Conference Proceeding Science and Technology, 3(1), 2020, 77–81

Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020).

Cofinitely eg-Supplemented Modules

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

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Abstract: Let M be an $R-$module. If every cofinite essential submodule of M has a g-supplement in M, then M is called a cofinitely essential g-supplemented (or briefly cofinitely eg-supplemented) module. In this work, some properties of these modules are investigated.

Keywords: Cofinite Submodules, Essential Submodules, g-Small Submodules, g-Supplemented Modules.

1 INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If there exists a submodule L of M such that $M = N + L$ and $N \cap L = 0$, then N is called a *direct summand* of M and denoted by $M = N \oplus L$. A submodule U of an $R-$module M is called a *cofinite submodule* of M if M/U is finitely generated. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A module M is said to be *hollow* if every proper submodule of M is small in M. M is said to be *local* if M has a proper submodule which contains all proper submodules. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$. A module M is said to be *generalized hollow* (briefly, *g-hollow*) if every proper submodule of M is g-small in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is said to be *cofinitely essential supplemented* (briefly, *cofinitely e-supplemented*) if every cofinite essential submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M. M is said to be *cofinitely g-supplemented* if every cofinite submodule of M has a g-supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of essential maximal submodules of an R-module M is called a *generalized radical* (briefly, *g-radical*) of M and denoted by $Rad_g M$. If M have no essential maximal submodules, then we denote $Rad_g M = M$.

More details about supplemented modules are in [2]-[10]. More informaitons about cofinitely supplemented modules are in [1]. More details about essential supplemented modules are in [9]. More details about cofinitely essential supplemented modules are in [4]. More informations about g-small submodules and g-supplemented modules are in [5]-[6]. The definition of cofinitely g-supplemented modules and more informations about these modules are in [3]. The definition of essential g-supplemented modules and some properties of them are in [7].

Lemma 1. *Let M be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof: See [10, 17.3]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*

(3) *If* $K \ll M$*, then* $\frac{K+L}{L} \ll \frac{M}{L}$ *for every* $L \leq M$.
(4) *If* $L \leq M$ *and* $K \ll L$*, then* $K \ll M$.
(5) *If* $K_1, K_2, ..., K_n \ll M$*, then* $K_1 + K_2 + ... + K_n \ll M$.
(6) *Let* $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. *If* $K_i \ll L_i$ *for every* $i = 1, 2, ..., n$*, then* $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.

Proof: See [2, 2.2] and [10, 19.3]. □

Lemma 3. *Let* M *be an* $R-$*module. The following assertions are hold.*
(1) *Every small submodule in* M *is g-small in* M.
(2) *If* $K \leq L \leq M$ *and* $L \ll_g M$*, then* $K \ll_g M$ *and* $L/K \ll_g M/K$.
(3) *Let* N *be an* $R-$*module and* $f : M \longrightarrow N$ *be an* $R-$*module homomorphism. If* $K \ll_g M$*, then* $f(K) \ll_g N$.
(4) *If* $K \ll_g M$*, then* $\frac{K+L}{L} \ll_g \frac{M}{L}$ *for every* $L \leq M$.
(5) *If* $L \leq M$ *and* $K \ll_g L$*, then* $K \ll_g M$.
(6) *If* $K_1, K_2, ..., K_n \ll_g M$*, then* $K_1 + K_2 + ... + K_n \ll_g M$.
(7) *Let* $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. *If* $K_i \ll_g L_i$ *for every* $i = 1, 2, ..., n$*, then* $K_1 + K_2 + ... + K_n \ll_g L_1 + L_2 + ... + L_n$.

Proof: See [5]-[6]-[8]. □

Lemma 4. *Let* M *be an* $R-$*module. The following assertions are hold.*
(1) $RadM \leq Rad_g M$.
(2) $Rad_g M = \sum_{L \ll_g M} L$.
(3) *Let* N *be an* $R-$*module and* $f : M \longrightarrow N$ *be an* $R-$*module homomorphism. Then* $f(Rad_g M) \leq Rad_g N$.
(4) *For* $K, L \leq M$, $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K+L}{L}$. *If* $L \leq Rad_g K$*, then* $Rad_g K/L \leq Rad(K/L)$.
(5) *If* $L \leq M$*, then* $Rad_g L \leq Rad_g M$.
(6) *For* $K, L \leq M$, $Rad_g K + Rad_g L \leq Rad_g (K + L)$.
(7) $Rx \ll_g M$ *for every* $x \in Rad_g M$.

