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r-Small Submodules

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Abstract: In this work, every ring have unity and every module is unital left module. Let M be an $R-$module and $N \leq M$. If $N \ll RadM$, then N is called a radical small (or briefly r-small) submodule of M and denoted by $N \ll_r M$. In this work, some properties of these submodules are given.

Keywords: Small Submodules, Maximal Submodules, Radical, Supplemented Modules.

1 Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If there exists $L \leq M$ such that $M = N + L$ and $N \cap L = 0$, then N is called a *direct summand of* M and denoted by $M = N \oplus L$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*)submodule of M and denoted by $N \ll M$. A module M is said to be *hollow* if every proper submodule of M is small in M. M is said to be *local* if there exists a proper submodule of M which contains all proper submodule of M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*Radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*Radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M.

More details about supplemented modules are in [1]-[2]-[5]-[6]. More details about generalized (Radical) supplemented modules are in [3]-[4].

Lemma 1. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*
(4) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(5) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(6) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

Proof: See [1, 2.2] and [5, 19.3]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) $RadM = \sum\limits_{L \ll M} L$.
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(RadM) \leq RadN$. If $Kef \leq RadM$, then $f(RadM) = Radf(M)$.*
(3) *If $N \leq M$, then $RadN \leq RadM$.*
(4) *For $K, L \leq M$, $RadK + RadL \leq Rad(K + L)$.*
(5) *$Rx \ll M$ for every $x \in RadM$.*

Proof: See [5, 21.5 and 21.6]. □

Lemma 3. *Let V be a supplement of U in M. Then*
(1) *If $W + V = M$ for some $W \leq U$, then V is a supplement of W in M.*
(2) *If M is finitely generated, then V is also finitely generated.*
(3) *If U is a maximal submodule of M, then V is cyclic and $U \cap V = RadV$ is the unique maximal submodule of V.*
(4) *If $K \ll M$, then V is a supplement of $U + K$ in M.*

(5) *For $K \ll M$, $K \cap V \ll V$ and hence $RadV = V \cap RadM$.*
(6) *Let $K \leq V$. Then $K \ll V$ if and only if $K \ll M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a supplement of U/L in M/L.*

Proof: See [5, 41.1]. □

2 r-Small Submodules

Definition 1. *Let M be an $R-$module and $N \leq M$. If $N \ll RadM$, then N is called a radical small (or briefly r-small) submodule of M and denoted by $N \ll_r M$.*

Lemma 4. *Let M be an $R-$module.*
(1) *If $M = U \oplus V$ then V is a supplement of U in M. Also U is a supplement of V in M.*
(2) *For $M_1, U \leq M$, if $M_1 + U$ has a supplement in M and M_1 is supplemented, then U also has a supplement in M.*
(3) *Let $M = M_1 + M_2$. If M_1 and M_2 are supplemented, then M is also supplemented.*
(4) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also supplemented.*
(5) *If M is supplemented, then M/L is supplemented for every $L \leq M$.*
(6) *If M is supplemented, then every homomorphic image of M is also supplemented.*
(7) *If M is supplemented, then $M/RadM$ is semisimple.*
(8) *Hollow and local modules are supplemented.*
(9) *If M is supplemented, then every finitely $M-$generated module is supplemented.*
(10) *${}_RR$ is supplemented if and only if every finitely generated $R-$module is supplemented.*

Proof: See [5, 41.2]. □

Lemma 5. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is Rad-supplemented.*
(2) *If V is a Rad-supplement of U in M and $W + V = M$ for some $W \leq U$, then V is a Rad-supplement of W in M.*
(3) *If U is a maximal submodule of M and V is a Rad-supplement of U in M, $U \cap V = RadV$ is the unique maximal submodule of V.*
(4) *If V is a Rad-supplement of U in M and $L \leq U$, then $\frac{V+L}{L}$ is a Rad-supplement of U/L in M/L.*
(5) *If V is a Rad-supplement of U in M, then $RadV = V \cap RadM$.*
(6) *Let $M = U + V$. Then V is a Rad-supplement of U in M if and only if $Rx \ll V$ for every $x \in U \cap V$.*
(7) *For $M_1, U \leq M$, if $M_1 + U$ has a Rad-supplement in M and M_1 is Rad-supplemented, then U also has a Rad-supplement in M.*

Proof: See [3]-[4]. □

Lemma 6. *Let M be an $R-$module.*
(1) *Let $M = M_1 + M_2$. If M_1 and M_2 are Rad-supplemented, then M is also Rad-supplemented.*
(2) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is Rad-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also Rad-supplemented.*
(3) *If M is Rad-supplemented, then M/L is Rad-supplemented for every $L \leq M$.*
(4) *If M is Rad-supplemented, then every homomorphic image of M is also Rad-supplemented.*
(5) *If M is Rad-supplemented, then $M/RadM$ is semisimple.*
(6) *If M is Rad-supplemented, then every finitely $M-$generated module is Rad-supplemented.*
(7) *${}_RR$ is Rad-supplemented if and only if every finitely generated $R-$module is Rad-supplemented.*

