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Finitely g-Supplemented Modules

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Abstract: Let M be an $R-$module. If every finitely generated submodule of M has a g-supplement in M, then M is called a finitely g-supplemented (or briefly fg-supplemented) module. In this work, some properties of these modules are investigated.

Keywords: Essential Submodules, g-Small Submodules, Supplemented Modules, g-Supplemented Modules.

1 Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A module M is said to be *hollow* if every proper submodule of M is small in M. M is said to be *local* if M has a proper submodule which contains all proper submodules. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [6] it is called an *e-small submodule* of M and denoted by $K \ll_e M$). A module M is said to be *generalized hollow* (briefly, *g-hollow*) if every proper submodule of M is g-small in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *finitely supplemented* (briefly, *f-supplemented*) if every finitely generated submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of essential maximal submodules of an R-module M is called a *generalized radical* (briefly, *g-radical* of M and denoted by $Rad_g M$ (in [6], it is denoted by $Rad_e M$). If M have no essential maximal submodules, then we denote $Rad_g M = M$. An $R-$module M is said to be *noetherian* if every submodule of M is finitely generated.

More details about supplemented modules are in [1]-[5]. More informations about g-small submodules and g-supplemented modules are in [2]-[3].

Lemma 1. *Let M be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof: See [5, 17.2]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*
(4) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(5) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(6) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

Proof: See [1, 2.2] and [5, 19.3]. □

Lemma 3. *Let M be an $R-$module. The following assertions are hold.*
(1) *Every small submodule in M is g-small in M.*
(2) *If $K \leq L \leq M$ and $L \ll_g M$, then $K \ll_g M$ and $L/K \ll_g M/K$.*
(3) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_g M$, then $f(K) \ll_g N$.*
(4) *If $K \ll_g M$, then $\frac{K+L}{L} \ll_g \frac{M}{L}$ for every $L \leq M$.*
(5) *If $L \leq M$ and $K \ll_g L$, then $K \ll_g M$.*
(6) *If $K_1, K_2, ..., K_n \ll_g M$, then $K_1 + K_2 + ... + K_n \ll_g M$.*
(7) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll_g L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll_g L_1 + L_2 + ... + L_n$.*

Proof: See [2]-[3]-[4]. □

Lemma 4. *Let M be an $R-$module. The following assertions are hold.*
(1) *$RadM \leq Rad_g M$.*
(2) *$Rad_g M = \sum\limits_{L \ll_g M} L$.*
(3) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(Rad_g M) \leq Rad_g N$.*
(4) *For $K, L \leq M$, $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K+L}{L}$.*
(5) *If $N \leq M$, then $Rad_g N \leq Rad_g M$.*
(6) *For $K, L \leq M$, $Rad_g K + Rad_g L \leq Rad_g (K + L)$.*
(7) *$Rx \ll_g M$ for every $x \in Rad_g M$.*

Proof: [2]-[3]-[4]. □

2 Finitely g-Supplemented Modules

Lemma 5. *Let V be a supplement of U in M. Then*
(1) *If $W + V = M$ for some $W \leq U$, then V is a supplement of W in M.*
(2) *If M is finitely generated, then V is also finitely generated.*
(3) *If U is a maximal submodule of M, then V is cyclic and $U \cap V = RadV$ is the unique maximal submodule of V.*
(4) *If $K \ll M$, then V is a supplement of $U + K$ in M.*
(5) *For $K \ll M$, $K \cap V \ll V$ and hence $RadV = V \cap RadM$.*
(6) *Let $K \leq V$. Then $K \ll V$ if and only if $K \ll M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a supplement of U/L in M/L.*

Proof: See [5, 41.1]. □

Lemma 6. *Let V be a g-supplement of U in M. Then*
(1) *If $W + V = M$ for some $W \leq U$, then V is a g-supplement of W in M.*
(2) *If every nonzero submodule of M is essential in M, then V is a supplement of U in M.*
(3) *If U is an essential maximal submodule of M, then $U \cap V = RadV$ is the unique essential maximal submodule of V.*
(4) *If $K \ll_g M$ and $U \trianglelefteq M$, then V is a g-supplement of $U + K$ in M.*
(5) *Let $U \trianglelefteq M$ and $K \ll_g M$. Then $K \cap V \ll_g V$ and hence $Rad_g V = V \cap Rad_g M$.*
(6) *Let $U \trianglelefteq M$ and $K \leq V$. Then $K \ll_g V$ if and only if $K \ll_g M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a g-supplement of U/L in M/L.*

