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Simpson Type Integral Inequalities for Harmonic Convex Functions via Conformable Fractional Integrals

Zeynep ŞANLI[1,*]

[1]*Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey*
zeynep.sanli@ktu.edu.tr, ORCID: 0000-0000-1564-2634

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Abstract

Fractional integral operators are very useful in the field of mathematical analysis and optimization theory. The main aim of this investigation is to establish a new Simpson type conformable fractional integral equality for harmonically convex functions. Using this identity, some new results related to Simpson-like type conformable fractional integral inequalities are obtained. Then, some interesting conclusions are attained for some special cases of conformable fractional integrals when $\alpha = 1$.

Keywords: Simpson inequality; Conformable fractional integral; Harmonic convex.

Conformable Kesirli İntegraller Aracılığıyla Harmonik Konveks Fonksiyonlar için Simpson Tipi İntegral Eşitsizlikleri

Öz

Kesirli integral operatörleri matematiksel analiz ve optimizasyon teorisi alanlarında oldukça kullanışlıdır. Bu araştırmanın temel amacı harmonik konveks fonksiyonlar için yeni bir Simpson tipi conformable kesirli integral eşitliği kurmaktır. Bu eşitliği kullanarak Simpson tipi conformable kesirli integral eşitsizlikleri ile ilgili bazı yeni sonuçlar elde edildi. Daha sonra, $\alpha =$ 1 olduğunda, conformable kesirli integrallerin bazı özel durumları için ilginç sonuçlara ulaşıldı.

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***Anahtar Kelimeler*:** Simpson eşitsizliği; Conformable kesirli integral; Harmonik konveks.

1. Introduction

We will start with the following inequality which is well known in the literature as Simpson's inequality.

Theorem 1. Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty=\sup|f^{(4)}(x)|<\infty$. Then, the following inequality holds:

$$\left|\int_a^b f(x)dx-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\leq\frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^4. \quad (1)$$

Inequality (1) has been studied by several authors (see [1-11]).

In [12], İşcan gave the definition of harmonically convex functions as follow:

Definition 2. [12] Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{uv}{tu+(1-t)v}\right)\leq tf(v)+(1-t)f(u), \quad (2)$$

for all $u,v\in I$ and $t\in[0,1]$. If the inequality (2) is reversed, then f is said to be harmonically concave.

Harmonic convex functions are important for mathematical inequalities. Many authors obtained several inequalities for harmonic convex functions [12-15]. The most famous inequality which has been used with harmonic convex functions is Hermite-Hadamard, which is stated as follow:

Theorem 3. [12] Let $f:I\subset\mathbb{R}\backslash\{0\}$ be a hormanically convex function and $u,v\in[a,b]$ with $u<v$. If $f\in L[u,v]$ then the following inequalities hold:

$$f\left(\frac{2uv}{u+v}\right)\leq\frac{uv}{v-u}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(u)+f(v)}{2}. \quad (3)$$

The aim of this paper is to establish Simpson type conformable fractional integral inequalities based on harmonically convexity.

2. Preliminaries

In this section, we give some definitions and basic results for later use.

Definition 4. Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ order $\alpha > 0$ are defined by

$$J_{a+}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \;\; x > a,$$

$$J_{b-}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \;\; x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ (see [16], p.69).

The following definition of conformable fractional integrals can be found in [13, 17, 15].

Definition 5. Let $\alpha \in (n, n+1], n = 0, 1, 2, \ldots, \beta = \alpha - n,\; a,\; b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right conformable fractional integrals $I_{\alpha}^{a} f$ and ${}^{b}I_{\alpha} f$ order $\alpha > 0$ are defined by

$$I_{\alpha}^{a} f = \frac{1}{n!} \int_a^x (x-t)^n (t-a)^{\beta-1} f(t) dt, \;\; x > a,$$

$${}^{b}I_{\alpha} f = \frac{1}{n!} \int_x^b (t-x)^n (b-t)^{\beta-1} f(t) dt, \;\; x < b,$$

respectively.

