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The Succesive Approximations Method for Solving Non-Newtonian Fredholm Integral Equations of the Second Kind

Nihan Güngör[1*]

[1] *Department of Mathematical Engineering,Faculty of Engineering and Naturel Science, Gumushane University, Gumushane, Turkey, ORCID: 0000-0003-1235-2700*
** Corresponding Author E-mail: nihanmath@gmail.com*

Abstract: In this study, the Fredholm integral equations are defined in the sense of non-Newtonian calculus by using the concept of $*-$integral. The main aim of the study to research the solution of the linear non-Newtonian Fredholm integral equations of the second kind by using the successive approximations method with respect to the non-Newtonian calculus. The necessary conditions for the $*-$continuity and uniqueness of the solution of these equations are investigated and finally given some numerical examples.
Keywords: Non-Newtonian calculus, Non-Newtonian Fredholm integral equations, Successive approximations method

1 Introduction and Preliminaries

Grossman and Katz [8] have built a new structure called non-Newtonian calculus consisting of the branches of geometric, harmonic, quadratic, bigeometric, biharmonic and biquadratic calculus, as an alternative to classic calculus created by Newton and Leibniz. Non-Newtonian calculus has many application areas as science, engineering and mathematics. Çakmak and Başar [3] investigated some characteristic features of complexs numbers and functions in terms of non-Newtonian calculus. Erdoğan and Duyar [6] introduced non-Newtonian improper integrals and investigated their convergence conditions. Sağır and Erdoğan [15] examined the function sequences and series in the non-Newtonian real numbers. Further details for the subject of non-Newtonian calculus can be found in [2]-[3]-[4]-[5]-[6]-[7]-[8]-[9]-[10]-[15].

Integral equations have used for the solution of several problems in engineering, applied mathematics and mathematical physics since the 18th century. The integral equations have begun to enter the problems of engineering and other fields because of the relationship with differential equations which have wide range of applications and so their importance has increased in recent years. One can find relevant terminology related to integral equations in [1]-[11]-[12]-[13]-[14]-[16]-[17]-[18].

A generator is one-to-one function whose domain is $\mathbb{R}$, the set of all real numbers and whose range is a subset of $\mathbb{R}$. The range of generator α is called non-Newtonian real line and it is denoted by $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$. $\alpha-$ arithmetic operations are descriped as indicated below:

$$\begin{array}{ll}
\alpha-\text{addition} & x\dot{+}y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\} \\
\alpha-\text{subtraction} & x\dot{-}y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\} \\
\alpha-\text{multiplication} & x\dot{\times}y = \alpha\left\{\alpha^{-1}(x) \cdot \alpha^{-1}(y)\right\} \\
\alpha-\text{division} & x\dot{/}y = \alpha\left\{\alpha^{-1}(x) \div \alpha^{-1}(y)\right\} \\
\alpha-\text{order} & x\dot{<}y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y).
\end{array}$$

for $x, y \in \mathbb{R}_\alpha$. $\left(\mathbb{R}_\alpha, \dot{+}, \dot{\times}\right)$ is complete field. We say that α generates α-arithmetic. In particular, if we choose α-generator as the identity function I , then α-arithmetic turns out to be classical arithmetic. If we choose α-generator as exp, $\alpha(x) = e^x$ for $x \in \mathbb{R}$, then $\alpha^{-1}(x) = \ln x$, α-arithmetic turns out to be geometric arithmetic. The geometric operations are descriped as indicated below:

$$\begin{array}{lll}
\alpha-\text{addition} & x \oplus y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\} = e^{(\ln x + \ln y)} = x.y & \textit{geometric addition} \\
\alpha-\text{subtraction} & x \ominus y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\} = e^{(\ln x - \ln y)} = x/y,\ y \neq 0 & \textit{geometric subtraction} \\
\alpha-\text{multiplication} & x \odot y = \alpha\left\{\alpha^{-1}(x) \cdot \alpha^{-1}(y)\right\} = e^{(\ln x \times \ln y)} = x^{\ln y} = y^{\ln x} & \textit{geometric multiplication} \\
\alpha-\text{division} & x \oslash y = \alpha\left\{\alpha^{-1}(x) \div \alpha^{-1}(y)\right\} = e^{(\ln x \div \ln y)} = x^{\frac{1}{\ln y}},\ y \neq 1 & \textit{geometric division}
\end{array}$$

An α-positive number is a number x such that $x\dot{>}\dot{0}$, similarly an α-negative number is a number x such that $x\dot{<}\dot{0}$. α-zero and α-one are denoted by $\alpha(0) = \dot{0}$ and $\alpha(1) = \dot{1}$, respectively. The α-integers are as the following:

$$..., \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), ...$$

For each integer n, we set $\alpha(n)=\dot{n}$ [8, 9]. The $\alpha-$absolute value of $x\in\mathbb{R}_\alpha$ determined by

$$|x|_\alpha=\begin{cases} x, & x\dot{>}\dot{0} \\ \dot{0}, & x=\dot{0} \\ \dot{0}\dot{-}x, & x\dot{<}\dot{0} \end{cases}.$$

and this is equivalent to the expression $\alpha\left(\left|\alpha^{-1}(x)\right|\right)$. For any $x,y\in\mathbb{R}_\alpha$, $\left|x\dot{+}y\right|_\alpha\dot{\leq}|x|_\alpha\dot{+}|y|_\alpha$ and $\left|x\dot{\times}y\right|_\alpha=|x|_\alpha\dot{\times}|y|_\alpha$. For $x\in\mathbb{R}_\alpha$, $x^{p_\alpha}=\alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$ and $\sqrt[p]{x}^\alpha=\alpha\left\{\sqrt[p]{\alpha^{-1}(x)}\right\}$ [8, 9]. A closed $\alpha-$interval on $\mathbb{R}_\alpha$ expressed with

$$\dot{[}a,b\dot{]}=\left\{x\in\mathbb{R}_\alpha\,|\,a\dot{\leq}x\dot{\leq}b\right\}=\left\{x\in\mathbb{R}_\alpha\,|\,\alpha^{-1}(a)\dot{\leq}\alpha^{-1}(x)\dot{\leq}\alpha^{-1}(b)\right\}=\alpha\left(\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]\right).$$

Similarly, an open α-interval on $\mathbb{R}_\alpha$ can be expressed [5].