Proof: See [2]-[3]-[4]. □

Lemma 7. *Let M be an $R-$module.*
(1) *If $M = U \oplus V$ then V is a supplement of U in M. Also U is a supplement of V in M.*
(2) *For $M_1, U \leq M$, if $M_1 + U$ has a supplement in M and M_1 is supplemented, then U also has a supplement in M.*
(3) *Let $M = M_1 + M_2$. If M_1 and M_2 are supplemented, then M is also supplemented.*
(4) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also supplemented.*
(5) *If M is supplemented, then M/L is supplemented for every $L \leq M$.*
(6) *If M is supplemented, then every homomorphic image of M is also supplemented.*
(7) *If M is supplemented, then $M/RadM$ is semisimple.*
(8) *Hollow and local modules are supplemented.*
(9) *If M is supplemented, then every finitely $M-$generated module is supplemented.*
(10) *${}_RR$ is supplemented if and only if every finitely generated $R-$module is supplemented.*

Proof: See [5, 41.2]. □

Lemma 8. *Let M be an $R-$module.*
(1) *If V is a supplement of U in M, then V is a g-suppement of U in M.*
(2) *If $M = U \oplus V$ then V is a g-supplement of U in M. Also U is a g-supplement of V in M.*
(3) *For $M_1, U \leq M$, if $M_1 + U$ has a g-supplement in M and M_1 is g-supplemented, then U also has a g-supplement in M.*
(4) *Let $M = M_1 + M_2$. If M_1 and M_2 are g-supplemented, then M is also g-supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is g-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-supplemented.*
(6) *If M is g-supplemented, then M/L is g-supplemented for every $L \leq M$.*
(7) *If M is g-supplemented, then every homomorphic image of M is also g-supplemented.*

(8) *If M is g-supplemented, then $M/Rad_g M$ is semisimple.*
(9) *Hollow, local and g-hollow modules are g-supplemented.*
(10) *If M is g-supplemented, then every finitely $M-$generated module is g-supplemented.*
(11) *${}_R R$ is g-supplemented if and only if every finitely generated $R-$module is g-supplemented.*

Proof: See [2]-[3]. □

Definition 1. *Let M be an $R-$module. If every finitely generated submodule of M has a g-supplement in M, then M is called a finitely g-supplemented (or briefly fg-supplemented) module.*

Clearly we can see that every f-supplemented module is fg-supplemented.

Lemma 9. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is f-supplemented.*
(2) *If M is f-supplemented and L is a finitely generated submodule of M, then M/L is also f-supplemented.*
(3) *If M is f-supplemented and L is a cyclic submodule of M, then M/L is also f-supplemented.*
(4) *If M is f-supplemented and $L \ll M$, then M/L is also f-supplemented.*
(5) *Let $f : M \longrightarrow N$ be an $R-$module epimorphism with Kef finitely generated. If M is f-supplemented, then N is also f-supplemented.*
(6) *Let $f : M \longrightarrow N$ be an $R-$module epimorphism with Kef cyclic. If M is f-supplemented, then N is also f-supplemented.*
(7) *Let $f : M \longrightarrow N$ be an $R-$module epimorphism with $Kef \ll M$. If M is f-supplemented, then N is also f-supplemented.*
(8) *If $RadM \ll M$, then every finitely generated submodule of $M/RadM$ is a direct summand of $M/RadM$.*
(9) *If $RadM$ is finitely generated, then every finitely generated submodule of $M/RadM$ is a direct summand of $M/RadM$.*
(10) *If M is noetherian and f-supplemented, then M is supplemented.*

Proof: See [5, 41.3]. □

Proposition 1. *Let M be an f-supplemented module. Then M is fg-supplemented.*

Proof: Let U be a finitely generated submodule of M. Since M is f-supplemented, then U has a supplement V in M. Since V is a supplement of U in M, by Lemma 8, V is a g-supplement of U in M. Hence M is fg-supplemented, as desired. □

Proposition 2. *Let M be an f-supplemented module and L be a finitely generated submodule of M. Then M/L is fg-supplemented.*

Proof: Since M is an f-supplemented module and L is a finitely generated submodule of M, by Lemma 9, M/L is f-supplemented. Then by Proposition 1, M/L is fg-supplemented. □