Proof: See [5]-[6]-[8]. □

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Lemma 5. *Let* V *be a supplement of* U *in* M. *Then*
(1) *If* $W + V = M$ *for some* $W \leq U$*, then* V *is a supplement of* W *in* M.
(2) *If* M *is finitely generated, then* V *is also finitely generated.*
(3) *If* U *is a maximal submodule of* M*, then* V *is cyclic and* $U \cap V = RadV$ *is the unique maximal submodule of* V.
(4) *If* $K \ll M$*, then* V *is a supplement of* $U + K$ *in* M.
(5) *For* $K \ll M$, $K \cap V \ll V$ *and hence* $RadV = V \cap RadM$.
(6) *Let* $K \leq V$. *Then* $K \ll V$ *if and only if* $K \ll M$.
(7) *For* $L \leq U$, $\frac{V+L}{L}$ *is a supplement of* U/L *in* M/L.

Proof: See [10, 41.1]. □

Lemma 6. *Let* V *be a g-supplement of* U *in* M. *Then*
(1) *If* $W + V = M$ *for some* $W \leq U$*, then* V *is a g-supplement of* W *in* M.
(2) *If every nonzero submodule of* M *is essential in* M*, then* V *is a supplement of* U *in* M.
(3) *If* U *is an essential maximal submodule of* M*, then* $U \cap V = RadV$ *is the unique essential maximal submodule of* V.
(4) *If* $K \ll_g M$ *and* $U \trianglelefteq M$*, then* V *is a g-supplement of* $U + K$ *in* M.
(5) *Let* $U \trianglelefteq M$ *and* $K \ll_g M$. *Then* $K \cap V \ll_g V$ *and hence* $Rad_g V = V \cap Rad_g M$.
(6) *Let* $U \trianglelefteq M$ *and* $K \leq V$. *Then* $K \ll_g V$ *if and only if* $K \ll_g M$.
(7) *For* $L \leq U$, $\frac{V+L}{L}$ *is a g-supplement of* U/L *in* M/L.

Proof: See [5]-[6]-[8]. □

Lemma 7. *Let* M *be an* $R-$*module.*
(1) *If* $M = U \oplus V$ *then* V *is a supplement of* U *in* M. *Also* U *is a supplement of* V *in* M.
(2) *For* $M_1, U \leq M$*, if* $M_1 + U$ *has a supplement in* M *and* M_1 *is supplemented, then* U *also has a supplement in* M.
(3) *Let* $M = M_1 + M_2$. *If* M_1 *and* M_2 *are supplemented, then* M *is also supplemented.*
(4) *Let* $M_i \leq M$ *for* $i = 1, 2, ..., n$. *If* M_i *is supplemented for every* $i = 1, 2, ..., n$*, then* $M_1 + M_2 + ... + M_n$ *is also supplemented.*
(5) *If* M *is supplemented, then* M/L *is supplemented for every* $L \leq M$.
(6) *If* M *is supplemented, then every homomorphic image of* M *is also supplemented.*
(7) *If* M *is supplemented, then* $M/RadM$ *is semisimple.*
(8) *Hollow and local modules are supplemented.*
(9) *If* M *is supplemented, then every finitely* $M-$*generated module is supplemented.*
(10) ${}_RR$ *is supplemented if and only if every finitely generated* $R-$*module is supplemented.*

Proof: See [10, 41.2]. □

Lemma 8. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is essential supplemented.*
(2) *For $M_1 \leq M$ and $U \trianglelefteq M$, if $M_1 + U$ has a supplement in M and M_1 is essential supplemented, then U also has a supplement in M.*
(3) *Let $M = M_1 + M_2$. If M_1 and M_2 are essential supplemented, then M is also essential supplemented.*
(4) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is essential supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also essential supplemented.*
(5) *If M is essential supplemented, then M/L is essential supplemented for every $L \leq M$.*
(6) *If M is essential supplemented, then every homomorphic image of M is also essential supplemented.*
(7) *If M is essential supplemented, then $M/RadM$ have no proper essential submodules.*
(8) *Hollow and local modules are essential supplemented.*
(9) *If M is essential supplemented, then every finitely $M-$generated module is essential supplemented.*
(10) *${}_RR$ is essential supplemented if and only if every finitely generated $R-$module is essential supplemented.*