Grosmann and Katz described the $*-$calculus with the help of two arbitrary selected generators. Let α and β are arbitrarily selected generators and $*$ is the ordered pair of arithmetic ($\alpha-$arithmetic, $\beta-$arithmetic). The following notions are used:

	$\alpha-$ arithmetic	$\beta-$ arithmetic
Realm	$A\,(\subset\mathbb{R}_\alpha)$	$B\,(\subset\mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$ (or $-\alpha$)	$\ddot{/}$(or $-\beta$)
Order	$\dot{<}$	$\ddot{<}$

If the generators α and β are chosen as one of I and $\exp$, the following special calculus are obtained:

Calculus	α	β
Classic	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp

The ι (iota) which is an isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic uniquely satisfying the following there properties:

1. ι is one to one,
2. ι is on A and onto B,
3. For any numbers x and y in A,
$\iota\left(x\dot{+}y\right)=\iota(x)\ddot{+}\iota(y)$
$\iota\left(x\dot{-}y\right)=\iota(x)\ddot{-}\iota(y)$
$\iota\left(x\dot{\times}y\right)=\iota(x)\ddot{\times}\iota(y)$
$\imath\left(x\dot{/}y\right)=\imath(x)\ddot{/}\imath(y)$
$x\dot{<}y\Leftrightarrow\iota(x)\ddot{<}\iota(y)$

It turns out that $\iota(x)=\beta\left\{\alpha^{-1}(x)\right\}$ for every x in A and $\iota(\dot{n})=\ddot{n}$ for every integer n [8].

Let $f:A\to\mathbb{R}_\beta$ be a function and $a\in A^{'}_\alpha$, $b\in\mathbb{R}_\beta$. If for every $\varepsilon\ddot{>}\ddot{0}$ there exists a number $\delta=\delta(\varepsilon)\dot{>}\dot{0}$ such that $\left|f(x)\ddot{-}b\right|_\beta\ddot{<}\varepsilon$ for all $x\in A$ whenever $\left|x\dot{-}a\right|_\alpha\dot{<}\delta$, then it is said that the $*-$limit function f at the point a is b and is denoted by $*-\lim\limits_{x\to a}f(x)=b$ [15].

Let $a\in A$ and $f:A\longrightarrow\mathbb{R}_\beta$ be function. If for every $\varepsilon\ddot{>}\ddot{0}$ there exists a number $\delta=\delta(\varepsilon)\dot{>}\dot{0}$ such that $\left|f(x)\ddot{-}f(a)\right|_\beta\ddot{<}\varepsilon$ for all $x\in A$ whenever $\left|x\dot{-}a\right|_\alpha\dot{<}\delta$, then it is said that f is $*-$continuous at the point $a\in A$.

If $*-\lim\limits_{x\to a}\left\{\left[f(x)\ddot{-}f(a)\right]\ddot{/}\left[\iota(x)\ddot{-}\iota(a)\right]\right\}$ exits, it is denoted by $[D^*f](a)$ and called the $*-$derivative of f at a and say that f is $*-$differentiable at a. If it exists, $[D^*f](a)$ is necessarily in B [8].

The $*-$average of a $*-$continuous function f on $\dot{[}a,b\dot{]}$ is denoted by $\overset{*}{M}{}_b^a f$ and defined to be the $\beta-$limit of the $\beta-$convergent sequence whose nth term is $\beta-$ average of $f(a_1),f(a_2),...,f(a_n)$ where $a_1,a_2,...,a_n$ is the $n-$fold $\alpha-$ partition of $\dot{[}a,b\dot{]}$. The $*-$integral of a $*-$continuous function f on $\dot{[}a,b\dot{]}$ is indicated by $*\int_a^b f(x)\,d^*x$ that is the number $\left[\iota(b)\ddot{-}\iota(b)\right]\ddot{\times}\overset{*}{M}{}_b^a f$ in B [8].

For $a\in A$, let $\bar{a}=\alpha^{-1}(a)$. Take f be a function with arguments in A and values in B and let $\bar{f}(t)=\beta^{-1}\{f(\alpha(t))\}$. Then the relationship with the classical calculus and $*-$calculus are occured in the following:

(1) The $*-\mathop{lim}\limits_{x\longrightarrow a}f(x)$ and $\mathop{lim}\limits_{t\longrightarrow\bar{a}}\bar{f}(t)$ coexist, and if they do exist $*-\mathop{lim}\limits_{x\longrightarrow a}f(x)=\beta\left\{\mathop{lim}\limits_{t\longrightarrow\bar{a}}\bar{f}(t)\right\}$. Moreover, f is $*-$continuous at a iff $\bar{f}$ is continuous at $\bar{a}$.
(2) The derivatives $[D^*f](a)$ and $\left[D\bar{f}\right](\bar{a})$ coexist and they do exist $[D^*f](a)=\beta\left\{\left[D\bar{f}\right](\bar{a})\right\}$.
(3) If f is $*-$continuous on $\dot{[}a,b\dot{]}$, then $\overset{*}{M}{}_b^a f=\beta\left(M_{\bar{b}}^{\bar{a}}\bar{f}\right)$ and $*\int_a^b f(x)\,d^*x=\beta\left(\int_{\bar{a}}^{\bar{b}}\bar{f}(t)\,dt\right)$ [8].

