Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 2, 232–239.
https://doi.org/10.31197/atnaa. 777775
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Generalised Picone's identity and some Qualitative properties of p-sub-Laplacian on Heisenberg groups

Gaurav Dwivedi[a]

[a]*Department of Mathematics, Birla Institute of Technology & Science Pilani, Pilani Campus Jhunjhunu, Rajasthan, India - 333031.*

Abstract

In this article, we derive a generalised nonlinear Picone's identity for p sub-Laplacian on the Heisenberg group. As an application of Picone's identity, we prove a Hardy type inequality and Picone's inequality. We also establish some qualitative results involving the system of nonlinear equations involving p-sub-Laplacian.

Keywords: Picone's identity; Heisenberg group; Hardy type inequality; Picone's inequality.
2010 MSC: 35R03; 35H20; 26D10.

1. Introduction

It is well known that Picone type identities play an important role in the study of qualitative properties of elliptic partial differential equations. The classical Picone's identity [25] is as follows: If $u \geq 0$ and $v > 0$ are sufficiently smooth functions, then

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0. \tag{1}$$

For some of the applications of this identity, we refer to [1, 2, 3, 22] and the references cited therein. W. Allegretto and Y.X. Huang [4] obtained Picone's identity for p-Laplace equations. Their identity is as follows:

$$|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v = |\nabla u|^p - \nabla(\frac{u^p}{v^{p-1}})|\nabla v|^{p-2}\nabla v. \tag{2}$$

Email address: `gaurav.dwivedi@pilani.bits-pilani.ac.in.` (Gaurav Dwivedi)

Received August 7, 2020; Accepted: March 31, 2021; Online: April 4, 2021.

J. Tyagi [26] generalised (1) and proved the following nonlinear Picone type identity:

$$\alpha|\nabla u|^2 - \frac{|\nabla u|^2}{f'(v)} + \left(\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}}\right)^2 = \alpha|\nabla u|^2 - \nabla\left(\frac{u^2}{f(v)}\right)\nabla v, \tag{3}$$

where $f(y) \neq 0$, $\forall\, 0 \neq y \in \mathbb{R}$ and $\alpha > 0$ is such that $f'(y) \geq \frac{1}{\alpha}$, $\forall\, 0 \neq y \in \mathbb{R}$.

K. Bal [5] established a nonlinear Picone's identity for p-Laplace operators. They showed that

$$|\nabla u|^p - \frac{pu^{p-1}\nabla u|\nabla v|^{p-2}\nabla v}{f(v)} + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2} = |\nabla u|^p - \nabla(\frac{u^p}{f(v)})|\nabla v|^{p-2}\nabla v.$$

where $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$ for all y.

T. Feng [14] further generalised Picone's identity for p-Laplace equations as follows:

$$|\nabla u|^p - \frac{g'(u)|\nabla v|^{p-2}\nabla v.\nabla u}{f(v)} + \frac{g(u)f'(v)|\nabla v|^p}{[f(v)]^2} = |\nabla u|^p - \nabla\left(\frac{g(u)}{f(v)}\right)|\nabla v|^{p-2}\nabla v.$$

where $v > 0$, $u \geq 0$, $g(u)$ and $f(v)$ satisfy that for $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{g(u)f'(v)|\nabla v|^p}{[f(v)]^2} \geq \frac{p}{q}\left[\frac{g'(u)|\nabla v|^{p-1}}{pf(v)}\right]^q,$$

where $g(u), g'(u) > 0$ for $u > 0$; $g(u), g'(u) = 0$ for $u = 0$; $f(v), f'(v) > 0$.

For some interesting Picone type identities and related results in Euclidean domains, we refer to [6, 11, 12, 15, 18, 19, 28].

