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Research Article

On the Van Der Laan Hybrid Sequence

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Abstract
In this paper the Van Der Laan hybrid sequence is introduced. Binet-like-formula, partial sum and generating function related to this sequence are obtained. Some interesting properties of Van Der Laan hybrid sequence are given. Finally, the eigenvalues and determinant of a circulant matrix involving Van Der Laan hybrid sequence are represented.

Keywords: Van Der Laan sequence, hybrid numbers, partial sum, generating function, determinant.

Van Der Laan Hibrit Dizileri Üzerine

Öz

Bu makalede Van Der Laan hibrit dizisi tanıtıldı. Bu dizi ile ilişkili Binet benzeri formül, kısmi toplam ve üreteç fonksiyonu elde edildi. Van Der Laan hibrit dizisinin bazı ilginç özellikleri verildi. Son olarak, Van Der Laan hibrit dizisini içeren bir sirkülant matrisin öz değerleri ve determinantı sunuldu.

Anahtar Kelimeler: Van Der Laan dizisi, hibrit sayılar, kısmi toplam, üreteç fonksiyonu, determinant.

1. Introduction

Many authors studied special recursion sequences such as Pell sequence, Pell Lucas sequence, Padovan and Perrin sequences, Jacobsthal sequence. They established new results about these sequences. Özdemir (2018) introduced the hybrid numbers as a generalization of complex hyperbolic and dual numbers. The set H of hybrid numbers Z is of the form

$$Z = a + bi + c\epsilon + dh,$$

Where $a, b, c \in \mathbb{R}$ and i, ϵ, h are operators such that

$$i^2 = -1 , \epsilon^2 = 0, ih = -hi = \epsilon + i.$$

For more results about the hybrid number, we refer to (Özdemir, 2018). The conjugate of hybrid number Z is defined by

$$\overline{Z} = \overline{a + bi + c\epsilon + dh} = a - bi - c\epsilon - dh .$$

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The real number $R(Z) = Z\overline{Z} = \overline{Z}Z = a^2 + b^2 - 2bc - d^2$ is called the character of the hybrid number Z. Liana and Wloch (2019) introduced the Jacobsthal and Jacobsthal Lucas hybrid numbers and investigated some of their properties. In this paper we introduce the Van Der Laan hybrid sequence. We obtain Binet-like formula of this sequence. Then we represent the partial sum and generating function of this sequence. We study some properties of this sequence. Finally we find the eigenvalues and determinant of particular circulant matrix involving van Der Laan hybrid sequence.

For more information about Van der Laan sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal-Lucas sequence, Padovan and Perrin sequences and related number sequences we refer to (Godase and Dhakne, 2014; He et al., 2018; Coskun and Taskara, 2018; Kaygisiz and Sahin, 2011;Morales, 2019; Petroudi and Pirouz, 2015; Petroudi and Pirouz, 2016; Petroudi, Pirouz and Ozkoc, 2020; Pirouz and Ozkoc, 2021; Shanon et al., 2006; Yilmaz and Bozkurt, 2011)

2. Van Der Laan sequence

The Van Der Laan (Coskun and Taskara, 2018) sequence $(V_n)$is defined by the recursion relation

$$V_n = V_{n-2} + V_{n-3} \text{ for all } n \geq 3,$$

With initial values $V_0 = 0,\ V_1 = 1, V_2 = 0$. The first values of (V_n) are

$$0,1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49$$

Also the Binet-like-formula (Coskun and Taskara, 2018) for the Van Der Laan sequences is:

$$V_n = \frac{{r_1}^n}{(r_1-r_2)(r_1-r_3)} + \frac{{r_2}^n}{(r_2-r_1)(r_2-r_3)} + \frac{{r_3}^n}{(r_1-r_3)(r_2-r_3)},$$

