Conference Proceeding Science and Technology, 3(1), 2020, 37–41

Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020).

eg-Radical Supplemented Modules

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Celil Nebiyev[1,*] *Hasan Hüseyin Ökten*[2]

[1] *Department of Mathematics, Ondokuz Mayıs University, 55270, Kurupelit, Atakum, Samsun, Turkey, ORCID:0000-0002-7992-7225*
[2] *Technical Sciences Vocational School, Amasya University, Amasya, Turkey, ORCID:0000-0002-7886-0815*
** Corresponding Author E-mail: cnebiyev@omu.edu.tr*

Abstract: In this work, R will denote an associative ring with unity and all module are unital left $R-$modules. Let M be an $R-$module. If every essential submodule of M has a g-radical supplement in M, then M is called an essential g-radical supplemented (or briefly eg-radical supplemented) module. In this work, some properties of these modules are investigated.

Keywords: Essential Submodules, g-Small Submodules, Generalized Radical, g-Supplemented Modules.

1 Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*)submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*Radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*Radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. The intersection of essential maximal submodules of an R-module M is called a *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM. If M have no essential maximal submodules, then we denote $Rad_gM = M$. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_gV$, then V is called a *g-radical supplement* of U in M. M is said to be *g-radical supplemented* if every submodule of M has a g-radical supplement in M.

More details about supplemented modules are in [1]-[9]. More details about essential supplemented modules are in [5]-[6]. More informations about g-small submodules and g-supplemented modules are in [2]-[3]. The definition of essential g-supplemented modules and some properties of them are in [4]. More details about generalized (Radical) supplemented modules are in [7, 8]. The definition of g-radical supplemented modules and some properties of them are in [3].

Lemma 1. *Let M be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof: See [9, 17.3]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*
(3) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(4) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(5) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

Proof: See [1, 2.2] and [9, 19.3]. □

Lemma 3. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$ and $L \ll_g M$, then $K \ll_g M$ and $L/K \ll_g M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_g M$, then $f(K) \ll_g N$.*
(3) *If $L \leq M$ and $K \ll_g L$, then $K \ll_g M$.*
(4) *If $K_1, K_2, ..., K_n \ll_g M$, then $K_1 + K_2 + ... + K_n \ll_g M$.*
(5) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll_g L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll_g L_1 + L_2 + ... + L_n$.*

Proof: See [2]-[3]. □

Lemma 4. *Let M be an $R-$module. The following assertions are hold.*
(1) $RadM = \sum_{L \ll M} L$.
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(RadM) \leq RadN$. If $Kef \leq RadM$, then $f(RadM) = Radf(M)$.*
(3) *If $N \leq M$, then $RadN \leq RadM$.*
(4) *For $K, L \leq M$, $RadK + RadL \leq Rad(K + L)$.*
(5) *$Rx \ll M$ for every $x \in RadM$.*

Proof: See [9, 21.5 and 21.6]. □

Lemma 5. *Let M be an $R-$module. The following assertions are hold.*
(1) $RadM \leq Rad_g M$.
(2) $Rad_g M = \sum_{L \ll_g M} L$.
(3) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(Rad_g M) \leq Rad_g N$.*
(4) *For $K, L \leq M$, $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K+L}{L}$.*
(5) *If $N \leq M$, then $Rad_g N \leq Rad_g M$.*
(6) *For $K, L \leq M$, $Rad_g K + Rad_g L \leq Rad_g (K + L)$.*
(7) *$Rx \ll_g M$ for every $x \in Rad_g M$.*

Proof: See [2] and [3]. □

2 eg-Radical Supplemented Modules

Definition 1. *Let M be an $R-$module. If every essential submodule of M has a g-radical supplement in M, then M is called an essential g-radical supplemented (or briefly eg-radical supplemented) module.*

Lemma 6. *Let V be a supplement of U in M. Then*
(1) *If $W + V = M$ for some $W \leq U$, then V is a supplement of W in M.*
(2) *If M is finitely generated, then V is also finitely generated.*
(3) *If U is a maximal submodule of M, then V is cyclic and $U \cap V = RadV$ is the unique maximal submodule of V.*
(4) *If $K \ll M$, then V is a supplement of $U + K$ in M.*
(5) *For $K \ll M$, $K \cap V \ll V$ and hence $RadV = V \cap RadM$.*
(6) *For $L \leq U$, $\frac{V+L}{L}$ is a supplement of U/L in M/L.*