It is easily seen that if one takes $\alpha = n+1$ in the Definition 5 (for the left and right conformable fractional integrals), then the Definition 4 is obtained (the left and right Riemann-Liouville fractional integrals) for $\alpha \in \mathbb{N}$.

3. Main Results

Throughout the paper, we will use the following notations for our results:

$$u_1(t) = \frac{2ab}{(1-t)a + (1+t)b},$$

$$u_2(t) = \frac{2ab}{(1+t)a + (1-t)b},$$

$$H = \frac{2ab}{a+b}.$$

Let's begin the following lemma which will help us to obtain the main results:

Lemma 6. Let $\varphi: I \subset (0, \infty) \to \mathbb{R}$, be a differentiable function on $I^{\circ}, a, b \in I^{\circ}$ and $a < b$. If $\varphi' \in L[a, b]$, then the following equality holds:

$$\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\begin{bmatrix} I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +\ {}^{\frac{1}{a}}I_{\alpha}\ (\varphi\circ\phi)\left(\frac{1}{H}\right)\end{bmatrix} \tag{4}$$

$$=\frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\int_0^1 p(t;\alpha,n)\left[(u_1(t))^2\varphi'\big((u_1(t)\big)-(u_2(t))^2\varphi'\big((u_2(t)\big)\right]dt,$$

where

$$p(t;\alpha,n)=\int_0^1\left(\frac{1}{3}\beta(n+1,\alpha-n)-\frac{1}{3}\beta_t(n+1,\alpha-n)\right)dt$$

and $\phi(x)=\frac{1}{x},\ \alpha>0$.

Proof. We begin by considering the following computations which follow from change of variables and using the definition of the conformable fractional integrals.

$$I_1=\frac{1}{n!}\int_0^1 p(t;\alpha,n)(u_1(t))^2\varphi'\big((u_1(t)\big)dt=\frac{1}{3}\frac{2ab}{a-b}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\varphi\big((u_1(t)\big)|_0^1$$

$$-\frac{1}{2.n!}\frac{2ab}{a-b}\begin{pmatrix}\int_0^1\left(\int_0^t x^n(1-x)^{\alpha-n-1}dx\right)\varphi\big((u_1(t)\big)dt \\ -\int_0^1 t^n(1-t)^{\alpha-n-1}\varphi\big((u_1(t)\big)dt\end{pmatrix}$$

$$=\frac{2ab}{a-b}\frac{1}{3}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\big(\varphi(a)-\varphi(H)\big)-\frac{1}{2}\frac{2ab}{a-b}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\varphi(a)$$

$$+\frac{1}{2}\left(\frac{2ab}{a-b}\right)^{\alpha+1}\ {}^{\frac{1}{a}}I_{\alpha}\ (\varphi\circ\phi)\left(\frac{1}{H}\right)=\frac{2ab}{a-b}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left(\frac{1}{6}\varphi(a)+\frac{1}{3}\varphi(H)\right)$$

$$-\frac{1}{2}\left(\frac{2ab}{a-b}\right)^{\alpha+1}\ {}^{\frac{1}{a}}I_{\alpha}\ (\varphi\circ\phi)\left(\frac{1}{H}\right)$$

and similarly

$$I_1=\frac{1}{n!}\int_0^1 p(t;\alpha,n)(u_1(t))^2\varphi'\big((u_1(t)\big)dt$$

$$=\frac{2ab}{b-a}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left(-\frac{1}{6}\varphi(a)-\frac{1}{3}\varphi(H)\right)+\frac{1}{2}\left(\frac{2ab}{b-a}\right)^{\alpha+1}I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right).$$

Thus, we have

$$\frac{b-a}{2ab}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}(I_1-I_2)=\frac{1}{6}\ [\varphi(a)+4\varphi(H)+\varphi(b)]$$

$$-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left[\begin{array}{c} I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +\ {}^{\frac{1}{a}}I_{\alpha}\ (\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right].$$