Where r_1, r_2, r_3 are the roots of the equation $x^3 - x - 1 = 0$. Also from (Faisant, 2019) we have

$$r_1 + r_2 + r_3 = 0, r_1r_2r_3 = 1, r_1r_2 + r_2r_3 + r_1r_3 = -1.$$

3. Van Der Laan hybrid sequences

Definition (3.1). We define the Van Der Laan hybrid sequence (VH_n) by the relation

$$VH_n = V_n + V_{n+1}i + V_{n+2}\epsilon + V_{n+3}h, \tag{3.1}$$

Where (V_n) is the Van Der Laan sequence. From the above definition, the first few values of Van Der Laan hybrid sequence are:

$$VH_0 = i + h,$$

$$VH_1 = 1 + \epsilon + h,$$

$$VH_2 = i + \epsilon + h,$$

$$VH_3 = 1 + i + \epsilon + 2h,$$

$$VH_4 = 1 + i + 2\epsilon + 2h.$$

Theorem (3.2). Let $n \geq 0$ be an integer. Then

$R(VH_n) = -2V_{n+1}(V_{n+2} + V_n).$

Proof. Let $V_n = V_{n-2} + V_{n-3}$. By definition we have

$R(VH_n) = {V_n}^2 + {V_{n+1}}^2 - 2V_{n+1}V_{n+2} - {V_{n+3}}^2.$

Therefore we get

$R(VH_n) = {V_n}^2 + {V_{n+1}}^2 - 2V_{n+1}V_{n+2} - (V_n + V_{n+1})^2 = {V_n}^2 + {V_{n+1}}^2 - 2V_{n+1}V_{n+2} - {V_n}^2 - {V_{n+1}}^2 - 2V_nV_{n+1} = -2V_{n+1}(V_{n+2} + V_n).$

Hence the proof is completed.

Theorem (3.3). Let $n \geq 0$ be an integer. Then

$VH_n =$

$$\frac{(1 + r_1 i + {r_1}^2\epsilon + {r_1}^3 h)}{(r_1 - r_2)(r_1 - r_3)}{r_1}^n + \frac{(1 + r_2 i + {r_2}^2\epsilon + {r_2}^3 h)}{(r_2 - r_1)(r_2 - r_3)}{r_2}^n + \frac{(1 + r_3 i + {r_3}^2\epsilon + {r_3}^3 h)}{(r_1 - r_3)(r_2 - r_3)}{r_3}^n.$$

Proof. From Binet-like-formula of Van Der Laan sequence we have

$$V_n = \frac{{r_1}^n}{(r_1 - r_2)(r_1 - r_3)} + \frac{{r_2}^n}{(r_2 - r_1)(r_2 - r_3)} + \frac{{r_3}^n}{(r_1 - r_3)(r_2 - r_3)}$$

In hence by taking

$$\alpha = \frac{1}{(r_1 - r_2)(r_1 - r_3)}, \beta = \frac{1}{(r_2 - r_1)(r_2 - r_3)}, \gamma = \frac{1}{(r_1 - r_3)(r_2 - r_3)},$$

we get

$$V_n = \alpha {r_1}^n + \beta {r_2}^n + \gamma {r_3}^n.$$

According to the definition of Van Der Laan hybrid sequence (VH_n) we have

$$VH_n = V_n + V_{n+1}i + V_{n+2}\epsilon + V_{n+3}h.$$

Thus we have

$VH_n = (\alpha {r_1}^n + \beta {r_2}^n + \gamma {r_3}^n) + (\alpha {r_1}^{n+1} + \beta {r_2}^{n+1} + \gamma {r_3}^{n+1})i + (\alpha {r_1}^{n+2} + \beta {r_2}^{n+2} + \gamma {r_3}^{n+2})\epsilon + (\alpha {r_1}^{n+3} + \beta {r_2}^{n+3} + \gamma {r_3}^{n+3})h$.