Research works available for Picone type identities in Heisenberg group are not as exhaustive as it is in the case of Euclidean domain. Niu et al. [24] obtained Picone's identity for p-sub-Laplacian in bounded domains of Heisenberg group. Their identity is as follows:

$$|\nabla_H u|^p + (p-1)\frac{u^p}{v^p}|\nabla_H v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla_H v|^{p-2}\nabla v = |\nabla_H u|^p - \nabla(\frac{u^p}{v^{p-1}})|\nabla_H v|^{p-2}\nabla v. \tag{4}$$

For some further results involving Picone's identity and its applications on the Heisenberg groups, we refer to [16, 17, 20, 21, 27] and references therein. For a nonlinear Picone identity for biharmonic operator on the Heisenberg group, see [13].

Motivated by the above research works, aim of this article is to prove a nonlinear analogue of Picone's identity for p-sub-Laplacian on the Heisenberg group. Our main result is stated below:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{H}^n$ and $u \geq 0, v > 0$ be differentiable functions. Suppose $f, g : \mathbb{R} \to (0, \infty)$ are continuously differentiable functions such that $f(y), f'(y) > 0$ if $y > 0$; $f(0) = 0, f'(0) = 0$ and $g(y) > 0, g'(y) > 0$. We further assume that*

$$\frac{f(u)g'(v)}{g^2(v)} \geq (p-1)\left(\frac{f'(u)}{pg(v)}\right)^{\frac{p}{p-1}}. \tag{5}$$

Let us denote

$$L(u,v) = |\nabla_{\mathbb{H}^n} u|^p - \frac{f'(u)|\nabla_{\mathbb{H}^n} v|^{p-2}\nabla_{\mathbb{H}^n} v.\nabla_{\mathbb{H}^n} u}{g(v)} + \frac{f(u)g'(v)|\nabla_{\mathbb{H}^n} v|^p}{(g(v))^2}.$$
$$R(u,v) = |\nabla_{\mathbb{H}^n} u|^p - \nabla_{\mathbb{H}^n}\left(\frac{f(u)}{g(v)}\right)|\nabla_{\mathbb{H}^n} v|^{p-2}\nabla_{\mathbb{H}^n} v.$$

Then

(i) $L(u,v) = R(u,v) \geq 0$;

(ii) $L(u,v) = 0$ *a.e. in* Ω *if and only if*

$$\nabla_{\mathbb{H}^n}\left(\frac{u}{v}\right) = 0, \tag{6}$$

$$|\nabla_{\mathbb{H}^n} u| = \left(\frac{f'(u)}{pg(v)}\right)^{\frac{1}{p-1}} |\nabla_{\mathbb{H}^n} v|, \tag{7}$$

$$(p-1)\left(\frac{f'(u)}{pg(v)}\right)^{\frac{p}{p-1}} = \frac{f(u)g'(v)}{(g(v))^2}. \tag{8}$$

Remark 1.1. *If we choose* $f(s) = s^p$ *and* $g(s) = s^{p-1}$, *then our result reduces to* (4).

The article is organized as follows: In Section 2, we recall some brief results on the Heisenberg group. Section 3 deals with the proof of Theorem 1.1. In section 4, we discuss some applications of the Theorem 1.1.

2. Preliminaries

In this section, we present some definitions related to the Heisenberg group. The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot)$, is a non-commutative group equipped with the product

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2,\, y_1 + y_2, t_1 + t_2 + 2(\langle y_1, x_2\rangle - \langle x_1,\, y_2\rangle)),$$

where $x_1,\, y_1,\, x_2,\, y_2 \in \mathbb{R}^n$, $t_1,\, t_2\, \in \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$. With this operation $\mathbb{H}^n$ is a Lie group and the Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t},\;\; X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t},\;\; Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t},\;\; i = 1,\, 2, .3,\, \ldots,\, n.$$

X_i, Y_i and T satisfy

$$[X_i, Y_j] = -4\delta_{ij}T,\;\; [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