Let $f,g:\dot{[}a,b\dot{]}\to\mathbb{R}_\beta$ be $*-$continuous. Then, the following statements are holds:

(1) $*\int_a^b\left[\lambda\ddot{\times}f(x)\ddot{+}\mu\ddot{\times}g(x)\right]d^*x=\lambda\ddot{\times}*\int_a^b f(x)\,d^*x\ddot{+}\mu\ddot{\times}*\int_a^b g(x)\,d^*x$ for all $\lambda,\mu\in\mathbb{R}_\beta$
(2) $*\int_a^b f(x)\,d^*x=*\int_a^c f(x)\,d^*x\ddot{+}*\int_c^b f(x)\,d^*x$ for any $c\in\dot{[}a,b\dot{]}$

(3) If $f(x)\ddot{\leq}g(x)$ for all $x\in[\dot{a},\dot{b}]$, then $*\int_a^b f(x)\,d^*x\ddot{\leq}*\int_a^b g(x)\,d^*x$

(4) The function f is $\beta-$bounded on $[\dot{a},\dot{b}]$

(5) $\left|*\int_a^b f(x)\,d^*x\right|_\beta\ddot{\leq}*\int_a^b |f(x)|_\beta\,d^*x$ [6, 8].

Let $n\in\mathbb{N}$ and A be nonempty subset of $\mathbb{R}_\alpha$. The sequence $(f_n)=(f_1,f_2,\ldots,f_n,\ldots)$ is called $*-$function sequence or non-Newtonian function sequence for functions $f_n:A\subset\mathbb{R}_\alpha\to\mathbb{R}_\beta$. Here all functions defined on same set. The sequence $(f_n(x_0))$ is β-sequence (or non-Newtonian sequence) in $\mathbb{R}_\beta$ for each $x_0\in A$. Let take the $*-$function sequence (f_n) where $f_n:A\subseteq\mathbb{R}_\alpha\to\mathbb{R}_\beta$. The $*-$function sequence (f_n) $*-$ uniform converges to the function f on the set A, if for any given $\varepsilon\ddot{>}\ddot{0}$, there is a naturel number n_0 depends on number ε but not depend on variable x such that $\left|f_n(x)\ddot{-}f(x)\right|_\beta\ddot{<}\varepsilon$ for all $n>n_0$ and each $x\in A$. We denote $*-$uniform convergence by $*-\lim\limits_{n\to\infty}f_n=f$ ($*-$uniform) or $f_n\overset{*}{\to}f$. Let take $*-$function sequence (f_n) with $f_n:A\subseteq\mathbb{R}_\alpha\to\mathbb{R}_\beta$. The infinite $\beta-$sum ${}_\beta\sum\limits_{n=1}^{\infty}f_n=f_1\ddot{+}f_2\ddot{+}\cdots\ddot{+}f_n+\cdots$ is called $*-$function series (or non-Newtonian function series). The $\beta-$sum $S_n={}_\beta\sum\limits_{k=1}^{n}f_k$ is called nth partial $\beta-$sum of the series ${}_\beta\sum\limits_{n=1}^{\infty}f_n$ for $n\in\mathbb{N}$. Let the $*-$function series ${}_\beta\sum\limits_{n=1}^{\infty}f_n$ with $f_n:A\subseteq\mathbb{R}_\alpha\to\mathbb{R}_\beta$ and the function $f:A\subseteq\mathbb{R}_\alpha\to\mathbb{R}_\beta$ be specified. If the $\beta-$partial sums sequence (S_n), where $S_n={}_\beta\sum\limits_{k=1}^{n}f_k$ is $*-$uniform convergent to the function f, then ${}_\beta\sum\limits_{n=1}^{\infty}f_n$ is called $*-$ uniform convergent to the function f on the set A and ${}_\beta\sum\limits_{n=1}^{\infty}f_n=f\,(*-uniform)$ is written [15].

Theorem 1 ($*-$Weierstrass M-criterian). *If there exist $\beta-$numbers M_n such that $|f_n(x)|_\beta\ddot{<}M_n$ for all $x\in A$ where $f_n:A\to\mathbb{R}_\beta$ and if the series ${}_\beta\sum\limits_{n=1}^{\infty}M_n$ is $\beta-$convergent, then the series ${}_\beta\sum\limits_{n=1}^{\infty}f_n$ is $*-$uniform convergent and $\beta-$absolutely convergent [15].*

Theorem 2. *The functions $f_n:A\to\mathbb{R}_\beta$ be $*-$continuous and the function $f:A\to\mathbb{R}_\beta$ be given. If ${}_\beta\sum\limits_{n=1}^{\infty}f_n=f$ ($*-$uniform), then the function f is $*-$continuous on the set A [15].*

Theorem 3. *Let the functions $f_n:[\dot{a},\dot{b}]\to\mathbb{R}_\beta$ be $*-$continuous on $[\dot{a},\dot{b}]$ for all $n\in\mathbb{N}$ and $f_n\overset{*}{\to}f$ ($*-$uniform) on $[\dot{a},\dot{b}]$. Then the function f is $*-$continuous on $[\dot{a},\dot{b}]$ and $*-\lim\limits_{n\to\infty}*\int_a^b f_n(x)\,d^*x=*\int_a^b f(x)\,d^*x$ [15].*