Consequently by some calculations we have

$$VH_n = \frac{(1 + r_1 i + {r_1}^2\epsilon + {r_1}^3 h)}{(r_1 - r_2)(r_1 - r_3)}{r_1}^n + \frac{(1 + r_2 i + {r_2}^2\epsilon + {r_2}^3 h)}{(r_2 - r_1)(r_2 - r_3)}{r_2}^n + \frac{(1 + r_3 i + {r_3}^2\epsilon + {r_3}^3 h)}{(r_1 - r_3)(r_2 - r_3)}{r_3}^n.$$

Lemma (3.4). (Shannon et al., 2006) Let $n \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} V_k = V_{n+5} - 1.$$

Theorem (3.5). Let $n \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} VH_k = VH_{n+5} - (1 + i + 2\epsilon + 2h).$$

Proof. As we know $\sum_{k=0}^{n} VH_k = VH_0 + VH_1 + VH_2 + \cdots + VH_n$. Thus we obtain

$$\sum_{k=0}^{n} VH_k = (V_0 + V_1 i + V_2\epsilon + V_3 h) + (V_1 + V_2 i + V_3\epsilon + V_4 h) + \cdots + (V_n + V_{n+1} i + V_{n+2}\epsilon + V_{n+3} h) = (V_0 + V_1 + V_2 + \cdots + V_n) + (V_1 + V_2 + \cdots + V_{n+1} + V_0 - V_0)i$$

$$+(V_2 + V_3 + \cdots + V_{n+2} + V_0 + V_1 - V_0 - V_1)\epsilon + (V_3 + V_4 + \cdots + V_{n+3} + V_0 + V_1 + V_2 - V_0 - V_1 - V_2)h$$

$=\sum_{k=0}^{n} V_k + (\sum_{k=0}^{n+1} V_k - 0)i + (\sum_{k=0}^{n+2} V_k - 1)\epsilon + (\sum_{k=0}^{n+3} V_k - 1)h$

Therefore by lemma (3.4) we get

$$\sum_{k=0}^{n} VH_k = (V_{n+5} - 1) + (V_{n+6} - 1)i + (V_{n+7} - 1 - 1)\epsilon + (V_{n+8} - 1 - 1)h$$

$$= V_{n+5} + V_{n+6} i + V_{n+7}\epsilon + V_{n+8} h - (1 + i + 2\epsilon + 2h) = VH_{n+5} - (1 + i + 2\epsilon + 2h).$$

Thus the proof is completed.

Theorem (3.6). Let $n \geq 0$ be an integer. Then

(a) $VH_{n+1} + VH_n = \alpha {r_1}^n(1 + r_1 i + {r_1}^2\epsilon + {r_1}^3 h)(1 + r_1) + \beta {r_2}^n(1 + r_2 i + {r_2}^2\epsilon + {r_2}^3 h)(1 + r_2) + \gamma {r_3}^n(1 + r_3 i + {r_3}^2\epsilon + {r_3}^3 h)(1 + r_3)$,

(b) $VH_{n+1} - VH_n = \alpha r_1{}^n(1 + r_1 i + r_1{}^2\epsilon + r_1{}^3 h)(1 - r_1) + \beta r_2{}^n(1 + r_2 i + r_2{}^2\epsilon + r_2{}^3 h)(1 - r_2) + \gamma r_3{}^n(1 + r_3 i + r_3{}^2\epsilon + r_3{}^3 h)(1 - r_3)$,

where

$$\alpha = \frac{1}{(r_1 - r_2)(r_1 - r_3)}, \beta = \frac{1}{(r_2 - r_1)(r_2 - r_3)}, \gamma = \frac{1}{(r_1 - r_3)(r_2 - r_3)}.$$

Proof. They can be proved by some calculations according to the theorem (3.3).