Remark 7. If we take $\alpha = n+1$ in Lemma 6, then we get

$$\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha+1)\left[\begin{array}{c} J_{\frac{1}{b}+}^{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +J_{\frac{1}{a}-}^{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right] \tag{5}$$

$$=\frac{b-a}{2ab}\int_0^1\left(\frac{1}{3}-\frac{t^{\alpha}}{2}\right)\left[(u_1(t))^2\varphi'\big((u_1(t)\big)-(u_2(t))^2\varphi'\big((u_2(t)\big)\right]dt.$$

Remark 8. If we take $\alpha = 1$ in Remark 7, then we have

$$\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-\frac{ab}{b-a}\int_a^b\frac{\varphi(x)}{x^2}dx \tag{6}$$

$$=\frac{b-a}{2ab}\int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)\left[(u_1(t))^2\varphi'\big((u_1(t)\big)-(u_2(t))^2\varphi'\big((u_2(t)\big)\right]dt.$$

Theorem 9. Let $\varphi: I\subset(0,\infty)\to\mathbb{R}$, be a differentiable function on $I^{\circ}, a, b\in I^{\circ}$ and $a<b$. If $\varphi'\in L[a,b]$, and $|\varphi'|$ is harmonic convex function on $[a,b]$, then the following inequality holds:

$$\left|\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left[\begin{array}{c} I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +\ {}^{\frac{1}{a}}I_{\alpha}\ (\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right| \tag{7}$$

$$\leq\frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left(\begin{array}{c}|\varphi'^{(a)|(\Im_1}(t;\alpha,n)+(\Im_2(t;\alpha,n)\big) \\ +|\varphi'(b)|(\Im_3(t;\alpha,n)+(\Im_4(t;\alpha,n))\end{array}\right),$$

where

$$\Im_1(t;\alpha,n)=\int_0^1|p(t;\alpha,n)|(u_1(t))^2\frac{1+t}{2}dt,$$

$$\Im_2(t;\alpha,n)=\int_0^1|p(t;\alpha,n)|(u_2(t))^2\frac{1-t}{2}dt,$$

$$\Im_3(t;\alpha,n)=\int_0^1|p(t;\alpha,n)|(u_1(t))^2\frac{1-t}{2}dt,$$

$\Im_4(t;\alpha,n) = \int_0^1 |p(t;\alpha,n)|(u_2(t))^2 \frac{1+t}{2} dt,$

where $n = 0,1,2, \dots$ and $\alpha \in (n, n+1]$.

Proof. From Lemma 6 and $|\varphi'|$ is harmonic convex, we have

$$\left|\frac{1}{6}[\varphi(a) + 4\varphi(H) + \varphi(b)] - 2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \begin{bmatrix} I_{\alpha}^{\frac{1}{b}}(\varphi \circ \phi)\left(\frac{1}{H}\right) \\ + \ {}^{\frac{1}{a}}I_{\alpha}\,(\varphi \circ \phi)\left(\frac{1}{H}\right) \end{bmatrix}\right|$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \int_0^1 |p(t;\alpha,n)|((u_1(t))^2 \varphi'(u_1(t)) + (u_2(t))^2 \varphi'(u_2(t)))dt$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \int_0^1 |p(t;\alpha,n)| \left((u_1(t))^2 \left(\frac{1+t}{2}|\varphi'(a)| + \frac{1-t}{2}|\varphi'(b)|\right)\right.$$

$$\left. + (u_2(t))^2 \left(\frac{1-t}{2}|\varphi'(a)| + \frac{1+t}{2}|\varphi'(b)|\right)\right) dt$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\Big(|\varphi'(a)|\big(\Im_1(t;\alpha,n) + \Im_2(t;\alpha,n)\big) + |\varphi'(b)|\big(\Im_3(t;\alpha,n)$$

$$+ \Im_4(t;\alpha,n)\big)\Big).$$

This completes the proof.