2 Non-Newtonian Fredholm Integral Equations

The equation that is the unkown function determined as $v(x)$ occurs under the $*-$integral sign is called non-Newtonian integral equation (NIE), if the integral exists. An equation of an unkown $\mathbb{R}_\beta-$valued function $v(x)$ is generated by fixed limits of $*-$integration in the form

$$v(x)=f(x)\ddot{+}\lambda\ddot{\times}*\int_a^b K(x,s)\,\ddot{\times}v(s)\,d^*s,\ a\dot{\leq}x\dot{\leq}b$$

is said to be a linear non-Newtonian Fredholm integral equation (LNFIE) of $v(x)$ where a,b are constants in $\mathbb{R}_\alpha$ and λ is a $\mathbb{R}_\beta-$parameter. Only if the unkown function $v(x)$ is under the $*-$integral sing in the form of

$$f(x)=*\int_a^b K(x,s)\,\ddot{\times}v(s)\,d^*s$$

the equation is called linear non-Newtonian Fredholm integral equation of the first kind (LNFIEFK). The equation of unkown function $v(x)$ occurs inside and outside the $*-$integral sing in the form

$$v(x)=f(x)\ddot{+}\lambda\ddot{\times}*\int_a^b K(x,s)\,\ddot{\times}v(s)\,d^*s \tag{1}$$

is called linear non-Newtonian Fredholm integral equation of the second kind (LNFIESK). If $f(x)=\ddot{0}$, the equation is called homogeneous. The functions $f(x)$ and $K(x,s)$ are known $\mathbb{R}_\beta-$valued functions. The function $K(x,s)$ defined in the rectangle $\mathbb{R}_\alpha$, for which $a\dot{\leq}x\dot{\leq}b$, $a\dot{\leq}s\dot{\leq}b$, is called the $*-$kernel of the NFIE.

2.1 The Succesive Approximations Method

In this method, the zeroth approximation which is taken as $v_0(x) = f(x)$ is replaced the unkown function $v(x)$ under the $*-$integral sign in (1). Because of this substituting, the first approximation $v_1(x)$ is obtained as

$$v_1(x) = f(x) \ddot{+} \lambda \ddot{\times} * \int_a^b K(x,s) \ddot{\times} v_0(s)\, d^* s. \tag{2}$$

The second approximation $v_2(x)$ is found by replacing $v_1(x)$ obtained in (2), instead of $v(x)$ on right side of the equation (1), then we get

$$v_2(x) = f(x) \ddot{+} \lambda \ddot{\times} * \int_a^b K(x,s) \ddot{\times} v_1(s)\, d^* s.$$

By proceeding likewise, the nth approximation is obtained in the following

$$v_n(x) = f(x) \ddot{+} \lambda \ddot{\times} * \int_a^b K(x,s) \ddot{\times} v_{n-1}(s)\, d^* s.$$

In others words, the approximations can be given in a repeated scheme as follows

$$\begin{aligned} v_0(x) &= f(x) \\ v_n(x) &= f(x) \ddot{+} \lambda \ddot{\times} * \int_a^b K(x,s) \ddot{\times} v_{n-1}(s)\, d^* s,\ n \geq 1. \end{aligned} \tag{3}$$

The convergence of $v_n(x)$ will be verified by theorem. Now, we will give the necessary proposition for the proof of the theorem that answers the question of convergence of $v_n(x)$.

Proposition 1. *Let*

$$\begin{aligned} \varphi_0(x) &= f(x) \\ \varphi_n(x) &= * \int_a^b K(x,s) \ddot{\times} \varphi_{n-1}(s)\, d^* s,\ n \geq 1. \end{aligned} \tag{4}$$

If $f(x)$ is $*-$continuous for the interval $\dot{[}a,b\dot{]}$ and $K(x,s)$ is $*-$continuous for $a \dot{\leq} x \dot{\leq} b$, $a \dot{\leq} s \dot{\leq} b$, then

$$_\beta \sum_{n=0}^{\infty} \lambda^{n_\beta} \ddot{\times} \varphi_n(x) \tag{5}$$

is $*-$uniform and $\beta-$absolutely convergent where $|\lambda|_\beta \ddot{<} \frac{\ddot{1}}{M \ddot{\times} (\iota(b) \ddot{-} \iota(a))} \beta$.

Proof: Since $K(x,s)$ is $*-$continuous for $a \dot{\leq} x \dot{\leq} b$, $a \dot{\leq} s \dot{\leq} b$, there is $M \ddot{\geq} \ddot{0}$ such that $|K(x,s)|_\beta \ddot{\leq} M$ for $a \dot{\leq} x \dot{\leq} b$, $a \dot{\leq} s \dot{\leq} b$. Because of $f(x)$ is $*-$continuous for the interval $\dot{[}a,b\dot{]}$, there is $F \ddot{\geq} \ddot{0}$ such that $|f(x)|_\beta \ddot{\leq} F$ for all $x \in \dot{[}a,b\dot{]}$. Consequently, we can write

$$M =_\beta \max_{a \dot{\leq} s \dot{\leq} x \dot{\leq} b} |K(x,s)|_\beta \tag{6}$$

and

$$F =_\beta \max_{a \dot{\leq} x \dot{\leq} b} |f(x)|_\beta . \tag{7}$$

Hence by (7),

$$|\varphi_0(x)|_\beta = |f(x)|_\beta \ddot{\leq} F \tag{8}$$

for all $x \in \dot{[}a,b\dot{]}$. Therefore, from (6) and (8) we find

$$\begin{aligned}
|\varphi_1(x)|_\beta &= \left| *\int_a^b K(x,s) \ddot{\times} \varphi_0(s)\, d^*s \right|_\beta \\
&= \left| *\int_a^b K(x,s) \ddot{\times} f(s)\, d^*s \right|_\beta \\
&\ddot{\leq} *\int_a^b |K(x,s)|_\beta \ddot{\times} |f(s)|_\beta\, d^*s \\
&= M \ddot{\times} F \ddot{\times} *\int_a^b \ddot{1} d^*s \\
&\ddot{\leq} M \ddot{\times} F \ddot{\times} \beta\left(\alpha^{-1}(b) - \alpha^{-1}(a)\right) \\
&= M \ddot{\times} F \ddot{\times} \left[\beta\left(\alpha^{-1}(b)\right) \ddot{-} \beta\left(\alpha^{-1}(a)\right)\right] \\
&= M \ddot{\times} F \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)
\end{aligned}$$

by replacing $\varphi_0(x)$ in (4). Therefore, we find

$$\begin{aligned}
|\varphi_2(x)|_\beta &= \left| *\int_a^b K(x,s) \ddot{\times} \varphi_1(s)\, d^*s \right|_\beta \\
&\ddot{\leq} *\int_a^b |K(x,s)|_\beta \ddot{\times} |\varphi_1(s)|_\beta\, d^*s \\
&\ddot{\leq} M \ddot{\times} M \ddot{\times} F \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right) \ddot{\times} *\int_a^b \ddot{1} d^*s \\
&= M^{2_\beta} \ddot{\times} F \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{2_\beta}.
\end{aligned}$$

