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Stability Analysis of A Linear Neutral Differential Equation

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Abstract: The main aim of this study is to give the stability analysis of linear neutral differential equations. The third order linear neutral differential equation is considered as a special case and its characteristic equation is examined for the stability properties. For the investigation of the roots of the third order characteristic equation the Sturm sequence method is used. The conditions obtained for getting positive real roots and the importance of the system parameters on the stability are given in the theorems.

Keywords: Linear neutral differential equation, Stability analysis, Sturm sequence

1 Introduction

Functional differential equations (FDEs) arise in many mathematical models of processes of delay effects, in physics, economics, engineering, and biology. FDEs including the present and past values of the unknown function are called delay differential equations and there are many studies on their solutions and the stability analysis. Moreover if they also include some derivatives to past values, they are called neutral differential equations (NDEs. The theory of neutral differential equations is important in theoretical and practical area in mathematics. For the details in the theory, one can refer to [1]-[2]-[3]. NDEs are one of the most important powerful tools for the modelling of systems depending on the present and the past state of it. There is a number of intriguing papers on applications of NDEs in engineering, biology and mechanics [6]-[7]-[8]-[9]. Since the effect of delay term plays an effective role on the persistence and stability of processes including memory, there are many interesting papers investigating the impact of delay on behaviour of the systems [7]-[8]. Systems becomes unstable even chaotic as delay increases and exceeds the critical delay margin. A detailed analysis of the stability can be made by using the characteristic equation of the neutral differential equations.

The general form of third order neutral differential equation is given below.

$$y^{'''}(t)+p_1y^{'''}(t-\tau)+p_2y^{''}(t)+p_3y^{''}(t-\tau)+p_4y^{'}(t)+p_5y^{'}(t-\tau)+p_6y(t)+p_7y(t-\tau)=0 \quad , \quad t\geq 0 \tag{1}$$

Here p_i , $1\leq i\leq 7$ are real numbers and τ is a positive real number. Some stability conditions are given in [5] by considering a real root for the characteristic equation and reducing its order. As a special case, considering the coefficient $p_1=p_5=p_6=0$, the stability results of a mechanical robotics model with damping and delay are given in [6]. The Pontryagin's theory is used in the analysis.

2 Stability Analysis

In this study, we consider the third order linear neutral differential equation and we make the stability analysis in an algebraic way. The effects of the system parameters on the stability are shown by using the Routh-Hurwitz criterion and the Sturm sequence.

We are looking for the solutions in form $y(t)=e^{\lambda t}$, $t\geq -\tau$.

Taking the derivatives and writing them, we get the following characteristic equation

$$\lambda^3+p_2\lambda^2+p_4\lambda+p_6+e^{-\lambda\tau}\left(p_1\lambda^3+p_3\lambda^2+p_5\lambda+p_7\right)=0. \tag{2}$$

Let us investigate the stability of the system having a characteristic equation (2) with no delay term $(\tau=0)$.

$$(1+p_1)\lambda^3+(p_2+p_3)\lambda^2+(p_4+p_5)\lambda+p_6+p_7=0.$$

Making the leading coefficient 1, we may write the following equation

$$\lambda^3+\left(\frac{p_2+p_3}{1+p_1}\right)\lambda^2+\left(\frac{p_4+p_5}{1+p_1}\right)\lambda+\frac{p_6+p_7}{1+p_1}=0. \tag{3}$$

This equation has roots with negative real parts if and only if

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$$\frac{p_2+p_3}{1+p_1}>0,$$

$$\frac{p_6+p_7}{1+p_1}>0$$

and

$$\left(\frac{p_2+p_3}{1+p_1}\right)\left(\frac{p_4+p_5}{1+p_1}\right)-\left(\frac{p_6+p_7}{1+p_1}\right)>0.$$

Theorem 1. *Consider the system having a characteristic equation (3) with no delay term ($\tau=0$). Hence the characteristic equation has roots with negative real parts if and only if*

$$\frac{p_2+p_3}{1+p_1}>0,$$

$$\frac{p_6+p_7}{1+p_1}>0$$

and

$$\left(\frac{p_2+p_3}{1+p_1}\right)\left(\frac{p_4+p_5}{1+p_1}\right)-\left(\frac{p_6+p_7}{1+p_1}\right)>0.$$

Proof: It is clear according to the Routh-Hurwitz criterion. □

Now we will analyze the stability of the third order neutral differential equation ($\tau\neq 0$) with the following characteristic equation (2)

$$\lambda^3+p_2\lambda^2+p_4\lambda+p_6+e^{-\lambda t}\left(p_1\lambda^3+p_3\lambda^2+p_5\lambda+p_7\right)=0.$$

In the previous theorem, we give some conditions for the case in which all roots of the equation have negative real parts. But here, because of the critical delays, roots may change to having negative real parts to having positive real parts. For the stability analysis in such a case, we consider purely imagiary roots first, as $\lambda=i\sigma$. Then writing it into the equation (2), we get

$$-i\sigma^3-p_2\sigma^2+p_4i\sigma+p_6+\left(\cos(\sigma\tau)-i\sin(\sigma\tau)\right)\left(-ip_1\sigma^3-p_3\sigma^2+ip_5\sigma+p_7\right)=0.$$

