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Existence of weak solutions for a nonlinear parabolic equations by Topological degree

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Abstract

We prove the existence of a weak solution for the nonlinear parabolic initial boundary value problem associated to the equation

$$u_t - div\ a(x,t,u,\nabla u) = f(x,t),$$

by using the Topological degree theory for operators of the form $L+S$, where L is a linear densely defined maximal monotone map and S is a bounded demicontinuous map of class (S_+) with respect to the domain of L.

We will therefore use the Topological degree theory to study a parabolic equation in the space $L^p(0,T;W_0^{1,p}(\Omega))$ where the exponent p is not necessarily equal to 2 ($p \geq 2$).

Keywords: Nonlinear parabolic equations, Topological degree, weak solution, map of class (S_+).
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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Fixing a final time $T > 0$, we denote by Q the cylinder $\Omega \times (0,T)$ and $\Gamma = \partial\Omega \times (0,T)$ its lateral surface. We consider the following nonlinear parabolic initial-boundary problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\ a(x,t,u,\nabla u) = f(x,t) & \text{in } Q, \\ u(x,t) = 0 & \text{in } \Gamma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

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where the terme $-div\ a(x, t, u, \nabla u)$ is a Leray-Lions operator acting from

$$\mathcal{H} := L^p(0, T; W_0^{1,p}(\Omega)), \ \ (p \geq 2).$$

to its dual $\mathcal{H}^* = L^q(0, T; W^{-1,q}(\Omega))$ where $\frac{1}{p} + \frac{1}{q} = 1$. The right-hand side f is assumed to belong to $L^q(Q)$.

Lions prove in [10] that there exists at least a solution $u \in \mathcal{H}$ of the following nonlinear parabolic Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\ a(x, t, u, \nabla u) = f(x, t), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & (x, t) \in \Gamma, \end{cases} \tag{2}$$

where $-div\ a(x, t, u, \nabla u)$ is a pseudomonotone, coercive, uniformly elliptic operator acting from $\mathcal{H}$ to $\mathcal{H}^*$ and f is a measurable function on Q that belongs to $\mathcal{H}^*$. Afterwards, Boccardo et al. in [6] concerned with certain results of existence and regularity for solutions of (2), according to the summability of the data f.

Asfaw in [1] proved the existence of at least one weak solution for problem (1) in $L^2(0, T; W_0^{1,2}(\Omega))$ using the topological degree theory.

In this paper, we will combine and generalize this works: using the topological degree and proving the existence of at lest one weak solution in the space $\mathcal{H}$ (p is not necessarily equal 2).

The theory of topological degree has been widely used in the study of nonlinear differential equations as a very effective tool, often those of elliptical type. For more details about the history of this theory and its use, the reader can refer, for example, to [1, 2, 3, 4, 5, 8, 9, 11] and references therein.

The rest of this paper is organized as follows. In Section 2, we state some mathematical preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators and then the associated topological degree. We will prove the main results in Section 4.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary. Let $p \geq 2$ and $q = \frac{p}{p-1}$. We denote by $L^p(\Omega)$ the Banach space of all measurable functions u defined in Ω for which

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p \, dx \Big)^{1/p} < \infty.$$

Let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the Sobolev space

$$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega) \Big\},$$

equipped with the norm

$$\|u\|_{1,p} = \Big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

Due to the Poincaré inequality, the norm $\|\cdot\|_{1,p}$ on $W_0^{1,p}(\Omega)$ is equivalent to the norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ given by

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} \text{ for } u \in W_0^{1,p}(\Omega).$$

Note that the Sobolev space $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space and the embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ is compact (see [12].)

Next, we consider the following functional space

$$\mathcal{H} := L^p(0, T; W_0^{1,p}(\Omega)), \ \ (T > 0)$$

which is a separable and reflexive Banach space endowed with the norm

$$|u|_{\mathcal{H}} = \Big(\int_0^T \|u\|^p_{W_0^{1,p}(\Omega)} \, dt \Big)^{1/p}$$

or, by Poincaré inequality, the equivalent norm

$$\|u\|_{\mathcal{H}} = \Big(\int_0^T \|\nabla u\|^p_{L^p(\Omega)} \, dt \Big)^{1/p}.$$

3. Classes of mappings and Topological degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle . \, , \, . \rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

We consider a multi-values mapping T from X to 2^{X^*} (i.e., with values subsets of X^*). With each such map, we associate its graph

$$G(T) = \{(u, w) \in X \times X^* : w \in T(u)\}.$$

The multi-values mapping T is said to be *monotone* if for any pair of elements $(u_1, w_1), (u_2, w_2)$ in $G(T)$, we have the inequality

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geq 0.$$

T is said to be *maximal monotone* if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from X to 2^{X^*}. An equivalent version of the last clause is that for any $(u_0, w_0) \in X \times X^*$ for which $\langle w_0 - w, u_0 - u \rangle \geq 0$, for all $(u, w) \in G(T)$, we have $(u_0, w_0) \in G(T)$.