The norm on $\mathbb{H}^n$ is given by

$$||\xi||_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}.$$

The distance between $\xi = (z,\, t)$ and $\xi' = (z',\, t')$ on $\mathbb{H}^n$ is defined as follows:

$$d(\xi, \xi') = d((z',\, t')^{-1} \,.\, (z,\, t)).$$

The Heisenberg gradient is defined as

$$\nabla_{\mathbb{H}^n} = (X_1,\, X_2,\, \ldots,\, X_n,\, Y_1,\, Y_2,\, \ldots,\, Y_n)$$

and hence the Heisenberg Laplacian is defined as

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} X_i^2 + Y_i^2 = \nabla_{\mathbb{H}} \cdot \nabla_{\mathbb{H}}.$$

The p-sub-Laplacian is defined as

$$\Delta_{\mathbb{H}^n, p} u = \nabla_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n}|^{p-2}\nabla_{\mathbb{H}^n} u).$$

Definition 2.1 ($S^{1,p}(\Omega)$ and $S_0^{1,p}(\Omega)$ Space)**.** *For an open subset $\Omega \subseteq \mathbb{H}^n$ and $1 < p < \infty$, we define*

$$S^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ such that } u, |\nabla_{\mathbb{H}^n} u| \in L^p(\Omega)\}.$$

The space $S^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

By $S_0^{1,p}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{S_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_{\mathbb{H}^n} u|^p dz dt\right)^{\frac{1}{p}}.$$

For further details on Heisenberg group, see [7, 9].

3. Proof of Theorem 1.1

It is easy to see that

$$\nabla_{\mathbb{H}^n}\left(\frac{f(u)}{g(v)}\right) = \frac{1}{g^2(v)}(g(v)f'(u)\nabla_{\mathbb{H}^n} u - g'(v)f(u)\nabla_{\mathbb{H}^n} v). \tag{9}$$

On using (9), we obtain

$$\begin{aligned}
R(u,v) &= |\nabla_{\mathbb{H}^n} u|^p - \nabla_{\mathbb{H}^n}\left(\frac{f(u)}{g(v)}\right)|\nabla_{\mathbb{H}^n} v|^{p-2}\nabla_{\mathbb{H}^n} v \\
&= |\nabla_{\mathbb{H}^n} u|^p - \frac{f'(u)}{g(v)}|\nabla_{\mathbb{H}^n} v|^{p-2}\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v + \frac{f(u)g'(v)}{g^2(v)}|\nabla_{\mathbb{H}^n} v|^p \\
&= L(u,v).
\end{aligned}$$

Next, we show that $L(u,v) \geq 0$. Let q be conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
L(u,v) &= |\nabla_{\mathbb{H}^n} u|^p - \frac{f'(u)}{g(v)}|\nabla_{\mathbb{H}^n} v|^{p-2}\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v + \frac{f(u)g'(v)}{g^2(v)}|\nabla_{\mathbb{H}^n} v|^p \\
&= \underbrace{p\left(\frac{1}{p}|\nabla_{\mathbb{H}^n} u|^p + \frac{1}{q}\left(\frac{f'(u)|\nabla_{\mathbb{H}^n} v|^{p-1}}{pg(v)}\right)^q\right) - \frac{f'(u)|\nabla_{\mathbb{H}^n} u||\nabla_{\mathbb{H}^n} v|^{p-1}}{g(v)}}_{T_1} \\
&\quad + \underbrace{\frac{f(u)g'(v)|\nabla_{\mathbb{H}^n} v|^p}{g^2(v)} - \frac{p}{q}\left(\frac{f'(u)|\nabla_{\mathbb{H}^n} v|^{p-1}}{pg(v)}\right)^q}_{T_2} \\
&\quad + \underbrace{\frac{f'(u)|\nabla_{\mathbb{H}^n} v|^{p-2}}{g(v)}(|\nabla_{\mathbb{H}^n} u||\nabla_{\mathbb{H}^n} v| - \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v)}_{T_3}.
\end{aligned}$$

Now, we will show that $T_i \geq 0$, $i = 1, 2, 3$. Let us recall Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{10}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Equality in (10) holds if and only if $a^p=b^q$. On choosing $a=|\nabla_{\mathbb{H}^n}u|$ and $b=\dfrac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-1}}{pg(v)}$ in (10), we obtain

$$\frac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-1}|\nabla_{\mathbb{H}^n}u|}{pg(v)}\le\frac{1}{p}|\nabla_{\mathbb{H}^n}u|^p+\frac{1}{q}\left(\frac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-1}}{pg(v)}\right)^q.$$

This shows that $T_1\ge 0$.