Lemma (3.7). Let $n \geq 0$ be an integer. Then

$VH_n - VH_{n-2} - VH_{n-3} = 0$,

Proof. By definition of Van Der Laan hybrid sequence we have

$VH_n - VH_{n-2} - VH_{n-3} = (V_n + V_{n+1}i + V_{n+2}\epsilon + V_{n+3}h) - (V_{n-2} + V_{n-1}i + V_n\epsilon + V_{n+1}h) - (V_{n-3} + V_{n-2}i + V_{n-1}\epsilon + V_n h) = (V_n - V_{n-2} - V_{n-3}) + (V_{n+1} - V_{n-1} - V_{n-2})i + (V_{n+2} - V_n - V_{n-1})\epsilon + (V_{n+3} - V_{n+1} - V_n)h = 0,$

As (V_n) is a Van Der Laan sequence hence we conclude that

$$VH_n - VH_{n-2} - VH_{n-3} = 0.$$

Theorem (3.8). The generating function for Van Der Laan hybrid sequence (VH_n) is

$$\sum_{n=0}^{\infty} VH_n x^n = \frac{VH_0 + VH_1 x + \epsilon x^2}{1 - x^2 - x^3}.$$

Proof. Suppose that the generating function of the Van Der Laan hybrid sequence (VH_n) has the form

$$f(x) = \sum_{n=0}^{\infty} VH_n x^n = VH_0 + VH_1 x + VH_2 x^2 + VH_3 x^3 + \cdots \quad .$$

Then we have

$$x^2 f(x) = VH_0 x^2 + VH_1 x^3 + VH_2 x^4 + VH_3 x^5 + \cdots \quad ,$$

and

$$x^3 f(x) = VH_0 x^3 + VH_1 x^4 + VH_2 x^5 + VH_3 x^6 + \cdots \, .$$

Thus we obtain

$f(x) - x^2 f(x) - x^3 f(x) =(VH_0 + VH_1x + VH_2x^2 + VH_3x^3 + \cdots) - (VH_0x^2 + VH_1x^3 + VH_2x^4 + VH_3x^5 + \cdots \quad) - (VH_0x^3 + VH_1x^4 + VH_2x^5 + VH_3x^6 + \cdots)=(VH_0 + VH_1x) + (VH_2 - VH_0)x^2 + (VH_3 - VH_1 - VH_0)x^2 + \cdots + (VH_n - VH_{n-2} - VH_{n-3})x^n + \cdots.$

By lemma (3.7) we have $VH_n - VH_{n-2} - VH_{n-3} = 0$. So we obtain

$f(x) - x^2 f(x) - x^3 f(x) = VH_0 + VH_1x + (VH_2 - VH_0)x^2.$

Thus we get

$f(x)(1 - x^2 - x^3) = VH_0 + VH_1x + (i + \epsilon + h - (i + h))x^2.$

Consequently we have

$$\sum_{n=0}^{\infty} VH_n x^n = \frac{VH_0 + VH_1x + \epsilon x^2}{1 - x^2 - x^3}.$$

Definition (3.9). Let $n \geq 0$ be an integer. Morales (2019) has defined sequences $Q_n = P_n + P_{-n}$ and $\acute{Q}_n = P_n - P_{-n}$ and gained interesting result about Padovan sequences using these sequences, where $P_{-n} = -P_{-(n-1)} + P_{-(n-3)}$.

According to these definitions we have the following table for Padovan numbers P_n , P_{-n}:

n	1	2	3	4	5	6	7	8	9	10
P_n	1	1	2	2	3	4	5	7	9	12
P_{-n}	0	1	0	0	1	-1	1	0	-1	2

By setting $V_{-n} = -V_{-(n-1)} + V_{-(n-3)}$ we can find the following table for the Van Der Laan sequence(V_n):

n	1	2	3	4	5	6	7	8	9	10
V_n	1	0	1	1	1	2	2	3	4	5
V_{-n}	0	1	-1	1	0	-1	2	-2	1	1

For example for $\mathrm{n} = 1$ we have

$V_{-1} = -V_{-(1-1)} + V_{-(1-3)} = -V_0 + V_2 = 0 + 0 = 0.$

Now we define the new sequences KH_n, $\acute{K}H_n$.

Definition (3.10). Let $n \geq 0$ be an integer. We define

$KH_n = VH_n + VH_{-n}$, $\acute{K}H_n = VH_n - VH_{-n}$

Where

$VH_{-n} = V_{-n} + V_{-(n+1)}i + V_{-(n+2)}\epsilon + V_{-(n+3)}h$.