We recall that a mapping $T : D(T) \subset X \to Y$ is *demicontinuous* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $T(u_n) \rightharpoonup T(u)$. T is said to be *of class* (S_+) if for any $(u_n) \subset D(T)$ with $u_n \rightharpoonup u$ and $limsup \langle Tu_n, u_n - u \rangle \leq 0$, it follows that $u_n \to u$.

Let L be a linear maximal monotone map from $D(L) \subset X$ to X^* such that $D(L)$ is dense in X. For each open and bounded subset G on X, we consider the following classes of operators:

$$\begin{aligned} \mathcal{F}_G \ &:= \ \{L + S : \bar{G} \cap D(L) \to X^* \mid S \text{ is bounded, demicontinuous map} \\ &\quad\ \text{of class } (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\}, \\ \mathcal{H}_G \ &:= \ \{L + S(t) : \bar{G} \cap D(L) \to X^* \mid S(t) \text{ is a bounded homotopy of class} \\ &\quad\ (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\}. \end{aligned}$$

Note that the class $\mathcal{H}_G$ (class of *admissible homotopies*) includes all affine homotopies $L + (1 - t)S_1 + tS_2$ with $(L + S_i) \in \mathcal{F}_G, i = 1, 2$.

We introduce the topological degree for the class $\mathcal{F}_G$ due to Berkovits and Mustonen [4].

Theorem 3.1. *Let L a linear maximal monotone densely defined map from $D(L) \subset X$ to X^*. There exists a topological degree function*

$$d : \{(F, G, h) : F \in \mathcal{F}_G, G \textit{ an open bounded subset in } X, h \notin F(\partial G \cap D(L))\} \to \mathbb{Z}$$

satisfying the following properties:

1. *(Existence) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G \cap D(L)$.*
2. *(Additivity) If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F[(\bar{G} \setminus (G_1 \cup G_2)) \cap D(L)]$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

3. *(Invariance under homotopies) If* $F(t) \in \mathcal{H}_G$ *and* $h(t) \notin F(t)(\partial G \cap D(L))$ *for all* $t \in [0,1]$*, where* $h(t)$ *is a continuous curve in* X^**, then*

$$d(F(t), G, h(t)) = constant, \forall t \in [0,1].$$

4. *(Normalization)* $L + J$ *is a normalising map, where* J *is the duality mapping of* X *into* X^**, that is,*

$$d(L + J, G, h) = 1, \text{ whenever } h \in (L + J)(G \cap D(L)).$$

Lemma 3.1. *Let* $L + S \in \mathcal{F}_X$ *and* $h \in X^*$*. Assume that there exists* $R > 0$ *such that*

$$\langle Lu + Su - h, u\rangle > 0, \tag{3}$$

for all $u \in \partial B_R(0) \cap D(L)$*. Then*

$$(L + S)(D(L)) = X^*.$$

Proof. Let $\varepsilon > 0$, $t \in [0,1]$ and

$$F_\varepsilon(t, u) = Lu + (1 - t)Ju + t(Su + \varepsilon Ju - h).$$

Since $0 \in L(0)$ and by using the boundary condition (3), we see that

$$\begin{aligned} \langle F_\varepsilon(t, u), u\rangle &= \langle t(Lu + Su - h, u\rangle + \langle (1 - t)Lu + (1 - t + \varepsilon)Ju, u\rangle \\ &\geq \langle (1 - t)Lu + (1 - t + \varepsilon)Ju, u\rangle \\ &= (1 - t)\langle Lu, u\rangle + (1 - t + \varepsilon)\langle Ju, u\rangle \\ &\geq (1 - t + \varepsilon) \parallel u \parallel^2 = (1 - t + \varepsilon)R^2 > 0. \end{aligned}$$