Proof: See [9, 41.1]. □

Lemma 7. *Let M be an $R-$module.*
(1) *For $M_1, U \leq M$, if $M_1 + U$ has a supplement in M and M_1 is supplemented, then U also has a supplement in M.*
(2) *Let $M = M_1 + M_2$. If M_1 and M_2 are supplemented, then M is also supplemented.*
(3) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also supplemented.*
(4) *If M is supplemented, then M/L is supplemented for every $L \leq M$.*
(5) *If M is supplemented, then every homomorphic image of M is also supplemented.*
(6) *If M is supplemented, then $M/RadM$ is semisimple.*
(7) *If M is supplemented, then every finitely $M-$generated module is supplemented.*

Proof: See [9, 41.2]. □

Lemma 8. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is Rad-supplemented.*
(2) *If V is a Rad-supplement of U in M and $W + V = M$ for some $W \leq U$, then V is a Rad-supplement of W in M.*
(3) *If U is a maximal submodule of M and V is a Rad-supplement of U in M, $U \cap V = RadV$ is the unique maximal submodule of V.*
(4) *If V is a Rad-supplement of U in M and $L \leq U$, then $\frac{V+L}{L}$ is a Rad-supplement of U/L in M/L.*
(5) *For $M_1, U \leq M$, if $M_1 + U$ has a Rad-supplement in M and M_1 is Rad-supplemented, then U also has a Rad-supplement in M.*
(6) *Let $M = M_1 + M_2$. If M_1 and M_2 are Rad-supplemented, then M is also Rad-supplemented.*
(7) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is Rad-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also Rad-supplemented.*

(8) *If M is Rad-supplemented, then M/L is Rad-supplemented for every $L \leq M$.*
(9) *If M is Rad-supplemented, then every homomorphic image of M is also Rad-supplemented.*
(10) *If M is Rad-supplemented, then $M/RadM$ is semisimple.*
(11) *If M is Rad-supplemented, then every finitely $M-$generated module is Rad-supplemented.*

Proof: See [7] and [8]. □

Lemma 9. *Let M be an $R-$module.*
(1) *If M is Rad-supplemented, then M is g-radical supplemented.*
(2) *If V is a g-radical supplement of U in M and $W + V = M$ for some $W \leq U$, then V is a g-radical supplement of W in M.*
(3) *If U is an essential maximal submodule of M and V is a g-radical supplement of U in M, then $U \cap V = Rad_g V$ is the unique essential maximal submodule of V.*
(4) *If V is a g-radical supplement of U in M and $L \leq U$, $\frac{V+L}{L}$ is a g-radical supplement of U/L in M/L.*
(5) *For $M_1, U \leq M$, if $M_1 + U$ has a g-radical supplement in M and M_1 is g-radical supplemented, then U also has a g-radical supplement in M.*
(6) *Let $M = M_1 + M_2$. If M_1 and M_2 are g-radical supplemented, then M is also g-radical supplemented.*
(7) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is g-radical supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-radical supplemented.*
(8) *If M is g-radical supplemented, then M/L is g-radical supplemented for every $L \leq M$.*
(9) *If M is g-radical supplemented, then every homomorphic image of M is also g-radical supplemented.*
(10) *If M is g-radical supplemented, then $M/Rad_g M$ is semisimple.*
(11) *If M is g-radical supplemented, then every finitely $M-$generated module is g-radical supplemented.*

Proof: See [3]. □

Proposition 1. *Every g-radical supplemented module is eg-radical supplemented.*

Proof: Clear from definitions. □

Proposition 2. *Let M be a g-radical supplemented $R-$module. Then every factor module is eg-radical supplemented.*

Proof: Let M/L be any factor module of M. Since M is g-radical supplemented, by Lemma 9, M/L is g-radical supplemented. Then by Proposition 1, M/L is eg-radical supplemented, as desired. □

Proposition 3. *Let M be a g-radical supplemented $R-$module. Then every homomorphic image of M is eg-radical supplemented.*

Proof: Let $f : M \longrightarrow N$ be an $R-$module epimorphism with N is an $R-$module. Since M is g-radical supplemented, by Lemma 9, N is g-radical supplemented. Then by Proposition 1, N is eg-radical supplemented, as desired. □