(5) shows that $T_2\ge 0$. Since $\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}v\le|\nabla_{\mathbb{H}^n}u||\nabla_{\mathbb{H}^n}v|$, we obtain $T_3\ge 0$. This completes the proof of (i).

It is easy to see that if (6) and (7) are satisfied then $T_2=0$ and $T_3=0$. By the equality case of Young's inequality (10), it is easy to see that $T_1=0$ if (7) is satisfied. Thus $L(u,v)=0$ if (6), (7) and (8) are satisfied.

Finally, we need to show that if $L(u,v)=0$ then (6), (7) and (8) are satisfied. If $L(u,v)=0$, then

$$p\left(\frac{1}{p}|\nabla_{\mathbb{H}^n}u|^p+\frac{1}{q}\left(\frac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-1}}{pg(v)}\right)^q\right)-\frac{f'(u)|\nabla_{\mathbb{H}^n}u||\nabla_{\mathbb{H}^n}v|^{p-1}}{g(v)}=0, \tag{11}$$

$$\frac{f(u)g'(v)|\nabla_{\mathbb{H}^n}v|^p}{g^2(v)}-\frac{p}{q}\left(\frac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-1}}{pg(v)}\right)^q=0 \tag{12}$$

and

$$\frac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-2}}{g(v)}(|\nabla_{\mathbb{H}^n}u||\nabla_{\mathbb{H}^n}v|-\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}v)=0. \tag{13}$$

From (11) and equality case of (10), we obtain

$$|\nabla_{\mathbb{H}^n}u|^p=\left(\frac{f'(u)|\nabla_{\mathbb{H}^n}v|^{p-1}}{pg(v)}\right)^q,$$

which gives (7). It is easy to see that (12) implies (8). If $u(x)\neq 0$, then $u=cv$, for some constant c. This shows that $\nabla_{\mathbb{H}^n}\left(\frac{u}{v}\right)=0$. If $u(x)=0$ for some $x\in\Omega$, then consider the set $N=\{x\in\Omega: u(x)=0\}$ and then $\nabla_{\mathbb{H}^n}u=0$, $f(u)=0$, $f'(u)=0$ in N. Thus (6) holds. This proves (ii). □

4. Applications of Theorem 1.1

Theorem 4.1. *Let $0<v\in C^2(\Omega)$ be such that*

$$-\Delta_{\mathbb{H}^n,p}v\ge\lambda h(x)g(v) \text{ in } \Omega,$$

where $h\in L^\infty(\Omega)$ is a nonnegative weight function. Let $0\le u\in S_0^{1,p}(\Omega)$ and $f(u)\in S_0^{1,p}(\Omega)$. Further, if f and g satisfy conditions of Theorem 1.1, we have

$$\int_\Omega|\nabla_{\mathbb{H}^n}u|^p dx\ge\lambda\int_\Omega h(x)f(u)dx. \tag{14}$$