Proof: See [3]-[4]. □

Proposition 1. *Let M be an $R-$module and $N \leq M$. If $N \ll_r M$, then $N \ll M$.*

Proof: Since $N \ll_r M$, $N \ll RadM$. Then by Lemma 1, $N \ll M$. □

Proposition 2. *Let M be an $R-$module and $K, L \leq M$. If $K \ll_r M$ and $L \ll_r M$, then $K + L \ll M$.*

Proof: Since $K \ll_r M$ and $L \ll_r M$, by Proposition 1, $K \ll M$ and $L \ll M$. Then by Lemma 1, $K + L \ll M$. □

Proposition 3. *Let M be an $R-$module and $K_i \ll_r M$ for $i = 1, 2, ..., n$. Then $K_1 + K_2 + ... + K_n \ll M$.*

Proof: Clear from Proposition 2. □

Proposition 4. *Let M be an $R-$module and $N \leq M$. If $N \ll_r M$, then $\frac{N+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*

Proof: Since $N \ll_r M$, by Proposition 2, $N \ll M$. Then by Lemma 1, $\frac{N+L}{L} \ll \frac{M}{L}$ for every $L \leq M$. □

Proposition 5. *Let $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_r M$, then $f(K) \ll N$.*

Proof: Since $K \ll_r M$, by Proposition 1, $K \ll M$. Then by Lemma 1, $f(K) \ll N$. □

Proposition 6. *Let M be an $R-$module and $K \leq N \leq M$. If $N \ll_r M$, then $K \ll M$.*

Proof: Since $N \ll_r M$, by Proposition 1, $N \ll M$. Then by Lemma 1, $K \ll M$. □

Proposition 7. *Let M be an $R-$module and $K \leq N \leq M$. If $K \ll_r N$, then $K \ll M$.*

Proof: Since $K \ll_r N$, by Proposition 1, $K \ll N$. Then by Lemma 1, $K \ll M$. □

Proposition 8. *Let M be an $R-$module and $K \leq L \leq M$. If $L \ll_r M$, then $L/K \ll M/K$.*

Proof: Since $L \ll_r M$, by Proposition 1, $L \ll M$. Then by Lemma 1, $L/K \ll M/K$. □

Proposition 9. *Let M be an $R-$module and $K \leq L \leq M$. If $K \ll_r M$ and $L/K \ll_r M/K$, then $L \ll M$.*

Proof: Since $K \ll_r M$ and $L/K \ll_r M/K$, by Proposition 1, $K \ll M$ and $L/K \ll M/K$. Then by Lemma 1, $L \ll M$. □

Proposition 10. *Let M be an $R-$module $K_1 \leq L_1 \leq M$ and $K_2 \leq L_2 \leq M$. If $K_1 \ll_r L_1$ and $K_2 \ll_r L_2$, then $K_1 + K_2 \ll L_1 + L_2$.*

Proof: Since $K_1 \ll_r L_1$ and $K_2 \ll_r L_2$, by Proposition 1, $K_1 \ll L_1$ and $K_2 \ll L_2$. Then by Lemma 1, $K_1 + K_2 \ll L_1 + L_2$. □

Proposition 11. *Let M be an $R-$module $K_i \leq L_i \leq M$ for $i = 1, 2, ..., n$. If $K_i \ll_r L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

Proof: Since $K_i \ll_r L_i$ for every $i = 1, 2, ..., n$, by Proposition 1, $K_i \ll L_i$. Then by Lemma 1, $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$. □

Proposition 12. *Let M be an $R-$module and $M = U + V$ for $U, V \leq M$. If $U \cap V \ll_r V$, then V is a supplement of U in M.*

Proof: Since $U \cap V \ll_r V$, by Proposition 1, $U \cap V \ll V$. Then by definition V is a supplement of U in M. □

Proposition 13. *Let V be a supplement of U in M. If $K \ll_r M$, then V is a supplement of $U + K$ in M.*

Proof: Since $K \ll_r M$, by Proposition 1, $K \ll M$. Since V is a supplement of U in M, by Lemma 3, V is a supplement of $U + K$ in M. □

Proposition 14. *Let V be a supplement of U in M and $K \ll_r M$. Then $K \cap V \ll V$.*

Proof: Since $K \ll_r M$, by Proposition 1, $K \ll M$. Since V is a supplement of U in M, by Lemma 3, $K \cap V \ll V$. □

Proposition 15. *Let V be a supplement of U in M and $K \leq V$. If$K \ll_r M$. Then $K \ll V$.*

Proof: Since $K \ll_r M$, by Proposition 1, $K \ll M$. Since V is a supplement of U in M, by Lemma 3, $K \ll V$. □

Proposition 16. *Let $M = U + V$ and $U \cap V \ll_r V$. Then $\frac{V+L}{L}$ is a supplement of $\frac{U}{L}$ in $\frac{M}{L}$ for every $L \leq U$.*

Proof: Since $M = U + V$ and $U \cap V \ll_r V$, by Proposition 12, V is a supplement of U in M. Then by Lemma 3, $\frac{V+L}{L}$ is a supplement of $\frac{U}{L}$ in $\frac{M}{L}$ for every $L \leq U$. □