Carrying in a similar manner, we get

$$|\varphi_n(x)|_\beta \ddot{\leq} F \ddot{\times} M^{n_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{n_\beta} \tag{9}$$

for $x \in [\dot{a}, \dot{b}]$ and $n \geq 0$. Hence we write

$$_\beta\sum_{n=0}^{\infty} \left|\lambda^{n_\beta} \ddot{\times} \varphi_n(x)\right|_\beta \ \ddot{\leq}_\beta \sum_{n=0}^{\infty} F \ddot{\times} |\lambda|_\beta^{n_\beta} \ddot{\times} M^{n_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{n_\beta}$$

from (9). Since $|\lambda|_\beta \ddot{<} \frac{\ddot{1}}{M \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)} \beta$, the $\beta-$geometric series $_\beta\sum_{n=0}^{\infty} F \ddot{\times} |\lambda|_\beta^{n_\beta} \ddot{\times} M^{n_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{n_\beta}$ is β-convergence. This implies via $*-$Weierstrass M-criterion, the series $_\beta\sum_{n=0}^{\infty} \lambda^{n_\beta} \ddot{\times} \varphi_n(x)$ is $*-$uniform and $\beta-$absolutely convergent. □

Theorem 4. *Assume that the following conditions are satisfied:*
(i) *f is $*-$continuous on the interval $[\dot{a}, \dot{b}]$,*
(ii) *$K(x,s)$ is $*-$ continuous on $a \dot{\leq} x \dot{\leq} b$, $a \dot{\leq} s \dot{\leq} b$,*
(iii) *λ is a $\mathbb{R}_\beta-$parameter and $|\lambda|_\beta \ddot{<} \frac{\ddot{1}}{M \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)} \beta$,*
then the sequence of succesive approximations $v_n(x)$ in (3), converges to the solution $v(x)$ of NFIE (1) and the solution $v(x)$ is $*-$continuous function on the interval $[\dot{a}, \dot{b}]$.

Proof: Let taken $v_0(x) = f(x) = \varphi_0(x)$. Hence we find

$$\begin{aligned}
v_1(x) &= f(x) \ddot{+} \lambda \ddot{\times} *\int_a^b K(x,s) \ddot{\times} v_0(s)\, d^*s \\
&= f(x) \ddot{+} \lambda \ddot{\times} *\int_a^b K(x,s) \ddot{\times} f(s)\, d^*s \\
&= f(x) \ddot{+} \lambda \ddot{\times} \varphi_1(x) \qquad (10)
\end{aligned}$$

where $\varphi_1(x) = *\int\limits_a^b K(x,s) \ddot{\times} f(s)\, d^*s$. The second approximation $v_2(x)$ is obtained as

$$\begin{aligned}
v_2(x) &= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} v_1(s)\, d^*s \\
&= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} \left[f(s) \ddot{+} \lambda \ddot{\times} \varphi_1(s)\right] d^*s \\
&= f(x) \ddot{+} \lambda \ddot{\times} \left[* \int\limits_a^b K(x,s) \ddot{\times} f(s)\, d^*s \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} \varphi_1(s)\, d^*s \right] \\
&= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} f(s)\, d^*s \ddot{+} \lambda^{2_\beta} \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} \varphi_1(s)\, d^*s \\
&= f(x) \ddot{+} \lambda \ddot{\times} \varphi_1(x) \ddot{+} \lambda^{2_\beta} \ddot{\times} \varphi_2(x)
\end{aligned}$$

where $\varphi_2(x) = *\int\limits_a^b K(x,s) \ddot{\times} \varphi_1(s)\, d^*s$ from (10). Proceeding this manner, we get

$$v_n(x) = f(x) \ddot{+}_\beta \sum_{k=1}^{n} \lambda^{k_\beta} \ddot{\times} \varphi_k(x) =_\beta \sum_{k=0}^{n} \lambda^{k_\beta} \ddot{\times} \varphi_k(x). \tag{11}$$

Under the hypothesis, the functions $v_n(x)$ are $*-$continuous on $\dot{[}a,b\dot{]}$ for all $n \geq 0$ and ${}_\beta\sum\limits_{n=0}^{\infty} \lambda^{n_\beta} \ddot{\times} \varphi_n(x)$ is $*-$uniform and $\beta-$absolutely convergent by Proposition 1. Hence there exists a $*-$continuous function $v(x)$ such that $* - \lim\limits_{n\to\infty} v_n(x) = v(x)$ ($*-$uniform) by Theorem 2. Now we will show that $v(x)$ is the solution of NFIE (1). If we take $*-$limit as $n \to \infty$ on both sides of the equation (11), we get

$$\begin{aligned}
v(x) &= * - \lim_{n\to\infty} v_n(x) = f(x) \ddot{+} \lambda \ddot{\times} \left(* - \lim_{n\to\infty} * \int\limits_a^b K(x,s) \ddot{\times} v_{n-1}(s)\, d^*s \right) \\
&= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} \left(* - \lim_{n\to\infty} v_{n-1}(s) \right) d^*s \\
&= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} v(s)\, d^*s
\end{aligned}$$

by Theorem 3. This completes the proof. □

Remark 1. *Under the hypothesis of Theorem 4, the series defined in (5) convergences and equals to the solution of NFIE (1). For this reason, the solution of (1) also can be determined by aid of the expressions in (4).*

Theorem 5. *Under the hypothesis of Theorem 4, the NFIE has an unique solution on* $\dot{[}a,b\dot{]}$.

