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Cofinitely Weak e-Supplemented Modules

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Abstract: In this work, R will denote an associative ring with unity and all module are unital left $R-$modules. Let M be an $R-$module. If every cofinite essential submodule of M has a weak supplement in M, then M is called a cofinitely weak e-supplemented (or briefly cwe-supplemented) module. In this work, some properties of these modules are investigated.

Keywords: Cofinite Submodules, Essential Submodules, Small Submodules, Supplemented Modules.

1 INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If there exists a submodule L of M such that $M = N + L$ and $N \cap L = 0$, then N is called a *direct summand* of M and denoted by $M = N \oplus L$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A module M is said to be *hollow* if every proper submodule of M is small in M. M is said to be *local* if M has a proper submodule which contains all proper submodules. A submodule N of an R -module M is called an *essential* submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. A submodule K of M is called a *cofinite* submodule of M if M/K is finitely generated. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. M is called an *essential supplemented* module if every essential submodule of M has a supplement in M. M is called an *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M. M is called a *cofinitely essential supplemented* module if every cofinite essential submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* module. If every cofinite submodule of M has ample supplements in M, then M is called an *amply cofinitely supplemented* module. If every cofinite essential submodule of M has ample supplements in M, then M is called an *amply cofinitely essential supplemented* module. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $U \cap V \ll M$, then V is called a *weak supplement* of U in M. M is said to be *weakly supplemented* if every submodule of M has a weak supplement in M. M is said to be *cofinitely weak supplemented* if every cofinite submodule of M has a weak supplement in M. M is called a *weakly essential supplemented* module if every essential submodule of M has a weak supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$.

More informations about (amply) supplemented modules are in [3]-[9]. The definitions of (amply) essential supplemented modules and some properties of them are in [7]-[8]. The definitions of (amply) cofinitely supplemented modules and some properties of them are in [1]. The definitions of (amply) cofinitely essential supplemented modules and some details of them are in [4]-[5]. Some details about weakly supplemented and cofinitely weak supplemented modules are in [2]-[3]. The definition of weakly essential supplemented modules and some properties of these modules are in [6].

Lemma 1. *Let M be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof: See [9, 17.3]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*

(4) *If* $L \leq M$ *and* $K \ll L$, *then* $K \ll M$.
(5) *If* $K_1, K_2, ..., K_n \ll M$, *then* $K_1 + K_2 + ... + K_n \ll M$.
(6) *Let* $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. *If* $K_i \ll L_i$ *for every* $i = 1, 2, ..., n$, *then* $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.

Proof: See [3, 2.2] and [9, 19.3]. □

Lemma 3. *Let* M *be an* $R-$*module. The following assertions are hold.*
(1) *If* $L \ll M$, *then* $L \leq T$ *for every maximal submodule* T *of* M.
(2) $RadM = \sum_{L \ll M} L$.
(3) *Let* N *be an* $R-$*module and* $f : M \longrightarrow N$ *be an* $R-$*module homomorphism. Then* $f(RadM) \leq RadN$.
(4) *For* $K, L \leq M$, $\frac{RadK+L}{L} \leq Rad\frac{K+L}{L}$. *If* $L \leq RadK$, *then* $RadK/L \leq Rad(K/L)$.
(5) *If* $L \leq M$, *then* $RadL \leq RadM$.
(6) *For* $K, L \leq M$, $RadK + RadL \leq Rad(K + L)$.
(7) $Rx \ll M$ *for every* $x \in RadM$.

Proof: See [9]. □

2 COFINITELY WEAK e-SUPPLEMENTED MODULES

Lemma 4. *Let* V *be a supplement of* U *in* M. *Then*
(1) *If* $W + V = M$ *for some* $W \leq U$, *then* V *is a supplement of* W *in* M.
(2) *If* M *is finitely generated, then* V *is also finitely generated.*
(3) *If* U *is a maximal submodule of* M, *then* V *is cyclic and* $U \cap V = RadV$ *is the unique maximal submodule of* V.
(4) *If* $K \ll M$, *then* V *is a supplement of* $U + K$ *in* M.
(5) *For* $K \ll M$, $K \cap V \ll V$ *and hence* $RadV = V \cap RadM$.
(6) *Let* $K \leq V$. *Then* $K \ll V$ *if and only if* $K \ll M$.
(7) *For* $L \leq U$, $\frac{V+L}{L}$ *is a supplement of* U/L *in* M/L.