That is $0 \notin F_\varepsilon(t, u)$. Since J and $S + \varepsilon J$ are continuous, bounded and of type (S_+), $\{F_\varepsilon(t, \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Therefore, by invariance under homotopy and normalisation, we obtain

$$d(F_\varepsilon(t, \cdot), B_R(0), 0) = d(L + J, B_R(0), 0) = 1.$$

Hence, there exists $u_\varepsilon \in D(L)$ such that $0 \in F_\varepsilon(t, \cdot)$. Letting $\varepsilon \to 0^+$ and $t = 1$, we have $h \in Lu + Su$ for some $u \in D(L)$. Since $h \in X^*$ is arbitrary, we conclude that $(L + S)(D(L)) = X^*$. □

4. Basic assumptions and Main result

We assume that $a(x, t, \eta, \xi) : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function; that is $(x, t) \mapsto a(x, t, \eta, \xi)$ is measurable for almost all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(\eta, \xi) \mapsto a(x, t, \eta, \xi)$ is continuous for almost all $(x, t) \in \Omega \times [0, T]$, such that there exist $c_1 > 0$, $c_2 > 0$ and $k \in L^q(Q)$ such that

$$|a(x, t, \eta, \xi)| \leq k(x, t) + c_1(|\eta|^{p-1} + |\xi|^{p-1}), \tag{4}$$

$$\Big(a(x, t, \eta, \xi) - a(x, t, \eta, \xi')\Big) \cdot \Big(\xi - \xi'\Big) > 0, \tag{5}$$

$$a(x, t, \eta, \xi) \cdot \xi \geq c_2|\xi|^p, \tag{6}$$

for all $(x, t) \in Q$ and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Lemma 4.1. *Under assumptions (4)–(6), the operator* S *defined from* $\mathcal{H}$ *to* $\mathcal{H}^*$ *by*

$$\langle Su, v\rangle = \int_Q a(x, t, u, \nabla u) \cdot \nabla v dxdt, \text{ for all } u, v \in \mathcal{H}$$

is bounded, continuous and of class (S_+)*.*

Proof. • Let's show that the operator S is bounded.
By using the Hölder's inequality, we have for all $u, v \in \mathcal{H}$

$$\begin{aligned} |\langle Su, v\rangle| &= \Big|\int_0^T \Big(\int_\Omega a(x,t,u,\nabla u)\cdot \nabla v \, dx\Big)dt\Big| \\ &\leq \int_0^T 2\|a(x,t,u,\nabla u)\|_{L^q(\Omega)}.\|\nabla v\|_{L^p(\Omega)}dt. \end{aligned}$$

Thanks to the growth condition (4) we can easily prove that $\|a(x,t,u,\nabla u)\|_{L^q(\Omega)}$ is bounded for all $u \in W_0^{1,p}(\Omega)$. Therefore

$$|\langle Su, v\rangle| \leq const \int_0^T \|\nabla v\|_{L^p(\Omega)}.$$

By the continuous embedding $\mathcal{H} \hookrightarrow L^1(0,T; W_0^{1,p}(\Omega))$, we concludes that

$$|\langle Su, v\rangle| \leq const\|v\|_{\mathcal{H}} \ ,$$

which means that the operator S is bounded.

- To show that S is continuous, let $u_n \to u$ in $\mathcal{H}$. Then $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^p(Q)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions α in $L^p(Q)$ and β in $(L^p(Q))^N$ such that

$$u_k \to u \text{ and } \nabla u_k \to \nabla u,$$

$$|u_k(x,t)| \leq \alpha(x,t) \text{ and } |\nabla u_k(x,t)| \leq |\beta(x,t)|$$

for a.e. $(x,t) \in Q$ and all $k \in \mathbb{N}$. Since a satisfies the Carathéodory condition, we obtain that

$$a(x,t,u_k(x,t),\nabla u_k(x,t)) \to a(x,t,u(x,t),\nabla u(x,t)) \text{ a.e. } (x,t) \in Q.$$

It follows from (4) that

$$|a(x,t,u_k(x,t),\nabla u_k(x,t))| \leq k(x,t) + c_1(|\alpha(x,t)|^{p-1} + |\beta(x,t)|^{p-1})$$

for a.e. $(x,t) \in Q$ and for all $k \in \mathbb{N}$.
Since

$$k_1 + c_1(|\alpha|^{p-1} + |\beta|^{p-1}) \in L^q(Q),$$

the dominated convergence theorem imply that

$$a(x,t,u,\nabla u_k) \to a(x,t,u,\nabla u) \text{ in } L^q(Q).$$

Thus the entire sequence $a(x,t,u_n,\nabla u_n)$ converges to $a(x,t,u,\nabla u)$ in $L^q(Q)$.
Then, $\forall v \in \mathcal{H}; \langle Su_n, v\rangle \to \langle Su, v\rangle$, which implies that the operator S is continuous.