Proposition 4. *Let $M = M_1 + M_2$. If M_1 and M_2 are g-radical supplemented, then M is eg-radical supplemented.*

Proof: Since M_1 and M_2 are g-radical supplemented, by Lemma 9, M is also g-radical supplemented. Then by Proposition 1, M is eg-radical supplemented, as desired. □

Proposition 5. *Let $M = M_1 + M_2 + ... + M_n$. If M_i is g-radical supplemented for every $i = 1, 2, ..., n$, then M is eg-radical supplemented.*

Proof: Clear from Proposition 4. □

Proposition 6. *Let M be a g-radical supplemented $R-$module. Then every finitely $M-$generated module is eg-radical supplemented.*

Proof: Since M is g-radical supplemented, by Lemma 9, every finitely $M-$generated module is g-radical supplemented. Then by Proposition 1, every finitely $M-$generated module is eg-radical supplemented. □

Proposition 7. *Let R be any ring. If ${}_R R$ is g-radical supplemented, then every finitely generated $R-$ module is eg-radical supplemented.*

Proof: Clear from Proposition 6. □

Proposition 8. *Every Rad-supplemented module is eg-radical supplemented.*

Proof: Let M be a Rad-supplemented module. Since M is Rad-supplemented, by Lemma 9, M is g-radical supplemented. Then by Proposition 1, M is eg-radical supplemented. □

Proposition 9. *Let M be a Rad-supplemented $R-$module. Then every factor module is eg-radical supplemented.*

Proof: Let M/L be any factor module of M. Since M is Rad-supplemented, by Lemma 8, M/L is Rad-supplemented. Then by Proposition 8, M/L is eg-radical supplemented, as desired. □

Proposition 10. *Let M be a Rad-supplemented $R-$module. Then every homomorphic image of M is eg-radical supplemented.*

Proof: Let $f : M \longrightarrow N$ be an $R-$module epimorphism with N is an $R-$module. Since M is Rad-supplemented, by Lemma 8, N is Rad-supplemented. Then by Proposition 8, N is eg-radical supplemented, as desired. □

Proposition 11. *Let $M = M_1 + M_2$. If M_1 and M_2 are Rad-supplemented, then M is eg-radical supplemented.*

Proof: Since M_1 and M_2 are Rad-supplemented, by Lemma 8, M is also Rad-supplemented. Then by Proposition 8, M is eg-radical supplemented, as desired. □

Proposition 12. *Let $M = M_1 + M_2 + ... + M_n$. If M_i is Rad-supplemented for every $i = 1, 2, ..., n$, then M is eg-radical supplemented.*

Proof: Clear from Proposition 11. □

Proposition 13. *Let M be a Rad-supplemented $R-$module. Then every finitely $M-$generated module is eg-radical supplemented.*

Proof: Since M is Rad-supplemented, by Lemma 8, every finitely $M-$generated module is Rad-supplemented. Then by Proposition 8, every finitely $M-$generated module is eg-radical supplemented. □

Proposition 14. *Let R be any ring. If $_RR$ is Rad-supplemented, then every finitely generated $R-$ module is eg-radical supplemented.*

Proof: Clear from Proposition 13. □

Proposition 15. *Every supplemented module is eg-radical supplemented.*

Proof: Let M be a supplemented module. Since M is supplemented, by Lemma 8, M is Rad-supplemented. Then by Proposition 8, M is eg-radical supplemented. □

Proposition 16. *Let M be a supplemented $R-$module. Then every factor module is eg-radical supplemented.*

Proof: Let M/L be any factor module of M. Since M is supplemented, by Lemma 7, M/L is supplemented. Then by Proposition 15, M/L is eg-radical supplemented, as desired. □

Proposition 17. *Let M be a supplemented $R-$module. Then every homomorphic image of M is eg-radical supplemented.*

Proof: Let $f : M \longrightarrow N$ be an $R-$module epimorphism with N is an $R-$module. Since M is supplemented, by Lemma 7, N is supplemented. Then by Proposition 15, N is eg-radical supplemented, as desired. □

Proposition 18. *Let $M = M_1 + M_2$. If M_1 and M_2 are supplemented, then M is eg-radical supplemented.*