According to these new sequences we have the following table about the first terms of sequences KH_n, $\acute{K}H_n$:

n	1	2	3	4	5
VH_n	1+ϵ+h	i+ϵ+h	1+i+ϵ+2h	1+i+2ϵ+2h	1+2i+2ϵ+3h
VH_{-n}	i- ϵ+h	1-i+ ϵ	-1+i-h	1- ϵ +2h	-i+2ϵ-2h
KH_n	1+i+2h	1+2ϵ+h	2i+ϵ+h	2+i+ϵ+4h	1+i+4ϵ+h
$\acute{K}H_n$	1-i+2ϵ	-1+2i+h	2+ϵ+3h	i+3ϵ	1+3i+5h

For example for $n = 2$ we have

$$VH_{-2} = V_{-2} + V_{-3}i + V_{-4}\epsilon + V_{-5}h = 1 - 1\mathrm{i} + 1\epsilon + 0\mathrm{h} = 1 - \mathrm{i} + \epsilon.$$

4. Circulant matrix involving Van Der Laan hybrid sequence

Definition (4.1). A matrix $C = [c_{i,j}] \in M_{n\times n}$ is called a Circulant matrix if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

A circulant matrix $C = [c_{i,j}]$ can be shown by $C = Circ(c_0, c_1, \cdots, c_{n-1})$.

Lemma (4.2). (He et al., 2018) Let $C = Circ(c_0, c_1, \cdots, c_{n-1})$ be an $n \times n$ circulant matrix. Then we have

$$\rho_j(C) = \sum_{k=0}^{n-1} c_k w^{-jk},$$

Where ρ_j for $j = 0,1,2,\cdots,n-1$ are the eigenvalues of the circulant matrix C and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$.

Lemma (4.3). Suppose that $\alpha = \frac{1}{(r_1-r_2)(r_1-r_3)}$, $= \frac{1}{(r_2-r_1)(r_2-r_3)}$, $\gamma = \frac{1}{(r_1-r_3)(r_2-r_3)}$, where r_1, r_2, r_3 are the roots of the equation $x^3 - x - 1 = 0$. Then

(a) $\alpha + \beta + \gamma = 0,$

(b) $\alpha r_1 + \beta r_2 + \gamma r_3 = 0,$

(c) $\alpha r_2 r_3 + \beta r_1 r_3 + \gamma r_1 r_2 = 0.$

Proof. They can be proved by direct calculation according to the definition of α, β, γ and following relations:

$r_1 + r_2 + r_3 = 0, \quad r_1 r_2 r_3 = 1, \qquad r_1 r_2 + r_2 r_3 + r_1 r_3 = -1.$

Theorem (4.5). Let $C = Cir(VH_0, VH_1, \cdots, VH_{n-1})$ be a $n \times n$ circulant matrix whose entries are the Van Der Laan hybrid sequence (VH_n). Then the eigenvalues of C are

$$\rho_j = \frac{-\left(VH_0 + VH_1 w^{-j}\right) + v w^{-2j} + \ VH_n + VH_{n+1} w^{-j} + VH_{n-1} w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where $v = (l_1 r_2 r_3 + l_2 r_3 r_1 + l_3 r_1 r_2)$. (for $j = 0,1,2,\cdots,n-1$)

