* , Results in Nonlinear Analysis 2 (2019): 102-112

[17] A.Ardjouni, A.Djoudi, *Existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations,* Results in Nonlinear Analysis 2 (2019): 136-142