Proof. Let K be a compact subset of Ω and $0\le\phi\in C_0^\infty(\Omega)$. By Theorem 1.1,

$$\begin{aligned}0&\le\int_K L(\phi,v)dx\le\int_\Omega L(\phi,v)dx=\int_\Omega R(\phi,v)dx\\&=\int_\Omega|\nabla_{\mathbb{H}^n}\phi|^p dx-\int\nabla_{\mathbb{H}^n}\left(\frac{f(\phi)}{g(v)}\right)|\nabla_{\mathbb{H}^n}v|^{p-2}\nabla_{\mathbb{H}^n}v dx\\&=\int_\Omega|\nabla_{\mathbb{H}^n}\phi|^p dx+\int_\Omega\frac{f(\phi)}{g(v)}\Delta_{\mathbb{H}^n,p}v dx\\&\le\int_\Omega|\nabla_{\mathbb{H}^n}\phi|^p dx-\lambda\int_\Omega h(x)f(\phi)dx.\end{aligned}$$

As ϕ tends to u, we obtain (14). □

Remark 4.1. *On choosing $f(u) = u^p$ and $g(v) = v^{p-1}$, we obtain Hardy type inequality proved by Niu et al. [23, Theorem 2.1].*

Theorem 4.2. *Suppose that $h_1(x)$ and $h_2(x)$ are continuous functions such that $h_1(x) < h_2(x)$ on $\Omega \subset \mathbb{R}^n$. If f and g satisfy conditions of Theorem 1.1 and there exists $u \in C^2(\Omega)$ such that*

$$\begin{aligned} -\Delta_{\mathbb{H}^n,p} u &= \frac{h_1(x) f(u)}{u} \text{ in } \Omega, \\ u > 0, g(u) &> 0 \text{ in } \Omega, \\ u = 0 &= g(u) \text{ on } \partial\Omega. \end{aligned} \tag{15}$$

Then any nontrivial solution v of

$$-\Delta_{\mathbb{H}^n,p} v = h_2(x) g(v) \text{ in } \Omega \tag{16}$$

changes sign.

Proof. Assume that v does not change sign, then

$$\begin{aligned} 0 &\leq \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \int_\Omega |\nabla_{\mathbb{H}^n} u|^p dx - \int_\Omega \nabla_{\mathbb{H}^n}\left(\frac{f(u)}{g(v)}\right) |\nabla_{\mathbb{H}^n} v|^{p-2} \nabla_{\mathbb{H}^n} v dx \\ &= \int_\Omega |\nabla_{\mathbb{H}^n} u|^p dx + \int_\Omega \frac{f(u)}{g(v)} \Delta_{\mathbb{H}^n,p} v dx \\ &= \int_\Omega (h_1(x) - h_2(x)) f(u) dx \\ &< 0, \end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 4.3. *Let f and g satisfy conditions of Theorem 1.1 and $(u,v) \in C^2(\Omega) \times C^2(\Omega)$ be a positive solution to the system*

$$\begin{aligned} -\Delta_{\mathbb{H}^n,p} u &= g(v) \text{ in } \Omega, \\ -\Delta_{\mathbb{H}^n,p} v &= \frac{(g(v))^2 u}{f(u)} \text{ in } \Omega, \\ u > 0, v > 0, g(u), f(v) &> 0 \text{ in } \Omega, \\ u = 0 &= g(u) \text{ on } \partial\Omega, \end{aligned} \tag{17}$$

then $|\nabla_{\mathbb{H}^n} u| = \left(\frac{f'(u)}{pg(v)}\right)^{1/p-1} |\nabla_{\mathbb{H}^n} v|$.

Proof. For any $\phi_1, \phi_2 \in S_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^{p-2} \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} \phi_1 dx = \int_\Omega g(v) \phi_1 dx, \tag{18}$$

$$\int_\Omega |\nabla_{\mathbb{H}^n} v|^{p-2} \nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} \phi_2 dx = \int_\Omega \frac{(g(v))^2 u}{f(u)} \phi_2 dx. \tag{19}$$

On choosing $\phi_1 = u$, $\phi_2 = \frac{f(u)}{g(v)}$, we get

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^p dx = \int_\Omega g(v) u dx \tag{20}$$

$$\int_\Omega |\nabla_{\mathbb{H}^n} v|^{p-2} \nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} \left(\frac{f(u)}{g(v)}\right) dx = \int_\Omega u g(v) dx. \tag{21}$$