Proposition 3. *Let M be an f-supplemented module and L be a cyclic submodule of M. Then M/L is fg-supplemented.*

Proof: Since M is an f-supplemented module and L is a cyclic submodule of M, by Lemma 9, M/L is f-supplemented. Then by Proposition 1, M/L is fg-supplemented. □

Proposition 4. *Let M be an f-supplemented module and $L \ll M$. Then M/L is fg-supplemented.*

Proof: Since M is an f-supplemented module and $L \ll M$, by Lemma 9, M/L is f-supplemented. Then by Proposition 1, M/L is fg-supplemented. □

Proposition 5. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and Kef be finitely generated. If M is f-supplemented, then N is fg-supplemented.*

Proof: Since M is f-supplemented and Kef is finitely generated, by Lemma 9, N is f-supplemented. Then by Proposition 9, N is fg-supplemented. □

Proposition 6. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and Kef be cyclic. If M is f-supplemented, then N is fg-supplemented.*

Proof: Since M is f-supplemented and Kef is cyclic, by Lemma 9, N is f-supplemented. Then by Proposition 9, N is fg-supplemented. □

Proposition 7. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $Kef \ll M$. If M is f-supplemented, then N is fg-supplemented.*

Proof: Since M is f-supplemented and $Kef \ll M$, by Lemma 9, N is f-supplemented. Then by Proposition 9, N is fg-supplemented. □

Proposition 8. *Every g-supplemented module is fg-supplemented.*

Proof: Clear from definitions. □

Proposition 9. *Every hollow module is fg-supplemented.*

Proof: By Lemma 8, every hollow module is g-supplemented. Then by Proposition 8, every hollow module is fg-supplemented. □

Proposition 10. *Every local module is fg-supplemented.*

Proof: By Lemma 8, every local module is g-supplemented. Then by Proposition 8, every local module is fg-supplemented. □

Proposition 11. *Every g-hollow module is fg-supplemented.*

Proof: By Lemma 8, every g-hollow module is g-supplemented. Then by Proposition 8, every g-hollow module is fg-supplemented. □

Proposition 12. *Let $M = M_1 + M_2$. If M_1 and M_2 are g-supplemented, then M is fg-supplemented.*

Proof: Since M_1 and M_2 are g-supplemented, by Lemma 8, M is g-supplemented. Then by Proposition 8, M is fg-supplemented. □

Proposition 13. *Let $M = M_1 + M_2 + ... + M_n$. If M_i is g-supplemented for every $i = 1, 2, ..., n$, then M is fg-supplemented.*

Proof: Since M_i is g-supplemented for every $i = 1, 2, ..., n$, by Lemma 8, M is g-supplemented. Then by Proposition 8, M is fg-supplemented. □

Proposition 14. *Let M be an $R-$module and $L \leq M$. If M is g-supplemented, then M/L is fg-supplemented.*

Proof: Since M is g-supplemented, by Lemma 8, M/L is g-supplemented. Then by Proposition 8, M/L is fg-supplemented. □

Proposition 15. *Let M be a g-supplemented module. Then every homomorphic image of M is fg-supplemented.*

Proof: Since M is g-supplemented, by Lemma 8, every homomorphic image of M is g-supplemented. Then by Proposition 8, every homomorphic image of M is fg-supplemented. □

Proposition 16. *Let M be a g-supplemented module. Then every finitely $M-$generated module is fg-supplemented.*

Proof: Since M is g-supplemented, by Lemma 8, every finitely $M-$generated module is g-supplemented. Then by Proposition 8, every finitely $M-$generated module is fg-supplemented. □

Proposition 17. *Let ${}_RR$ be g-supplemented. Then every finitely generated $R-$module is fg-supplemented.*

Proof: Since ${}_RR$ is g-supplemented, by Lemma 8, every finitely generated $R-$module is g-supplemented. Then by Proposition 8, every finitely generated $R-$module is fg-supplemented. □

Proposition 18. *Every supplemented module is fg-supplemented.*

Proof: By Lemma 8, every supplemented module is g-supplemented. Then by Proposition 8, every supplemented module is fg-supplemented. □

Proposition 19. *Let $M = M_1 + M_2$. If M_1 and M_2 are supplemented, then M is fg-supplemented.*

Proof: Since M_1 and M_2 are supplemented, by Lemma 7, M is supplemented. Then by Proposition 18, M is fg-supplemented. □