Proof: See [9, 41.1]. □

Lemma 5. *Let* V *be a weak supplement of* U *in* M. *Then*
(1) *If* $W + V = M$ *for some* $W \leq U$, *then* V *is a weak supplement of* W *in* M.
(2) U *is also a weak supplement of* V *in* M.
(3) $U \cap V \leq RadM$.
(4) *If* $K \ll M$, *then* V *is a g-supplement of* $U + K$ *in* M.
(5) *If* $K \ll M$ *and* V *is a weak supplement of* $X + K$ *in* M, *then* V *is a weak supplement of* X *in* M.
(6) *If* $K \ll M$, *then* $V + K$ *is a g-supplement of* U *in* M.
(7) *For* $L \leq U$, $\frac{V+L}{L}$ *is a weak supplement of* U/L *in* M/L.

Proof: See [2]-[3]. □

Lemma 6. *Let* M *be an* $R-$*module.*
(1) *If* $M = U \oplus V$ *then* V *is a supplement of* U *in* M. *Also* U *is a supplement of* V *in* M.
(2) *For* $M_1, U \leq M$, *if* $M_1 + U$ *has a supplement in* M *and* M_1 *is supplemented, then* U *also has a supplement in* M.
(3) *Let* $M = M_1 + M_2$. *If* M_1 *and* M_2 *are supplemented, then* M *is also supplemented.*
(4) *Let* $M_i \leq M$ *for* $i = 1, 2, ..., n$. *If* M_i *is supplemented for every* $i = 1, 2, ..., n$, *then* $M_1 + M_2 + ... + M_n$ *is also supplemented.*
(5) *If* M *is supplemented, then* M/L *is supplemented for every* $L \leq M$.
(6) *If* M *is supplemented, then every homomorphic image of* M *is also supplemented.*
(7) *If* M *is supplemented, then* $M/RadM$ *is semisimple.*
(8) *Hollow and local modules are supplemented.*
(9) *If* M *is supplemented, then every finitely* $M-$*generated module is supplemented.*
(10) ${}_RR$ *is supplemented if and only if every finitely generated* $R-$*module is supplemented.*

Proof: See [9, 41.2]. □

Lemma 7. *Let* M *be an* $R-$*module.*
(1) *If* M *is supplemented, then* M *is essential supplemented.*
(2) *For* $M_1 \leq M$ *and* $U \trianglelefteq M$, *if* $M_1 + U$ *has a supplement in* M *and* M_1 *is essential supplemented, then* U *also has a supplement in* M.
(3) *Let* $M = M_1 + M_2$. *If* M_1 *and* M_2 *are essential supplemented, then* M *is also essential supplemented.*
(4) *Let* $M_i \leq M$ *for* $i = 1, 2, ..., n$. *If* M_i *is essential supplemented for every* $i = 1, 2, ..., n$, *then* $M_1 + M_2 + ... + M_n$ *is also essential supplemented.*
(5) *If* M *is essential supplemented, then* M/L *is essential supplemented for every* $L \leq M$.
(6) *If* M *is essential supplemented, then every homomorphic image of* M *is also essential supplemented.*
(7) *If* M *is essential supplemented, then* $M/RadM$ *have no proper essential submodules.*
(8) *Hollow and local modules are essential supplemented.*
(9) *If* M *is essential supplemented, then every finitely* $M-$*generated module is essential supplemented.*
(10) ${}_RR$ *is essential supplemented if and only if every finitely generated* $R-$*module is essential supplemented.*

Proof: See [7]-[8]. □

Lemma 8. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is cofinitely supplemented.*
(2) *If M is finitely generated and cofinitely supplemented, then M is supplemented.*
(3) *For $M_1 \leq M$ and U cofinite submodule of M, if $M_1 + U$ has a supplement in M and M_1 is cofinitely supplemented, then U also has a supplement in M.*
(4) *Let $M = \sum\limits_{i \in I} M_i$. If M_i is cofinitely supplemented for every $i \in I$, then M is also cofinitely supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely supplemented.*
(6) *If M is cofinitely supplemented, then M/L is cofinitely supplemented for every $L \leq M$.*
(7) *If M is cofinitely supplemented, then every homomorphic image of M is also cofinitely supplemented.*
(8) *If M is cofinitely supplemented, then every cofinite submodule of $M/RadM$ is a direct summand of $M/RadM$.*
(9) *Hollow and local modules are cofinitely supplemented.*
(10) *If M is cofinitely supplemented, then every $M-$generated module is cofinitely supplemented.*
(11) *${}_RR$ is supplemented if and only if every generated $R-$module is cofinitely supplemented.*