Proposition 17. *Let M be an $R-$module. If every proper submodule of M is r-small in M, then M is hollow.*

Proof: Since every proper submodule of M is r-small in M, by Proposition 1, every proper submodule of M is small in M. Then by definition M is hollow. □

Proposition 18. *Let M be an $R-$module. If every proper submodule of M is r-small in M, then M is supplemented.*

Proof: Since every proper submodule of M is r-small in M, by Proposition 17, M is hollow. Then by Lemma 4, M is supplemented. □

Proposition 19. *Let M be an $R-$module. If every proper submodule of M is r-small in M, then M is Rad-supplemented.*

Proof: Since every proper submodule of M is r-small in M, by Proposition 18, M is supplemented. Then by Lemma 5, M is Rad-supplemented. □

Proposition 20. *Let R be any ring. If every proper submodule of ${}_RR$ is r-small in ${}_RR$, then every finitely generated $R-$module is supplemented.*

Proof: Since every proper submodule of ${}_RR$ is r-small in ${}_RR$, by Proposition 18, ${}_RR$ is supplemented. Then by Lemma 4, every finitely generated $R-$module is supplemented. □

Proposition 21. *Let R be any ring. If every proper submodule of ${}_RR$ is r-small in ${}_RR$, then every finitely generated $R-$module is Rad-supplemented.*

Proof: Since every proper submodule of ${}_RR$ is r-small in ${}_RR$, by Proposition 20, every finitely generated $R-$module is supplemented. Then by Lemma 5, every finitely generated $R-$module is Rad-supplemented. □

Proposition 22. *Let M be an $R-$module and $M = U + V$ with $U, V \leq M$. If $Rx \ll_r V$ for every $x \in U \cap V$, then V is a Rad-supplement of U in M.*

Proof: Since $Rx \ll_r V$ for every $x \in U \cap V$, by Proposition 1, $Rx \ll V$. Then by Lemma 5, V is a Rad-supplement of U in M. □

Lemma 7. *Let $N \leq M$. If $N \ll M$ and $RadM$ is a supplement submodule in M, then $N \ll_r M$.*

Proof: Since $N \ll M$ and $RadM$ is a supplement submodule in M, by Lemma 3, $N = N \cap RadM \ll RadM$. Hence $N \ll_r M$, as desired. □

Corollary 1. *Let $N \leq M$. If $N \ll M$ and $RadM$ is a direct summand of M, then $N \ll_r M$.*

Proof: Clear from Lemma 7. □

Proposition 23. *If $N \ll_r M$, then $N \ll K$ for every maximal submodule K of M.*

Proof: Since $N \ll_r M$, $N \ll RadM$ and since $RadM \leq K$ for every maximal submodule K of M, by Lemma 1, $N \ll K$. □

Proposition 24. *Let M be an $R-$module and $N \leq K \leq M$. If $N \ll_r K$, then $N \ll_r M$.*

Proof: Since $N \ll_r K$, $N \ll RadK$. By Lemma 2, $RadK \leq RadM$. Then by Lemma 1, $N \ll RadM$ and $N \ll_r M$. □

Proposition 25. *Let M be an $R-$module and $N \leq K \leq M$. If $K \ll_r M$, then $N \ll_r M$.*

Proof: Since $K \ll_r M$, $K \ll RadM$. Then by Lemma 1, $N \ll RadM$. Hence $N \ll_r M$, as desired. □

Proposition 26. *Let M be an $R-$module and $N, K \leq M$. If $N \ll_r M$, then $(N+K)/K \ll_r M/K$.*

Proof: Since $N \ll_r M$, $N \ll RadM$. By Lemma 1, $(N+K)/K \ll (RadM + K)/K$. By Lemma 2, $(RadM + K)/K \leq Rad(M/K)$. Then by Lemma 1, $(N+K)/K \ll Rad(M/K)$. Hence $(N+K)/K \ll_r M/K$, as desired. □

Proposition 27. *Let $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_r M$, then $f(K) \ll_r N$.*

Proof: Since $K \ll_r M$, $K \ll RadM$. By Lemma 1 and Lemma 2, $f(K) \ll f(RadM) \leq RadN$. Hence $f(K) \ll_r N$, as desired. □

Lemma 8. *Let M be an $R-$module and $K, L \leq M$. If $N \ll_r K$ and $T \ll_r L$, then $N + T \ll_r K + L$.*

Proof: Since $N \ll_r K$ and $T \ll_r L$, $N \ll RadK$ and $T \ll RadL$. By Lemma 1 and Lemma 2, $T + N \ll RadK + RadL \leq Rad(K+L)$. Hence $N + T \ll_r K + L$, as desired. □

Corollary 2. *Let $M_1, M_2, ..., M_k \leq M$. If $N_1 \ll_r M_1$, $N_2 \ll_r M_2$, ..., $N_k \ll_r M_k$, then $N_1 + N_2 + ... + N_k \ll_r M_1 + M_2 + ... + M_k$.*

Proof: Clear from Lemma 8. □

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