Proof: See [9]. □

Lemma 9. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is cofinitely supplemented.*
(2) *If M is finitely generated and cofinitely supplemented, then M is supplemented.*
(3) *For $M_1 \leq M$ and U cofinite submodule of M, if $M_1 + U$ has a supplement in M and M_1 is cofinitely supplemented, then U also has a supplement in M.*
(4) *Let $M = \sum_{i \in I} M_i$. If M_i is cofinitely supplemented for every $i \in I$, then M is also cofinitely supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely supplemented.*
(6) *If M is cofinitely supplemented, then M/L is cofinitely supplemented for every $L \leq M$.*
(7) *If M is cofinitely supplemented, then every homomorphic image of M is also cofinitely supplemented.*
(8) *If M is cofinitely supplemented, then every cofinite submodule of $M/RadM$ is a direct summand of $M/RadM$.*
(9) *Hollow and local modules are cofinitely supplemented.*
(10) *If M is cofinitely supplemented, then every $M-$generated module is cofinitely supplemented.*
(11) *${}_RR$ is supplemented if and only if every generated $R-$module is cofinitely supplemented.*

Proof: See [1]. □

Lemma 10. *Let M be an $R-$module.*
(1) *If M is essential supplemented, then M is cofinitely essential supplemented.*
(2) *If M is supplemented, then M is cofinitely essential supplemented.*
(3) *If M is finitely generated and cofinitely essential supplemented, then M is essential supplemented.*
(4) *For $M_1 \leq M$ and U cofinite essential submodule of M, if $M_1 + U$ has a supplement in M and M_1 is cofinitely essential supplemented, then U also has a supplement in M.*
(5) *Let $M = \sum_{i \in I} M_i$. If M_i is cofinitely essential supplemented for every $i \in I$, then M is also cofinitely essential supplemented.*
(6) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely essential supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely essential supplemented.*
(7) *If M is cofinitely essential supplemented, then M/L is cofinitely essential supplemented for every $L \leq M$.*
(8) *If M is cofinitely essential supplemented, then every homomorphic image of M is also cofinitely essential supplemented.*
(9) *If M is cofinitely essential supplemented, then $M/RadM$ have no proper essential submodules.*
(10) *Hollow and local modules are cofinitely essential supplemented.*
(11) *If M is cofinitely essential supplemented, then every $M-$generated module is cofinitely essential supplemented.*
(12) *${}_RR$ is essential supplemented if and only if every generated $R-$module is cofinitely essential supplemented.*

Proof: See [4]. □

Lemma 11. *Let M be an $R-$module.*
(1) *If V is a supplement of U in M, then V is a g-suppement of U in M.*
(2) *If $M = U \oplus V$ then V is a g-supplement of U in M. Also U is a g-supplement of V in M.*
(3) *For $M_1, U \leq M$, if $M_1 + U$ has a g-supplement in M and M_1 is g-supplemented, then U also has a g-supplement in M.*
(4) *Let $M = M_1 + M_2$. If M_1 and M_2 are g-supplemented, then M is also g-supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is g-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-supplemented.*
(6) *If M is g-supplemented, then M/L is g-supplemented for every $L \leq M$.*
(7) *If M is g-supplemented, then every homomorphic image of M is also g-supplemented.*
(8) *If M is g-supplemented, then M/Rad_gM is semisimple.*
(9) *Hollow, local and g-hollow modules are g-supplemented.*
(10) *If M is g-supplemented, then every finitely $M-$generated module is g-supplemented.*
(11) *${}_RR$ is g-supplemented if and only if every finitely generated $R-$module is g-supplemented.*
(12) *If M is g-supplemented and every nonzero submodule of M is essential in M, then M is supplemented.*

Proof: See [5]-[6]. □

Lemma 12. *Let M be an $R-$module.*
(1) *If M is g-supplemented, then M is cofinitely g-supplemented.*
(2) *If M is supplemented, then M is cofinitely g-supplemented.*