Proof: See [1]. □

Lemma 9. *Let M be an $R-$module.*
(1) *If M is essential supplemented, then M is cofinitely essential supplemented.*
(2) *If M is supplemented, then M is cofinitely essential supplemented.*
(3) *If M is finitely generated and cofinitely essential supplemented, then M is essential supplemented.*
(4) *For $M_1 \leq M$ and U cofinite essential submodule of M, if $M_1 + U$ has a supplement in M and M_1 is cofinitely essential supplemented, then U also has a supplement in M.*
(5) *Let $M = \sum\limits_{i \in I} M_i$. If M_i is cofinitely essential supplemented for every $i \in I$, then M is also cofinitely essential supplemented.*
(6) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely essential supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely essential supplemented.*
(7) *If M is cofinitely essential supplemented, then M/L is cofinitely essential supplemented for every $L \leq M$.*
(8) *If M is cofinitely essential supplemented, then every homomorphic image of M is also cofinitely essential supplemented.*
(9) *If M is cofinitely essential supplemented, then $M/RadM$ have no proper essential submodules.*
(10) *Hollow and local modules are cofinitely essential supplemented.*
(11) *If M is cofinitely essential supplemented, then every $M-$generated module is cofinitely essential supplemented.*
(12) *${}_RR$ is essential supplemented if and only if every generated $R-$module is cofinitely essential supplemented.*

Proof: See [4]-[5]. □

Lemma 10. *Let M be an $R-$module.*
(1) *If V is a supplement of U in M, then V is a weak suppement of U in M.*
(2) *If $M = U \oplus V$ then V is a weak supplement of U in M.*
(3) *For $M_1, U \leq M$, if $M_1 + U$ has a weak supplement in M and M_1 is weakly supplemented, then U also has a weak supplement in M.*
(4) *Let $M = M_1 + M_2$. If M_1 and M_2 are weakly supplemented, then M is also weakly supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is weakly supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also weakly supplemented.*
(6) *If M is weakly supplemented, then M/L is weakly supplemented for every $L \leq M$.*
(7) *If M is weakly supplemented, then every homomorphic image of M is also weakly supplemented.*
(8) *If M is weakly supplemented, then $M/RadM$ is semisimple.*
(9) *Hollow and local modules are weakly supplemented.*
(10) *If M is weakly supplemented, then every finitely $M-$generated module is weakly supplemented.*
(11) *${}_RR$ is weakly supplemented if and only if every finitely generated $R-$module is weakly supplemented.*

Proof: See [2]-[3]. □

Lemma 11. *Let M be an $R-$module.*
(1) *If M is weakly supplemented, then M is cofinitely weak supplemented.*
(2) *If M is supplemented, then M is cofinitely weak supplemented.*
(3) *If M is cofinitely supplemented, then M is cofinitely weak supplemented.*
(4) *If M is finitely generated and cofinitely weak supplemented, then M is weakly supplemented.*
(5) *For $M_1 \leq M$ and U cofinite submodule of M, if $M_1 + U$ has a weak supplement in M and M_1 is cofinitely weak supplemented, then U also has a weak supplement in M.*
(6) *Let $M = \sum\limits_{i \in I} M_i$. If M_i is cofinitely weak supplemented for every $i \in I$, then M is also cofinitely weak supplemented.*
(7) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely weak supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely weak supplemented.*
(8) *If M is cofinitely weak supplemented, then M/L is cofinitely weak supplemented for every $L \leq M$.*
(9) *If M is cofinitely weak supplemented, then every homomorphic image of M is also cofinitely weak supplemented.*
(10) *If M is cofinitely weak supplemented, then every cofinite submodule of $M/RadM$ is a direct summand of $M/RadM$.*
(11) *Hollow and local modules are cofinitely weak supplemented.*
(12) *If M is cofinitely weak supplemented, then every $M-$generated module is cofinitely weak supplemented.*

(13) ${}_RR$ *is weakly supplemented if and only if every* $R-$*module is cofinitely weak supplemented.*