Proof. By lemma (4.2) for the eigenvalues of circulant matrix $C = Circ(c_0, c_1, \cdots, c_{n-1})$ we have

$$\rho_j(C) = \sum_{k=0}^{n-1} c_k w^{-jk}.$$

Hence for the circulant matrix $C = Cir(VH_0, VH_1, \cdots, VH_{n-1})$ we have

$$\begin{aligned}\rho_j(C) = \sum_{k=0}^{n-1} VH_k w^{-jk} \\ &= \sum_{k=0}^{n-1} \left[\frac{(1 + r_1 i + {r_1}^2 \epsilon + {r_1}^3 h)}{(r_1 - r_2)(r_1 - r_3)} {r_1}^n + \frac{(1 + r_2 i + {r_2}^2 \epsilon + {r_2}^3 h)}{(r_2 - r_1)(r_2 - r_3)} {r_2}^n \right. \\ &\left. + \frac{(1 + r_3 i + {r_3}^2 \epsilon + {r_3}^3 h)}{(r_1 - r_3)(r_2 - r_3)} {r_3}^n \right] w^{-jk} = \sum_{k=0}^{n-1} [l_1 {r_1}^n + l_2 {r_2}^n + l_3 {r_3}^n] w^{-jk},\end{aligned}$$

where $l_1 = \frac{(1+r_1 i + {r_1}^2 \epsilon + {r_1}^3 h)}{(r_1-r_2)(r_1-r_3)}$, $l_2 = \frac{(1+r_2 i + {r_2}^2 \epsilon + {r_2}^3 h)}{(r_2-r_1)(r_2-r_3)}$, $l_3 = \frac{(1+r_3 i + {r_3}^2 \epsilon + {r_3}^3 h)}{(r_1-r_3)(r_2-r_3)}$.

Therefore we have

$$\rho_j = l_1\left(\frac{\left(r_1w^{-j}\right)^n - 1}{r_1w^{-j} - 1}\right) + l_2\left(\frac{\left(r_2w^{-j}\right)^n - 1}{r_2w^{-j} - 1}\right) + l_3\left(\frac{\left(r_3w^{-j}\right)^n - 1}{r_3w^{-j} - 1}\right)$$
$$= l_1\left(\frac{{r_1}^n - 1}{r_1w^{-j} - 1}\right) + l_2\left(\frac{{r_2}^n - 1}{r_2w^{-j} - 1}\right) + l_3\left(\frac{{r_3}^n - 1}{r_3w^{-j} - 1}\right)$$

After some computations we get

$$\rho_j = \frac{-(l_1 + l_2 + l_3) + (l_1{r_1}^n + l_2{r_2}^n + l_3{r_3}^n) - (l_1r_1 + l_2r_2 + l_3r_3)w^{-j}}{w^{-3j} + w^{-2j} - 1}$$
$$+ \frac{(l_1{r_1}^{n+1} + l_2{r_2}^{n+1} + l_3{r_3}^{n+1})w^{-j}}{w^{-3j} + w^{-2j} - 1} + \frac{(l_1r_2r_3 + l_2r_3r_1 + l_3r_1r_2)w^{-2j}}{w^{-3j} + w^{-2j} - 1}$$
$$+ \frac{(l_1{r_1}^{n-1} + l_2{r_2}^{n-1} + l_3{r_3}^{n-1})w^{-2j}}{w^{-3j} + w^{-2j} - 1}.$$

Consequently by lemma (4.3) and the definition of Van Der Laan hybrid sequence we obtain

$$\rho_j = \frac{-\left(VH_0 + VH_1w^{-j}\right) + vw^{-2j} + VH_n + VH_{n+1}w^{-j} + VH_{n-1}w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where $v = (l_1r_2r_3 + l_2r_3r_1 + l_3r_1r_2)$. Thus the proof is completed.

Lemma (4.6). Let x, y, z be real numbers and $n > 0$ be an integer. Then

$\prod_{k=0}^{n-1}(x - yw^{-k} + zw^{-2k}) = x^n\left(1 - \left(\frac{y-\sqrt{y^2-4xz}}{2x}\right)^n - \left(\frac{y+\sqrt{y^2-4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right) = x^n + z^n - \left[\left(\frac{y-\sqrt{y^2-4xz}}{2}\right)^n + \left(\frac{y+\sqrt{y^2-4xz}}{2}\right)^n\right],$

Where $w = e^{\frac{2\pi i}{n}}$.

Proof. See (Coskun and Taskara, 2018).