On using (20) and (21), we get

$$\begin{aligned}\int_\Omega |\nabla_{\mathbb{H}^n} v|^{p-2} \nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} \left(\frac{f(u)}{g(v)}\right) dx &= \int_\Omega u g(v) dx \\ &= \int_\Omega |\nabla_{\mathbb{H}^n} u|^p dx,\end{aligned}$$

which gives

$$\int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx = 0.$$

On applying Theorem 1.1, we get $|\nabla_{\mathbb{H}^n} u| = \left(\frac{f'(u)}{pg(v)}\right)^{1/p-1} |\nabla_{\mathbb{H}^n} v|$ a.e. in Ω. □

Next, we prove a generalised Picone type inequality in the spirit of [10].

Theorem 4.4. *Let Ω be a bounded domain in $\mathbb{H}^n$ and f, g satisfy the conditions in Theorem 1.1. Let $0 \le u \in S_0^{1,p}(\Omega)$, and $0 \le v \in S_0^{1,p}(\Omega)$ be such that $-\Delta_{\mathbb{H}^n} v \ge 0$ is a bounded Radon measure. We further assume that $v \not\equiv 0$ in Ω and $v = 0$ on $\partial\Omega$. Then*

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^p dx \ge \int_\Omega \frac{f(u)}{g(v)} (-\Delta_{\mathbb{H}^n} v) dx. \tag{22}$$

Proof. Since $v \ge 0$ and $v = 0$ on $\partial\Omega$, therefore by strong maximum principle [8] either $v > 0$ or $v \equiv 0$ in Ω. Since $v \not\equiv 0$ in Ω, $v > 0$ in Ω. Let $v_m(\xi) = v(\xi) + \frac{1}{m}$, then $-\Delta_{\mathbb{H}^n} v_m = -\Delta_{\mathbb{H}^n} v$ and $v_m \to v$ in $S^{1,p}(\Omega)$ and almost everywhere. Now, we consider $0 \le u \in S_0^{1,p}(\Omega)$, then there exists a sequence $\{u_n\}$ in $C_0^\infty(\Omega)$ such that $u_n \ge 0$ for each n and $u_n \to u$ in $S_0^{1,p}(\Omega)$. By using Theorem 1.1, we obtain

$$\int_\Omega |\nabla_{\mathbb{H}^n} u_n|^p dx \ge \int_\Omega \frac{f(u_n)}{g(v_m) + \frac{1}{m}} (-\Delta_{\mathbb{H}^n} v_m) dx. \tag{23}$$

Fatou's lemma and Lebesgue dominated convergence theorem implies that as $n, m \to \infty$, we obtain

$$\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx \ge \int_\Omega \frac{f(u)}{g(v)} (-\Delta_{\mathbb{H}^n} v) dx. \tag{24}$$

This completes the proof. □

Remark 4.2. *Theorem 4.4 reduces to the classical Picone's inequality for p-sub-Laplacian on the Heisenberg in case of $f(u) = u^2$ and $g(v) = v^{p-1}$. See [23, Corollary 3.1] for further details.*

Acknowledgement

The author is supported by Science and Engineering Research Board, India under the grant MTR/2018/000233.

References

[1] W. Allegretto, Positive solutions and spectral properties of weakly coupled elliptic systems, J. Math. Anal. Appl. 120(2) (1986) 723–729.
[2] W. Allegretto, On the principal eigenvalues of indefinite elliptic problems, Math. Z. 195(1) (1987) 29–35.
[3] W. Allegretto, Sturmian theorems for second order systems, Proc. Amer. Math. Soc. 94(2) (1985) 291–296.
[4] W. Allegretto, Y.X.Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal. 32(7) (1998) 819–830.
[5] K.Bal, Generalized Picone's identity and its applications, Electron. J. Diff. Equations 243 (2013) 1–6.

[6] G. Bognár, O. Došlý, Picone-type identity for pseudo p-Laplacian with variable power, Electron. J. Diff. Equations 174 (2012) 1–8.