Theorem 10. Let $\varphi: I \subset (0,\infty) \to \mathbb{R}$, be a differentiable function on I°, $a, b \in I^{\circ}$ and $a < b$. If $\varphi' \in L[a,b]$, and $|\varphi'|^q$ is harmonic convex function on $[a,b]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{p} = 1$, then the following inequality holds:

$$\left|\frac{1}{6}[\varphi(a) + 4\varphi(H) + \varphi(b)] - 2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \begin{bmatrix} I_{\alpha}^{\frac{1}{b}}(\varphi \circ \phi)\left(\frac{1}{H}\right) \\ + \ {}^{\frac{1}{a}}I_{\alpha}\,(\varphi \circ \phi)\left(\frac{1}{H}\right) \end{bmatrix}\right| \tag{8}$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left(\frac{1}{2}\right)^{2q+1}\left(\int_0^1 |p(t;\alpha,n)|^p dt\right)^{\frac{1}{p}}$$

$$\times \begin{pmatrix} (\gamma_1(q;a,b)|\varphi'(a)|^q + \gamma_2(q;a,b)|\varphi'(b)|^q)^{\frac{1}{q}} \\ +(\gamma_3(q;a,b)|\varphi'(a)|^q + \gamma_4(q;a,b)|\varphi'(b)|^q)^{\frac{1}{q}} \end{pmatrix},$$

where

$$\gamma_1(q;a,b)=-\frac{\begin{array}{c}a^{2q}b^{2q}\left(\left((8b^2-8ab)q-4b^2+8ab\right)e^{2\ln(b+a)q}\right.\\ \left.+\left((2a^2-2b^2)q+b^2-2ab-3a^2\right)e^{2\ln(2b)q}\right).2^{2q-1}e^{-2\ln(b+a)q-2\ln(2b)q}\end{array}}{(b-a)^2(2q^2-3q+1)},$$

$$\gamma_2(q;a,b)=\frac{\begin{array}{c}a^{2q}b^{2q}\left(4b^2e^{2\ln(b+a)q}+((2b^2-2a^2)q-3b^2-2ab+a^2\right)e^{2\ln(2b)q}\\ .2^{2q-1}e^{2\ln(b+a)q-2\ln(2b)q}\end{array}}{(b-a)^2(2q^2-3q+1)},$$

$$\gamma_3(q;a,b)=\frac{\begin{array}{c}a^{2q}b^{2q}\left(4b^2e^{2\ln(b+a)q}+((2a^2-2b^2)q+b^2-2ab-3a^2\right)e^{2\ln(2a)q}\\ .2^{2q-1}e^{2\ln(b+a)q-2\ln(2a)q}\end{array}}{(b-a)^2(2q^2-3q+1)},$$

$$\gamma_4(q;a,b)=\frac{\begin{array}{c}a^{2q}b^{2q}\left(\left((8ab-8a^2)q-8ab+4a^2\right)e^{2\ln(b+a)q}\right.\\ \left.+\left((2a^2-2b^2)q+3b^2+2ab-a^2\right)e^{2\ln(2a)q}\right).2^{2q-1}e^{-2\ln(b+a)q-2\ln(2a)q}\end{array}}{(b-a)^2(2q^2-3q+1)},$$

$n=0,1,2,\ldots$ *and* $\alpha\in(n,n+1]$.