Proof: Assume that $v(x)$ and $u(x)$ are different solutions of NFIE (1). Hence, we can write

$$\begin{aligned}
v(x) &= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} v(s)\, d^*s \\
u(x) &= f(x) \ddot{+} \lambda \ddot{\times} * \int\limits_a^b K(x,s) \ddot{\times} u(s)\, d^*s.
\end{aligned}$$

If we set $\psi(x) = v(x) \ddot{-} u(x)$, then

$$\begin{aligned} |\psi(x)|_\beta &= \left| \lambda \ddot{\times} * \int_a^b K(x,s) \ddot{\times} \psi(s) \, d^*s \right|_\beta \\ &= |\lambda|_\beta \ddot{\times} \left| * \int_a^b K(x,s) \ddot{\times} \psi(s) \, d^*s \right|_\beta \\ &\ddot{\leq} |\lambda|_\beta \ddot{\times} * \int_a^b |K(x,s)|_\beta \ddot{\times} |\psi(s)|_\beta \, d^*s \\ &\ddot{\leq} |\lambda|_\beta \ddot{\times} \left(* \int_a^b |K(x,s)|_\beta^{2_\beta} \, d^*s \right)^{\left(\frac{1}{2}\right)_\beta} \ddot{\times} \left(* \int_a^b |\psi(s)|_\beta^{2_\beta} \, d^*s \right)^{\left(\frac{1}{2}\right)_\beta} \end{aligned}$$

by using Cauchy-Schwartz inequality in the sense of non-Newtonian. Consequently, we obtain

$$\begin{aligned} |\psi(x)|_\beta^{2_\beta} &\ddot{\leq} |\lambda|_\beta^{2_\beta} \ddot{\times} * \int_a^b |K(x,s)|_\beta^{2_\beta} \, d^*s \ddot{\times} * \int_a^b |\psi(s)|_\beta^{2_\beta} \, d^*s \\ &\ddot{\leq} |\lambda|_\beta^{2_\beta} \ddot{\times} M^{2_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right) \ddot{\times} * \int_a^b |\psi(s)|_\beta^{2_\beta} \, d^*s \end{aligned}$$

from (6). By integration both sides of this inequality according to x, we find

$$* \int_a^b |\psi(x)|_\beta^{2_\beta} \, d^*x \ddot{\leq} |\lambda|_\beta^{2_\beta} \ddot{\times} M^{2_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{2_\beta} \ddot{\times} * \int_a^b |\psi(s)|_\beta^{2_\beta} \, d^*s$$

$$\left(\ddot{1} - |\lambda|_\beta^{2_\beta} \ddot{\times} M^{2_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{2_\beta} \right) \ddot{\times} * \int_a^b |\psi(x)|_\beta^{2_\beta} \, d^*x \ddot{\leq} \ddot{0}.$$

Since $\ddot{1} - |\lambda|_\beta^{2_\beta} \ddot{\times} M^{2_\beta} \ddot{\times} \left(\iota(b) \ddot{-} \iota(a)\right)^{2_\beta} \ddot{>} \ddot{0}$, it is obtained that $* \int_a^b |\psi(x)|_\beta^{2_\beta} \, dx \ddot{\leq} \ddot{0}$. Hence it must be $* \int_a^b |\psi(x)|_\beta^{2_\beta} \, d^*x = \ddot{0}$.

$$\begin{aligned} * \int_a^b |\psi(x)|_\beta^{2_\beta} \, d^*x &= \ddot{0} \\ \beta \left\{ \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1} \left(\beta \left\{ \left(\beta^{-1} \left\{ \beta \left(\left| \beta^{-1} \left(\psi\left(\alpha(x)\right) \right) \right| \right) \right\} \right)^2 \right\} \right) dx \right\} &= \beta(0) \\ \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \left| \beta^{-1} \left(\psi\left(\alpha(x)\right) \right) \right|^2 dx &= 0 \end{aligned}$$

It implies that $\beta^{-1}\left(\psi\left(\alpha(x)\right)\right) = 0$ on $\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$. Consequently, it is obtained that $\psi(x) = \ddot{0}$ on the interval $\dot{[}a, b\dot{]}$. Hence, we get $v(x) = u(x)$ for all $x \in \dot{[}a, b\dot{]}$. In that case the solution of NFIE is unique. □

2.2 Numerical Examples

Example 1. *For geometric calculus, since $\alpha(x) = I(x) = x$ and $\beta(x) = e^x$, $\mathbb{R}_\alpha = \mathbb{R}$ and $\mathbb{R}_\beta = (0, +\infty)$. According to this, the geometric Fredholm integral equation is*

$$v(x) = f(x) \oplus \lambda \odot {}^G\int_a^b K(x,s) \odot v(s) \, d^G s = f(x) \oplus \left(\lambda \odot e^{\int_a^b \ln K(x,s) . \ln v(s) ds} \right) = f(x) . \left(e^{\ln \lambda . \int_a^b \ln K(x,s) . \ln v(s) ds} \right).$$

Let the Fredholm integral equation as

$$v(x) = e^{\cos x} \oplus e^{\frac{1}{2}} \odot {}^{G}\int_{0}^{\frac{\pi}{2}} e^{\cos x} \odot e^{\sin s} \odot v(s)\, d^{G}s$$

in geometric calculus. Find the solution of this equation by using the succesive approximations method.