Separating the real and imaginary part of this equation we get,

$$-p_2\sigma^2+p_6-p_3\sigma^2\cos(\sigma\tau)+p_7\cos(\sigma\tau)-p_1\sigma^3\sin(\sigma\tau)+p_5\sigma\sin(\sigma\tau)=0$$

and

$$-\sigma^3+p_4\sigma-p_1\sigma^3\cos(\sigma\tau)+p_5\sigma\cos(\sigma\tau)+p_3\sigma^2\sin(\sigma\tau)-p_7\sin(\sigma\tau)=0.$$

To complete the analysis, we need a polynomial getting from the sum of the squares of these two parts,

$$-p_2\sigma^2+p_6=p_3\sigma^2\cos(\sigma\tau)-p_7\cos(\sigma\tau)+p_1\sigma^3\sin(\sigma\tau)-p_5\sigma\sin(\sigma\tau),$$

$$-\sigma^3+p_4\sigma=p_1\sigma^3\cos(\sigma\tau)-p_5\sigma\cos(\sigma\tau)-p_3\sigma^2\sin(\sigma\tau)+p_7\sin(\sigma\tau).$$

Hence, we have

$$\left(1-p_1^2\right)\sigma^6+\left(p_2^2-2p_4-p_3^2+2p_1p_5\right)\sigma^4+\left(p_4^2-2p_2p_6+2p_3p_7-p_5^2\right)\sigma^2+p_6^2-p_7^2=0$$

in the form of a polynomial equation. The roots may be investigated in a simple way since there is no trigonometric terms and some general stability results for $\tau>0$ may be given.

Let $\mu=\sigma^2$ to write the following equation

$$\left(1-p_1^2\right)\mu^3+\left(p_2^2-2p_4-p_3^2+2p_1p_5\right)\mu^2+\left(p_4^2-2p_2p_6+2p_3p_7-p_5^2\right)\mu+p_6^2-p_7^2=0.$$

And writing it with leading coefficient 1, we get

$$\mu^3+\left(\frac{p_2^2-2p_4-p_3^2+2p_1p_5}{1-p_1^2}\right)\mu^2+\left(\frac{p_4^2-2p_2p_6+2p_3p_7-p_5^2}{1-p_1^2}\right)\mu+\frac{p_6^2-p_7^2}{1-p_1^2}=0. \quad (4)$$

We will call the coefficients of this polynomial as

$$A=\frac{p_2^2-2p_4-p_3^2+2p_1p_5}{1-p_1^2}\quad,\quad B=\frac{p_4^2-2p_2p_6+2p_3p_7-p_5^2}{1-p_1^2}\quad,\quad C=\frac{p_6^2-p_7^2}{1-p_1^2} \quad (5)$$

Since the leading coefficient of (4) is 1 and hence positive, a positive real root may occur in two cases [4].

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i) If $C < 0$ then the positive real root occurs.
ii) If $C > 0$ then a negative real root is guaranteed. For the possibility of having two positive real roots, we use the Sturm sequence of the polynomial (4).

Sturm sequence method is used to determine whether a positive real root exists. Considering the functions in Sturm sequence, the sign changes in endpoints of the interval on which we study are determined. The number of sign changes gives the number of real roots.

The details of the method are given in [4]. As a summary of the procedure, let us consider the polynomials

$$f_0 = \mu^3 + A\mu^2 + B\mu + C$$

and

$$f_1 = 3\mu^2 + 2A\mu + B$$

where $f_1 = f_0^{'}$. And then applying the division algorithm as

$$f_0 = q_0 f_1 + f_2$$

$$f_1 = q_1 f_2 + f_3$$

f_2 and f_3 may be found in the following way

$$f_2 = \left(\frac{2}{9}A^2 - \frac{2}{3}B\right)\mu + C - \frac{1}{9}AB$$

$$f_3 = -\frac{9}{4}\,\frac{4B^3 - A^2B^2 - 18ABC + 4CA^3 + 27C^2}{\left(A^2 - 3B\right)^2}.$$

In order to show that there are three real roots for the case (ii), we need three sign changes at endpoint of the interval $(-\infty, \infty)$. Hence we have to get the Table 1 given in [4].

	$-\infty$	∞
f_0	$-$	$+$
f_1	$+$	$+$
f_2	$-$	$+$
f_3	$+$	$+$

Table 1 Sign changes of Sturm functions in endpoints

In order to complete this table, we need the following conditions,

$$A^2 - 3B > 0$$

and

$$4\left(B^2 - 3AC\right)\left(A^2 - 3B\right) - (9C - AB)^2 > 0$$

where the constants A, B, C are determined as in (5).
And for the case (ii), there exists one positive real root if $A < 0$ or $A > 0$ and $B < 0$ [4].
As a conclusion, we give the following theorem based on the theorem in [4].

Theorem 2. *Consider the neutral differential equation (1) with its characteristic equation (2). The equation (2) has positive real root if and only if A, B and C are not all positive and either $C < 0$, or $C > 0$, $A^2 - 3B > 0$ and $4\left(B^2 - 3AC\right)\left(A^2 - 3B\right) - (9C - AB)^2 > 0$ are satisfied where A, B and C are given in (5).*

Proof: In the existence of the conditions given above, there are three real characteristic roots and one of them is positive. □

3 Conclusion

In this work, we study on the stability properties of a third order linear neutral differential equation by using its characteristic equation. We analyze under which conditions, the equation has positive roots. The results obtained by the Sturm sequence are given in the last theorem and hence the effects of the system parameters on the stability are shown.

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