Proof. From Lemma 6 and using Hölder's integral inequality and the harmonic convexity of $|\varphi'|^q$, we have

$$\left|\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left[\begin{array}{c}I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\ +\ {}^{\frac{1}{a}}I_{\alpha}\,(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|$$

$$\leq\frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left(\int_0^1|p(t;\alpha,n)|^p dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1((u_1(t))^{2q}|\varphi'(u_1(t))|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1((u_2(t))^{2q}|\varphi'(u_2(t))|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left\{\left(\int_0^1 |p(t;\alpha,n)|^p dt\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\int_0^1 ((u_1(t))^{2q}\left[|\varphi'(a)|^q\left(\frac{1+t}{2}\right)+|\varphi'(b)|^q\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1 ((u_2(t))^{2q}\left[|\varphi'(a)|^q\left(\frac{1-t}{2}\right)+|\varphi'(b)|^q\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\right]\right\}$$

$$\leq \frac{b-a}{ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left(\frac{1}{2}\right)^{\frac{1}{2q+1}}\left(\int_0^1 |p(t;\alpha,n)|^p dt\right)^{\frac{1}{p}}$$

$$\times\left[(\gamma_1(q;a,b)|\varphi'(a)|^q+\gamma_2(q;a,b)|\varphi'(b)|^q)^{\frac{1}{q}}\right.$$

$$\left.+(\gamma_3(q;a,b)|\varphi'(a)|^q+\gamma_4(q;a,b)|\varphi'(b)|^q)^{\frac{1}{q}}\right].$$

This completes the proof.

Remark 11. If we take $\alpha = n+1$, after that if we take $\alpha = 1$ in Theorem 10, we obtain the following inequality

$$\left|\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-\frac{ab}{b-a}\int_0^1 \frac{\varphi(x)}{x^2}dx\right|$$

$$\leq \frac{b-a}{12ab}\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{4}\right)^{\frac{1}{q}}$$

$$\times\left[(\gamma_1(q;a,b)|\varphi'(a)|^q+\gamma_2(q;a,b)|\varphi'(b)|^q)^{\frac{1}{q}}\right.$$

$$\left.+(\gamma_3(q;a,b)|\varphi'(a)|^q+\gamma_4(q;a,b)|\varphi'(b)|^q)^{\frac{1}{q}}\right]$$

with

$$\int_0^1 \left|\frac{1}{2}t-\frac{1}{3}\right|^p dt = \frac{2^{p+2}+2}{(p+1)6^{p+1}}.$$

Theorem 12. Let $\varphi: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^{\circ}, a, b \in I^{\circ}$ and $a < b$. If $\varphi' \in L[a,b]$ and $|\varphi'|^q$ is harmonic convex function on $[a,b]$ for $q \geq 1$, then the following inequality holds:

$$\left| \frac{1}{6}[\varphi(a) + 4\varphi(H) + \varphi(b)] - 2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \begin{bmatrix} I_{\alpha}^{\frac{1}{b}}(\varphi \circ \phi)\left(\frac{1}{H}\right) \\ + \ {}^{\frac{1}{a}}I_{\alpha}\,(\varphi \circ \phi)\left(\frac{1}{H}\right) \end{bmatrix} \right| \tag{9}$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left(\int_0^1 |p(t;\alpha,n)|^p dt\right)^{1-\frac{1}{q}}$$

$$\times \begin{pmatrix} (\zeta_1(q,t;\alpha,n)|\varphi'(a)|^q + \zeta_2(q,t;\alpha,n)|\varphi'(b)|^q)^{\frac{1}{q}} \\ +(\zeta_3(q,t;\alpha,n)|\varphi'(a)|^q + \zeta_4(q,t;\alpha,n)|\varphi'(b)|^q)^{\frac{1}{q}} \end{pmatrix}$$

where

$$\zeta_1(q,t;\alpha,n) = \int_0^1 |p(t;\alpha,n)|(u_1(t))^{2q}\,\frac{1+t}{2}\,dt,$$

$$\zeta_2(q,t;\alpha,n) = \int_0^1 |p(t;\alpha,n)|(u_1(t))^{2q}\,\frac{1-t}{2}\,dt,$$

$$\zeta_3(q,t;\alpha,n) = \int_0^1 |p(t;\alpha,n)|(u_2(t))^{2q}\,\frac{1-t}{2}\,dt,$$

$$\zeta_4(q,t;\alpha,n) = \int_0^1 |p(t;\alpha,n)|(u_2(t))^{2q}\,\frac{1-t}{2}\,dt,$$

$n = 0,1,2,\ldots$ and $\alpha \in (n, n+1]$.