Solution 1. *Taking the zeroth approximation as*

$$v_0(x) = e^{\cos x}. \tag{12}$$

We will use the iteration formula

$$v_n(x) = e^{\cos x} \oplus e^{\frac{1}{2}} \odot {}^{G}\int_{0}^{\frac{\pi}{2}} e^{\cos x} \odot e^{\sin s} \odot v_{n-1}(s)\, d^{G}s,\ n \geq 1. \tag{13}$$

Substituting (12) into (13), we obtain

$$\begin{aligned}
v_1(x) &= e^{\cos x} \oplus e^{\frac{1}{2}} \odot {}^{G}\int_{0}^{\frac{\pi}{2}} e^{\cos x} \odot e^{\sin s} \odot e^{\cos s} d^{G}s \\
&= e^{\cos x} \oplus e^{\frac{1}{2}} \odot e^{\cos x} \odot {}^{G}\int_{0}^{\frac{\pi}{2}} e^{\sin s} \odot e^{\cos s} d^{G}s \\
&= e^{\cos x} \oplus e^{\frac{1}{2}} \odot e^{\cos x} \odot {}^{G}\int_{0}^{\frac{\pi}{2}} \left(e^{\sin s}\right)^{\ln e^{\cos s}} d^{G}s \\
&= e^{\cos x}. \left(\left(e^{\frac{1}{2}}\right)^{\ln e^{\cos x}} \odot \left(e^{\int_0^{\frac{\pi}{2}} \ln\left(e^{\sin s}\right)^{\ln e^{\cos s}} ds} \right) \right) \\
&= e^{\cos x}. \left(\left(e^{\frac{1}{2}}\right)^{\cos x} \odot \left(e^{\int_0^{\frac{\pi}{2}} \sin s. \cos s\, ds} \right) \right) \\
&= e^{\cos x}. \left(\left(e^{\frac{1}{2}}\right)^{\cos x} \odot e^{\frac{1}{2}} \right) \\
&= e^{\cos x}. e^{\frac{1}{4}\cos x} \\
&= e^{\frac{5}{4}\cos x}
\end{aligned}$$

$$\begin{aligned}
v_2(x) &= e^{\cos x} \oplus e^{\frac{1}{2}} \odot {}^{G}\int_{0}^{\frac{\pi}{2}} e^{\cos x} \odot e^{\sin s} \odot e^{\frac{5}{4}\cos s} d^{G}s \\
&= e^{\frac{21}{16}\cos x}
\end{aligned}$$

$$\vdots$$

$$v_n(x) = e^{\frac{1+4+4^2+\cdots+4^n}{4^n}\cos x} = e^{\frac{1}{3}\left(4-\frac{1}{4^n}\right)\cos x}.$$

The solution $v(x)$ of equation is found as

$$v(x) = G - \lim_{n\to\infty} v_n(x) = G - \lim_{n\to\infty} e^{\frac{1}{3}\left(4-\frac{1}{4^n}\right)\cos x} = e^{\lim_{n\to\infty}\left(\ln e^{\frac{1}{3}\left(4-\frac{1}{4^n}\right)\cos x}\right)} = e^{\lim_{n\to\infty}\frac{1}{3}\left(4-\frac{1}{4^n}\right)\cos x} = e^{\frac{4}{3}\cos x}.$$

Example 2. *Solve the NFIE*

$$v(x) = \iota(x) \ddot{+} \lambda \ddot{\times} * \int_{\dot{0}\dot{-}\dot{1}}^{\dot{1}} \iota\left(x \dot{\times} s\right) \ddot{\times} v(s)\, d^{*}s$$

by using the succesive approximations method.

Solution 2. *Taking the zeroth approximation as*

$$v_0(x) = \iota(x). \tag{14}$$

We will use the iteration formula

$$v_n(x) = \iota(x) \ddot{+} \lambda \ddot{\times} * \int_{\dot{0}\dot{-}\dot{1}}^{\dot{1}} \iota(x) \ddot{\times} \iota(s) \ddot{\times} v_{n-1}(s) \, d^*s, \; n \geq 1. \tag{15}$$

Substituting (14) into (15), we obtain

$$\begin{aligned} v_1(x) &= \iota(x) \ddot{+} \lambda \ddot{\times} \iota(x) \ddot{\times} * \int_{\dot{0}\dot{-}\dot{1}}^{\dot{1}} \iota(s) \ddot{\times} \iota(s) \, d^*s \\ &= \iota(x) \ddot{+} \lambda \ddot{\times} \iota(x) \ddot{\times} \beta \left\{ \int_{\alpha^{-1}(\dot{0}\dot{-}\dot{1})}^{\alpha^{-1}(\dot{1})} \beta^{-1} \left\{ \beta \left[\beta^{-1} \left(\iota(\alpha(s)) \right)^2 \right] \right\} ds \right\} \\ &= \iota(x) \ddot{+} \lambda \ddot{\times} \iota(x) \ddot{\times} \beta \left\{ \int_{-1}^{1} s^2 ds \right\} \\ &= \iota(x) \ddot{+} \frac{\ddot{2}}{\ddot{3}} \beta \ddot{\times} \lambda \ddot{\times} \iota(x) \end{aligned}$$

$$\begin{aligned} v_2(x) &= \iota(x) \ddot{+} \lambda \ddot{\times} \iota(x) \ddot{\times} * \int_{\dot{0}\dot{-}\dot{1}}^{\dot{1}} \iota(s) \ddot{\times} \left(\iota(s) \ddot{+} \frac{\ddot{2}}{\ddot{3}} \beta \ddot{\times} \lambda \ddot{\times} \iota(s) \right) d^*s \\ &= \iota(x) \ddot{+} \lambda \ddot{\times} \iota(x) \ddot{\times} \left[* \int_{\dot{0}\dot{-}\dot{1}}^{\dot{1}} \iota(s)^{2_\beta} \, d^*s \ddot{\times} \ddot{+} \frac{\ddot{2}}{\ddot{3}} \beta \ddot{\times} \lambda \ddot{\times} \; * \int_{\dot{0}\dot{-}\dot{1}}^{\dot{1}} \iota(s)^{2_\beta} \, d^*s \right] \\ &= \iota(x) \ddot{+} \frac{\ddot{2}}{\ddot{3}} \beta \ddot{\times} \lambda \ddot{\times} \iota(x) \ddot{+} \left(\frac{\ddot{2}}{\ddot{3}} \beta \right)^{2_\beta} \ddot{\times} \lambda^{2_\beta} \ddot{\times} \iota(x) \end{aligned}$$