Proof: See [2]-[3]. □

Lemma 12. *Let* M *be an* $R-$*module.*
(1) *If* M *is weakly supplemented, then* M *is weakly essential supplemented.*
(2) *For* $M_1 \leq M$ *and* $U \trianglelefteq M$*, if* $M_1 + U$ *has a weak supplement in* M *and* M_1 *is weakly essential supplemented, then* U *also has a weak supplement in* M.
(3) *Let* $M = M_1 + M_2$*. If* M_1 *and* M_2 *are weakly essential supplemented, then* M *is also weakly essential supplemented.*
(4) *Let* $M_i \leq M$ *for* $i = 1, 2, ..., n$*. If* M_i *is weakly essential supplemented for every* $i = 1, 2, ..., n$*, then* $M_1 + M_2 + ... + M_n$ *is also weakly essential supplemented.*
(5) *If* M *is weakly essential supplemented, then* M/L *is weakly essential supplemented for every* $L \leq M$.
(6) *If* M *is weakly essential supplemented, then every homomorphic image of* M *is also weakly essential supplemented.*
(7) *If* M *is weakly essential supplemented, then* $M/RadM$ *have no proper essential submodules.*
(8) *Hollow and local modules are weakly essential supplemented.*
(9) *If* M *is weakly essential supplemented, then every finitely* $M-$*generated module is weakly essential supplemented.*
(10) ${}_RR$ *is weakly essential supplemented if and only if every finitely generated* $R-$*module is weakly essential supplemented.*

Proof: See [6]. □

Definition 1. *Let* M *be an* $R-$*module. If every cofinite essential submodule of* M *has a weak supplement in* M*, then* M *is called a cofinitely weak e-supplemented (or briefly cwe-supplemented) module.*

Proposition 1. *Every cofinitely essential supplemented module is cwe-supplemented.*

Proof: Let M be a cofinitely essential supplemented module and U be a cofinite essential submodule of M. Then U has a supplement V in M. Here $M = U + V$ and $U \cap V \ll V$. Since $U \cap V \ll V$, $U \cap V \ll M$. Then V is a weak supplement of U in M. Hence M is cwe-supplemented. □

Proposition 2. *Every essential supplemented module is cwe-supplemented.*

Proof: Since every essential supplemented module cofinitely essential supplemented, by Proposition 1, every essential supplemented module is cwe-supplemented. □

Proposition 3. *Every weakly essential supplemented module is cwe-supplemented.*

Proof: Let M be a weakly essential supplemented module and U be a cofinite essential submodule of M. Since M is weakly essential supplemented and $U \leq M$, U has a weak supplement in M. Hence M is cwe-supplemented. □

Proposition 4. *Every finitely generated cwe-supplemented module is weakly essential supplemented.*

Proof: Let M be a finitely generated cwe-supplemented module and $U \trianglelefteq M$. Since M is finitely generated, M/U is also finitely generated. Then U is a cofinite essential submodule of M and since M is cwe-supplemented, U has a weak supplement in M. Hence M is weakly essential supplemented. □

Proposition 5. *Every cofinitely weak supplemented module is cwe-supplemented.*

Proof: Let M be a cofinitely weak supplemented module and U be a cofinite essential submodule of M. Since M is cofinitely weak supplemented and U is a cofinite essential submodule of M, U has a weak supplement in M. Hence M is cwe-supplemented. □

Proposition 6. *Every weakly supplemented module is cwe-supplemented.*

Proof: Since every weakly supplemented module is cofinitely weak supplemented, by Proposition 5, every weakly supplemented module is cwe-supplemented. □

Proposition 7. *Every cofinitely supplemented module is cwe-supplemented.*

Proof: Clear from Proposition 5, since every cofinitely supplemented module is cofinitely weak supplemented. □

Proposition 8. *Every supplemented module is cwe-supplemented.*

Proof: Clear from Proposition 7, since every supplemented module is cofinitely supplemented. □

Proposition 9. *Let* M *be a cwe-supplemented module. If every nonzero submodule of* M *is essential in* M*, then* M *is cofinitely weak supplemented.*

Proof: Let U be a cofinite submodule of M. If $U = 0$, M is a weak supplement of U in M. Let $U \neq M$. Then by hypothesis, $U \trianglelefteq M$. Since U is a cofinite essential submodule of M and M is cwe-supplemented, U has a weak supplement in M. Hence M is cofinitely weak supplemented. $\square$

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