Lemma (4.7). Let $n > 0$ be an integer. Then

$$\prod_{j=0}^{n-1}\left(w^{-3j} + w^{-2j} - 1\right) = (-1)^n(-Q_{-n} - Q_n).$$

Where (Q_n) is the Perrin sequence that is defined by the recursive relation $Q_{n+3} = Q_{n+1} + Q_n$ with initial values $Q_0 = 3, Q_1 = 0, Q_2 = 2$ and sequence (Q_{-n}) is defined by recursive relation $Q_{-n} = Q_{-(n-1)} + Q_{-(n-3)}$.

Proof. See (Coskun and Taskara, 2018).

Theorem (4.9). Let $C = Cir(VH_0, VH_1, \cdots, VH_{n-1})$ be a $n \times n$ circulant matrix whose entries are the Van Der Laan hybrid sequence(VH_n). Then determinant of C is

$$\begin{aligned}&\det(C)\\&=\Bigg((VH_n - VH_0)^n + (VH_{n-1}+v)^n\\&-\Bigg[\left(\frac{(VH_{n+1}-VH_1)-\sqrt{(VH_{n+1}-VH_1)^2-4(VH_n-VH_0)(VH_{n-1}+v)}}{2}\right)^n\\&+\left(\frac{(VH_{n+1}-VH_1)-\sqrt{(VH_{n+1}-VH_1)^2-4(VH_n-VH_0)(VH_{n-1}+v)}}{2}\right)^n\Bigg]\Bigg)\\&\times\frac{1}{(-1)^n(Q_{-n}-Q_n)}.\end{aligned}$$

Where $v = (l_1r_2r_3 + l_2r_3r_1 + l_3r_1r_2)$.

Proof. Let $\rho_0, \rho_1, \cdots, \rho_{r-1}$ are the eigenvalues of circulant matrix C. From a basic theorem in matrix algebra about the determinant of a matrix we have

$$\det(C) = \prod_{j=0}^{n-1} \rho_j$$

Therefore by theorem (4.5) we get

$$\det(C) = \prod_{j=0}^{n-1} \rho_j = \prod_{j=0}^{n-1} \frac{-(VH_0+VH_1w^{-j})+vw^{-2j}+ VH_n+VH_{n+1}w^{-j}+VH_{n-1}w^{-2j}}{w^{-3j}+w^{-2j}-1} = \prod_{j=0}^{n-1} \frac{(VH_n-VH_0)+ (VH_{n+1}-VH_1)w^{-j}+(VH_{n-1}+v)w^{-2j}}{w^{-3j}+w^{-2j}-1} = \left(\prod_{j=0}^{n-1}(VH_n - VH_0) + (VH_{n+1} - VH_1)w^{-j} + (VH_{n-1}+v)w^{-2j}\right) \times \left(\frac{1}{\prod_{j=0}^{n-1} w^{-3j}+w^{-2j}-1}\right).$$

Therefore by lemma (4.6) and lemma (4.7) we realize that

$$\begin{aligned}&\det(C)\\&=\Bigg((VH_n - VH_0)^n + (VH_{n-1}+v)^n\\&-\Bigg[\left(\frac{(VH_{n+1}-VH_1)-\sqrt{(VH_{n+1}-VH_1)^2-4(VH_n-VH_0)(VH_{n-1}+v)}}{2}\right)^n\\&+\left(\frac{(VH_{n+1}-VH_1)-\sqrt{(VH_{n+1}-VH_1)^2-4(VH_n-VH_0)(VH_{n-1}+v)}}{2}\right)^n\Bigg]\Bigg)\\&\times\frac{1}{(-1)^n(Q_{-n}-Q_n)}.\end{aligned}$$

Thus the proof is completed.

5. Conclusion

In this paper we introduced the Van Der Laan hybrid sequence. We obtained Binet-like formula of this sequence. We represented the partial sum and generating function of this sequence. We investigated some properties of this sequence. Finally we found the eigenvalues and determinant of particular circulant matrix involving Van Der Laan hybrid sequence.

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