[7] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians, Springer Science & Business Media, 2007.

[8] J.M. Bony, Principe du maximum, inégalité de Harnack et unicité du probleme de Cauchy pour les opÃĺrateurs elliptiques dégénérés, Ann. Inst. Fourier 19(1) (1969) 277–304.

[9] L. Capogna, D. Danielli, S. D. Pauls, J. Tyson, An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem, Springer Science & Business Media, 2007.

[10] J. Dou, Picone inequalities for p-sublaplacian on the Heisenberg group and its applications, Commun. Contem. Math. 12(2) (2010) 295–307.

[11] G. Dwivedi, J. Tyagi, Remarks on the qualitative questions for biharmonic operators, Taiwan. J. Math. 19(6) (2015) 1743–1758.

[12] G.Dwivedi, J.Tyagi, A note on the Caccioppoli inequality for biharmonic operators, Mediterr. J. Math. 13(4) (2016) 1823–1828.

[13] G. Dwivedi, J. Tyagi, Picone's identity for biharmonic operators on Heisenberg group and its applications, Nonlinear Differential Equations Appl. NoDEA 23(2) (2016).

[14] T. Feng, A new nonlinear Picone identity and applications, Math. Appl. 30(2) (2017) 278–283.

[15] T. Feng, A generalized nonlinear Picone identity for the p-biharmonic operator and its applications, J. Inequal. Appl. 2019 (2019) 56.

[16] Y. Han, P. Niu, Some Hardy type inequalities in the Heisenberg group, JIPAM. J. Inequal. Pure Appl. Math 4 (2003) 1–5.

[17] J. Han, P. Niu, W. Qin, Hardy inequalities in half spaces of the Heisenberg group, Bull. Korean Math. Soc. 45(3) (2008) 405–417.

[18] J. Jaroš, The higher-order Picone identity and comparison of half-linear differential equations of even order, Nonlinear Anal. 74(18) (2011) 7513–7518.

[19] J. Jaroš, Caccioppoli estimates through an anisotropic PiconeâĂŹs identity, Proc. Amer. Math. Soc. 143(3) (2015) 1137–1144.

[20] B.S. Lian, Q.H. Yang, F. Yang, Some weighted Hardy-type inequalities on anisotropic Heisenberg groups, J. Inequal. Appl. 2011 (2011) 924840.

[21] H.X. Liu, J.W. Luan, Hardy-type inequalities on a half-space in the Heisenberg group, J. Inequal. Appl. 2013 (2013) 291.

[22] A. Manes, A.M. Micheletti, Un'estensione della teoria variazionale classica degli autovalori per operatoriellittici del secondo ordine, Boll. Unione Mat. Ital. 7 (1973) 285–301.

[23] P. Niu, H. Zhang, Y. Wang, Hardy type and Rellich type inequalities on the Heisenberg group, Proc. Amer. Math. Soc. 129(12) (2001) 3623–3630.

[24] P. Niu, H. Zhang, X. Luo, Hardy's inequalities and PohozaevâĂŹs identities on the Heisenberg group, Acta Math. Sinica. 46(2) (2003) 279–290.

[25] M. Picone, Un teorema sulle soluzioni delle equazioni lineari ellittiche autoaggiunte alle derivate parziali del secondo-ordine, Atti Accad. Naz. Lincei Rend. 20 (1911) 213–219.

[26] J.Tyagi, A nonlinear Picone's identity and applications, Appl. Math. Lett. 26 (2013) 624–626.

[27] Y. Xiao, Hardy inequalities with Aharonov-Bohm type magnetic field on the Heisenberg group, J. Inequal. Appl. 2015 (2015) 95.

[28] N. Yoshida, Picone identities for half-linear elliptic operators with p(x)-Laplacians and applications to Sturmian comparison theory, Nonlinear Anal. 74(16) (2011) 5631–5642.