- It remains to show that the operator S is of class (S_+). Let $(u_n)_n$ be a sequence in $D(S)$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } \mathcal{H} \\ \limsup\limits_{k\to\infty} \langle Su_n, u_n - u\rangle \leq 0. \end{cases} \tag{7}$$

We will prove that

$$u_n \longrightarrow u \text{ in } \mathcal{H}.$$

We have

$$\begin{aligned}\limsup_{k\to\infty}\langle Su_n, u_n - u\rangle &= \limsup_{k\to\infty}\langle Su_n - Su, u_n - u\rangle \\ &= \limsup_{k\to\infty}\int_Q (a(x,t,u_n,\nabla u_n) - a(x,t,u,\nabla u)).(\nabla u_n - \nabla u)\, dx\, dt \\ &\leq 0.\end{aligned}$$

From the Theorem 5.1 of Lions [10] we know that $\mathcal{H}$ embeds compactly in $L^p(Q)$. So, there is a subsequence still denoted by (u_n) such that

$$u_n \to u \text{ in } L^p(Q) \text{ and a.e in } Q.$$

So, the above inequality becomes

$$\limsup_{n\to\infty}\int_Q (a(x,t,u,\nabla u_n) - a(x,t,u,\nabla u)).(\nabla u_n - \nabla u)\, dx\, dt \leq 0.$$

But not that the hypothesis (5) implies that $\xi \mapsto a(x,t,\eta,\xi)$ is strictly increasing, and by hypothesis (6) and a result of Browder [7], we can get that

$$\nabla u_n \to \nabla u \text{ in } L^p(Q)\ ,$$

and so conclude that S is of type (S_+).

□

Our main result is the following existence theorem:

Theorem 4.1. *Let $f \in \mathcal{H}^*$ and $u_0 \in L^2(\Omega)$. There exists at least one weak solution $u \in D(L)$ of problem* (1) *in the following sense*

$$-\int_Q uv_t dxdt + \int_Q a(x,t,u,\nabla u)\cdot\nabla v dxdt = \int_Q fv dxdt$$

for all $v \in \mathcal{H}$.

Proof. Let L be the operator defined from $\mathcal{H} \supset D(L)$ to $\mathcal{V}^*$, where

$$D(L) = \{v \in \mathcal{H} : v' \in \mathcal{H}^*, v(0) = 0\},$$

by

$$\langle Lu, v\rangle = -\int_Q uv_t dxdt, \text{ for all } u \in D(L), v \in \mathcal{H}.$$

The operator L is generated by $\partial/\partial t$ via the relation

$$\langle Lu, v\rangle = \int_0^T \langle u'(t), v(t)\rangle dt, \text{ for all } u \in D(L), v \in \mathcal{H}.$$

One can verify, as in [12] that L is a densely defined maximal monotone operator.
By the monotonicity of L ($\langle Lu, u\rangle \geq 0$ for all $u \in D(L)$) and the condition (6), we get

$$\begin{aligned}\langle Lu + Su, u\rangle &\geq \langle Su, u\rangle \\ &= \int_Q a(x,t,u,\nabla u)\cdot\nabla u\, dxdt \\ &\geq \int_Q c_2|\nabla u|^p\, dxdt \\ &= c_2\|u\|^p_{\mathcal{H}}\end{aligned}$$

for all $u \in \mathcal{H}$.
Since the right side of the above inequality approaches ∞ as $\|u\|_{\mathcal{H}} \to \infty$, then for each $h \in \mathcal{H}^*$ there exists $R = R(f)$ such that $\langle Lu + Su - f, u\rangle > 0$ for all $u \in B_R(0) \cap D(L)$. By applying Lemma 3.1, we conclude that the equation $Lu + Su = f$ is solvable in $D(L)$; that is, (1) admits at least one-weak solution. □

Conclusion

We can then say that the Topological degree theory is an effective tool to solve nonlinear problems, not only elliptical, but also parabolic. We hope in a future work to solve other similar problems in spaces with variable exponents under suitable conditions (growth, monotony, coercivity, ...) on the Leray-Lions operator $-div\ a(x,t,u,\nabla u)$.

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