Proof: Since M_1 and M_2 are supplemented, by Lemma 7, M is also supplemented. Then by Proposition 15, M is eg-radical supplemented, as desired. □

Proposition 19. *Let $M = M_1 + M_2 + ... + M_n$. If M_i is supplemented for every $i = 1, 2, ..., n$, then M is eg-radical supplemented.*

Proof: Clear from Proposition 18. □

Proposition 20. *Let M be a supplemented $R-$module. Then every finitely $M-$generated module is eg-radical supplemented.*

Proof: Since M is supplemented, by Lemma 7, every finitely $M-$generated module is supplemented. Then by Proposition 15, every finitely $M-$generated module is eg-radical supplemented. □

Proposition 21. *Let R be any ring. If $_RR$ is supplemented, then every finitely generated $R-$ module is eg-radical supplemented.*

Proof: Clear from Proposition 20. □

Proposition 22. *Every g-supplemented module is eg-radical supplemented.*

Proof: Let M be a g-supplemented module. Since M is g-supplemented, then M is g-radical supplemented. Then by Proposition 1, M is eg-radical supplemented. □

Proposition 23. *Every essential g-supplemented module is eg-radical supplemented.*

Proof: Let M be an essential g-supplemented module and $U \trianglelefteq M$. Then U has a g-supplement V in M. Here $M = U + V$ and $U \cap V \ll_g V$. Since $U \cap V \ll_g V$, by [2, Lemma 5], $U \cap V \leq Rad_g V$. Hence V is a g-radical supplement of U in M. Therefore, M is eg-radical supplemented. □

Proposition 24. *Let M be an eg-radical supplemented module. If every nonzero submodule of M is essential in M, then M is g-radical supplemented.*

Proof: Let $U \leq M$. If $U = 0$, then M is a g-radical supplement of U in M. If $U \neq 0$, then by hypothesis, $U \trianglelefteq M$ and since M is eg-radical supplemented, U has a g-radical supplement in M. Hence M is g-radical supplemented. □

Proposition 25. *Every essential supplemented module is eg-radical supplemented.*

Proof: Let M be an essential supplemented module. Then M is essential g-supplemented and by Proposition 23, M is eg-radical supplemented. □

Corollary 1. *Let $M = M_1 + M_2 + ... + M_n$. If M_i is essential supplemented for every $i = 1, 2, ..., n$, then M is eg-radical supplemented.*

Proof: Since M_i is essential supplemented for every $i = 1, 2, ..., n$, by [5, Corollary 2.8], M is essential supplemented. Then by Proposition 25, M is eg-radical supplemented. □

Corollary 2. *Let M be an essential supplemented module. Then every finitely $M-$generated module is eg-radical supplemented.*

Proof: Clear from Corollary 1. □

Corollary 3. *Let R be a ring. If ${}_RR$ is essential supplemented, then every finitely generated $R-$module is eg-radical supplemented.*

Proof: Clear from Corollary 2. □

3 References

1 J. Clark, C. Lomp, N. Vanaja, R. Wisbauer, *Lifting Modules Supplements and Projectivity In Module Theory, Frontiers in Mathematics*, Birkhauser, Basel, 2006.
2 B. Koşar, C. Nebiyev, N. Sökmez, *g-Supplemented Modules*, Ukrainian Math. J., **67**(6) (2015), 861-864.
3 B. Koşar, C. Nebiyev, A. Pekin, *A Generalization of g-Supplemented Modules*, Miskolc Math. Notes, **20**(1) (2019), 345-352.
4 C. Nebiyev, H. H. Ökten, *Essential g-Supplemented Modules*, Turkish St. Inf. Tech. and Appl. Sci., **14**(1) (2019), 83-89.
5 C. Nebiyev, H. H. Ökten, A. Pekin, *Essential Supplemented Modules*, Int. J. of Pure and Appl. Math., **120**(2) (2018), 253-257.
6 C. Nebiyev, H. H. Ökten, A. Pekin, *Amply Essential Supplemented Modules*, J. of Sci. Res. and Rep., **21**(4) (2018), 1-4.
7 W. Xue, *Characterizations of Semiperfect and Perfect Rings*, Publ. Mat., **40** (1996), 115-125.
8 Y. Wang, N. Ding, *Generalized Supplemented Modules*, Taiwanese J. of Math., **10**(6) (2006), 1589-1601.
9 R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.