(3) *If M is cofinitely supplemented, then M is cofinitely g-supplemented.*
(4) *If M is cofinitely g-supplemented and every nonzero submodule of M is essential in M, then M is cofinitely supplemented.*
(5) *If M is finitely generated and cofinitely g-supplemented, then M is g-supplemented.*
(6) *For $M_1 \leq M$ and U cofinite submodule of M, if $M_1 + U$ has a g-supplement in M and M_1 is cofinitely g-supplemented, then U also has a g-supplement in M.*
(7) *Let $M = \sum_{i \in I} M_i$. If M_i is cofinitely g-supplemented for every $i \in I$, then M is also cofinitely g-supplemented.*
(8) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely g-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely g-supplemented.*
(9) *If M is cofinitely g-supplemented, then M/L is cofinitely g-supplemented for every $L \leq M$.*
(10) *If M is cofinitely g-supplemented, then every homomorphic image of M is also cofinitely g-supplemented.*
(11) *If M is cofinitely g-supplemented, then every cofinite submodule of $M/Rad_g M$ is a direct summand of $M/Rad_g M$.*
(12) *Hollow, g-hollow and local modules are cofinitely g-supplemented.*
(13) *If M is cofinitely g-supplemented, then every $M-$generated module is cofinitely g-supplemented.*
(14) *${}_R R$ is g-supplemented if and only if every $R-$module is cofinitely g-supplemented.*

Proof: See [3]. □

Definition 1. *Let M be an $R-$module. If every cofinite essential submodule of M has a g-supplement in M, then M is called a cofinitely essential g-supplemented (or briefly cofinitely eg-supplemented) module.*

Proposition 1. *Every cofinitely essential supplemented module is cofinitely eg-supplemented.*

Proof: Let M be a cofinitely essential supplemented module and U be a cofinite essential submodule of M. Then U has a supplement V in M. Here $M = U + V$ and $U \cap V \ll V$. Since $U \cap V \ll V$, $U \cap V \ll_g V$. Hence V is a g-supplement of U in M. Therefore, M is cofinitely eg-supplemented. □

Lemma 13. *Every essential g-supplemented module is cofinitely eg-supplemented.*

Proof: Let M be an essential g-supplemented module and U be a cofinite essential submodule of M. Since M is essential g-supplemented and $U \trianglelefteq M$, U has a g-supplement in M. Hence every cofinite essential submodule of M has a g-supplement in M and M is cofinitely eg-supplemented. □

Corollary 1. *Every g-supplemented module is cofinitely eg-supplemented.*

Proof: Since every g-supplemented module is essential g-supplemented, by Lemma 13, every g-supplemented module is cofinitely eg-supplemented. □

Lemma 14. *Every cofinitely g-supplemented module is cofinitely eg-supplemented.*

Proof: Let M be a cofinitely g-supplemented module and U be a cofinite essential submodule of M. Since M is cofinitely g-supplemented and U is a cofinite submodule of M, U has a g-supplement in M. Hence every cofinite essential submodule of M has a g-supplement in M and M is cofinitely eg-supplemented. □

Corollary 2. *Let $M = \sum_{i \in I} M_i$. If M_i is cofinitely g-supplemented for every $i \in I$, then M is cofinitely eg-supplemented.*

Proof: Since M_i is cofinitely g-supplemented for every $i \in I$, by [3, Lemma 2.6], $M = \sum_{i \in I} M_i$ cofinitely g-supplemented. Then by Lemma 14, M is cofinitely eg-supplemented. □

Corollary 3. *Let M be a cofinitely g-supplemented module. Then every $M-$generated module is cofinitely eg-supplemented.*

Proof: Clear from Corollary 2. □

Corollary 4. *Let R be a ring. If ${}_R R$ is g-supplemented, then every $R-$module is cofinitely eg-supplemented.*

Proof: Since ${}_R R$ is g-supplemented, ${}_R R$ is cofinitely g-supplemented. Then by Corollary 3, every $R-$module is cofinitely eg-supplemented. □

Proposition 2. *Let M be an cofinitely eg-supplemented module. If every nonzero submodule of M is essential in M, then M is cofinitely g-supplemented.*

Proof: Let U be a cofinite submodule of M. If $U = 0$, then M is a g-supplement of U in M. Let $U \neq 0$. Then $U \trianglelefteq M$ and since M is cofinitely eg-supplemented, U has a g-supplement in M. Hence M is cofinitely g-supplemented. □

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