Proposition 20. *Let $M = M_1 + M_2 + ... + M_n$. If M_i is supplemented for every $i = 1, 2, ..., n$, then M is fg-supplemented.*

Proof: Since M_i is supplemented for every $i = 1, 2, ..., n$, by Lemma 7, M is supplemented. Then by Proposition 18, M is fg-supplemented. □

Proposition 21. *Let M be an $R-$module and $L \leq M$. If M is supplemented, then M/L is fg-supplemented.*

Proof: Since M is supplemented, by Lemma 7, M/L is supplemented. Then by Proposition 18, M/L is fg-supplemented. □

Proposition 22. *Let M be a supplemented module. Then every homomorphic image of M is fg-supplemented.*

Proof: Since M is supplemented, by Lemma 7, every homomorphic image of M is supplemented. Then by Proposition 18, every homomorphic image of M is fg-supplemented. □

Proposition 23. *Let M be a supplemented module. Then every finitely $M-$generated module is fg-supplemented.*

Proof: Since M is supplemented, by Lemma 7, every finitely $M-$generated module is supplemented. Then by Proposition 18, every finitely $M-$generated module is fg-supplemented. □

Proposition 24. *Let ${}_RR$ be supplemented. Then every finitely generated $R-$module is fg-supplemented.*

Proof: Since ${}_RR$ is supplemented, by Lemma 7, every finitely generated $R-$module is supplemented. Then by Proposition 18, every finitely generated $R-$module is fg-supplemented. □

Proposition 25. *Let M be an fg-supplemented $R-$module. If M is noetherian, then M is g-supplemented.*

Proof: Let $U \leq M$. Since M is noetherian, U is finitely generated and since M is fg-supplemented, U has a g-supplement in M. Hence M is g-supplemented. □

Lemma 10. *Let M be an fg-supplemented $R-$module and N be a finitely generated submodule of M. Then M/N is fg-supplemented.*

Proof: Let U/N be a finitely generated submodule of M/N. Since U/N finitely generated, there exists a finitely generated submodule K of M such that $U = K + N$. Since K and N are finitely generated, $U = K + N$ is also finitely generated. By hypothesis, U has a g-supplement V in M. Then by [2, Lemma 4], $(V + N)/N$ is a g-supplement of U/N in M/N. Hence M/N is fg-supplemented. □

Corollary 1. *Let M be an fg-supplemented $R-$module and N be a cyclic submodule of M. Then M/N is fg-supplemented.*

Proof: Clear from Lemma 10. □

Corollary 2. *Let $f : M \longrightarrow N$ be an $R-$module epimomorphism and Kef be finitely generated. If M is fg-supplemented, then N is also fg-supplemented.*

Proof: Since M is fg-supplemented and Kef is finitely generated, by Lemma 10, M/Kef is fg-supplemented. Then by $M/Kef \cong N$, N is also fg-supplemented. □

Corollary 3. *Let $f : M \longrightarrow N$ be an $R-$module epimomorphism with cyclic kernel. If M is fg-supplemented, then N is also fg-supplemented.*

Proof: Clear from Corollary 2. □

Lemma 11. *Let M be an fg-supplemented $R-$module and $N \ll M$. Then M/N is fg-supplemented.*

Proof: Let U/N be a finitely generated submodule of M/N. Then there exists a finitely generated submodule K of M such that $U = K + N$. Since M is fg-supplemented, K has a g-supplement V in M. Here $M = K + V$ and $K \cap V \ll_g V$. Since $K \leq U$, $M = K + V = U + V$. Let $M = U + T$ with $T \trianglelefteq V$. Then $M = U + T = K + N + T$ and since $N \ll M$, $K + T = M$. Since V is a g-supplement of K in M and $T \trianglelefteq V$, by definition, $T = V$. Hence V is a g-supplement of U in M. By [2, Lemma 4], $(V + N)/N$ is a g-supplement of U/N in M/N. Hence M/N is fg-supplemented. □

Corollary 4. *Let $f : M \longrightarrow N$ be an $R-$module epimomorphism with small kernel. If M is fg-supplemented, then N is also fg-supplemented.*

Proof: Since M is fg-supplemented and $Kef \ll M$, by Lemma 11, M/Kef is fg-supplemented. Then by $M/Kef \cong N$, N is also fg-supplemented. □

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