$$\vdots$$

$$v_n(x) = \iota(x) \ddot{+} \frac{\ddot{2}}{\ddot{3}} \beta \ddot{\times} \lambda \ddot{\times} \iota(x) \ddot{+} \left(\frac{\ddot{2}}{\ddot{3}} \beta \right)^{2_\beta} \ddot{\times} \lambda^{2_\beta} \ddot{\times} \iota(x) \ddot{+} \cdots \ddot{+} \left(\frac{\ddot{2}}{\ddot{3}} \beta \right)^{n_\beta} \ddot{\times} \lambda^{n_\beta} \ddot{\times} \iota(x).$$

Since the $*-$geometric series $\beta \sum_{n=0}^{\infty} \left(\frac{\ddot{2}}{\ddot{3}} \beta \ddot{\times} \lambda \right)^{n_\beta}$ *convergences to* $\frac{\ddot{3}}{\ddot{3} \ddot{-} \ddot{2} \ddot{\times} \lambda} \beta$ *for* $\left| \frac{\ddot{2} \ddot{\times} \lambda}{\ddot{3}} \beta \right|_\beta \ddot{<} \ddot{1}$, *the solution of NFIE is obtained as*

$$v(x) = * - \lim_{n \to \infty} v_n(x) = \frac{\ddot{3}}{\ddot{3} \ddot{-} \ddot{2} \ddot{\times} \lambda} \beta \ddot{\times} \iota(x)$$

for $\ddot{0} \ddot{-} \frac{\ddot{3}}{\ddot{2}} \beta \ddot{<} \lambda \ddot{<} \frac{\ddot{3}}{\ddot{2}} \beta$.

Example 3. *Solve the NFIE*

$$v(x) = \ddot{e}^{\left(\alpha^{-1}(x) \right)_\beta} \ddot{-} \iota(x) \ddot{+} \lambda \ddot{\times} * \int_{\dot{0}}^{\dot{1}} \iota\left(x \dot{\times} s \right) \ddot{\times} v(s) \, ds.$$

with the aid of the series in (5).

Solution 3. *Proceeding with the recurrence relation in (4), that gives*

$$\varphi_0(x) = \ddot{e}^{\left(\alpha^{-1}(x) \right)_\beta} \ddot{-} \iota(x)$$

$$\varphi_1(x) = *\int_{\dot{0}}^{\dot{1}} \iota(x \dot{\times} s) \ddot{\times} \left(\ddot{e}^{(\alpha^{-1}(s))_\beta} \ddot{-} \iota(s) \right) d^* s$$

$$= \iota(x) \ddot{\times} *\int_{\dot{0}}^{\dot{1}} \iota(s) \ddot{\times} \left(\ddot{e}^{(\alpha^{-1}(s))_\beta} \ddot{-} \iota(s) \right) d^* s$$

$$= \iota(x) \ddot{\times} \frac{\ddot{2}}{\ddot{3}} \beta$$

$$\varphi_2(x) = \iota(x) \ddot{\times} *\int_{\dot{0}}^{\dot{1}} \iota(s) \ddot{\times} \left(\iota(s) \ddot{\times} \frac{\ddot{2}}{\ddot{3}} \beta \right) d^* s$$

$$= \iota(x) \ddot{\times} \frac{\ddot{2}}{\ddot{9}} \beta$$

$$\vdots$$

$$\varphi_n(x) = \iota(x) \ddot{\times} \frac{\ddot{2}}{\ddot{3}^{n_\beta}} \beta$$

and so on. The solution of the NIE is obtained as a series form is given by

$$v(x) = \beta \sum_{n=0}^{\infty} \lambda^{n_\beta} \ddot{\times} \varphi_n(x)$$

$$= \varphi_0(x) \ddot{+} \beta \sum_{n=1}^{\infty} \lambda^{n_\beta} \ddot{\times} \varphi_n(x)$$

$$= \ddot{e}^{(\alpha^{-1}(x))_\beta} \ddot{-} \iota(x) \ddot{+} \iota(x) \ddot{\times} \beta \sum_{n=1}^{\infty} \left(\ddot{2} \ddot{\times} \left(\frac{\lambda}{\ddot{3}} \beta \right)^{n_\beta} \right).$$

Since the $*-$*geometric series* $\beta \sum_{n=1}^{\infty} \left(\ddot{2} \ddot{\times} \left(\frac{\lambda}{\ddot{3}} \beta \right)^{n_\beta} \right)$ *convergences to* $\frac{\ddot{2} \ddot{\times} \lambda}{\ddot{3} \ddot{-} \lambda} \beta$ *for* $\left| \frac{\lambda}{\ddot{3}} \beta \right|_\beta \ddot{<} \ddot{1}$*. Consequently, the solution of the NIE is found as*

$$v(x) = \ddot{e}^{(\alpha^{-1}(x))_\beta} \ddot{-} \frac{\ddot{2} \ddot{\times} \lambda}{\ddot{3} \ddot{-} \lambda} \beta \ddot{\times} \iota(x)$$

for $\ddot{0} \ddot{-} \ddot{3} \ddot{<} \lambda \ddot{<} \ddot{3}$*.*

3 Conclusion

The Fredholm integral equations are defined in the sense of non-Newtonian calculus by using the $*-$integral. The successive approximations method is applied to solve LNFIESK and the conditions for the uniqueness of the solution are given.

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