Proof. From Lemma 6 and using the power mean inequality, we see that the following inequality holds:

$$\left|\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)]-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left[\begin{array}{c} I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +\ {}^{\frac{1}{a}}I_{\alpha}\,(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\int_0^1 |p(t;\alpha,n)|\left(\left(u_1(t)\right)^2|\varphi'(u_1(t))|\right.$$

$$\left.+\left(u_2(t)\right)^2|\varphi'(u_2(t))|\right)dt$$

$$\leq \frac{b-a}{2.n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left\{\left(\int_0^1 |p(t;\alpha,n)|^p dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1 |p(t;\alpha,n)|\left(u_1(t)\right)^{2q}|\varphi'(u_1(t))|^q dt\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\int_0^1 |p(t;\alpha,n)|\left(u_2(t)\right)^{2q}|\varphi'(u_2(t))|^q dt\right)^{\frac{1}{q}}\right\}.$$

By the harmonic convexity of $|\varphi'|^q$ we write

$$\int_0^1 |p(t;\alpha,n)|\left(u_1(t)\right)^{2q}|\varphi'(u_1(t))|^q dt$$

$$\leq |\varphi'(a)|^q\int_0^1 |p(t;\alpha,n)|\left(u_1(t)\right)^{2q}\frac{1+t}{2}dt$$

$$+|\varphi'(b)|^q\int_0^1 |p(t;\alpha,n)|\left(u_1(t)\right)^{2q}\frac{1-t}{2}dt$$

and

$$\int_0^1 |p(t;\alpha,n)|\left(u_2(t)\right)^{2q}|\varphi'(u_2(t))|^q dt$$

$$\leq |\varphi'(a)|^q\int_0^1 |p(t;\alpha,n)|\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt$$

$$+|\varphi'(b)|^q \int_0^1 |p(t;\alpha,n)|\left(u_2(t)\right)^{2q} \frac{1+t}{2} dt.$$

Using the last two inequalities, we obtain

$$\left|\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)] - 2^{\alpha-1}\left(\frac{ab}{b-a}\right)^{\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \left[\begin{array}{c} I_{\alpha}^{\frac{1}{b}}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ + \ {}^{\frac{1}{a}}I_{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|$$

$$\leq \frac{b-a}{2ab}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}\left(\int_0^1 |p(t;\alpha,n)|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(|\varphi'(a)|^q \int_0^1 |p(t;\alpha,n)|\left(u_1(t)\right)^{2q}\frac{1+t}{2}dt\right.\right.$$

$$\left.+\,|\varphi'(b)|^q \int_0^1 |p(t;\alpha,n)|\left(u_1(t)\right)^{2q}\frac{1-t}{2}dt\right)^{\frac{1}{q}}$$

$$+\left(|\varphi'(a)|^q \int_0^1 |p(t;\alpha,n)|\left(u_2(t)\right)^{2q}\frac{1+t}{2}dt\right.$$

$$\left.\left.+\,|\varphi'(b)|^q \int_0^1 |p(t;\alpha,n)|\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt\right)^{\frac{1}{q}}\right].$$

Remark 13. If we take $\alpha = n+1$, after that if we take $\alpha = 1$ in Theorem 12, we obtain the following inequality

$$\left|\frac{1}{6}[\varphi(a)+4\varphi(H)+\varphi(b)] - \frac{ab}{b-a}\int_a^b \frac{\varphi(x)}{x^2}dx\right|$$

$$\leq \frac{b-a}{2ab}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[\left(\zeta_1(q,t;\alpha,n)|\varphi'(a)|^q + \zeta_2(q,t;\alpha,n)|\varphi'(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\zeta_3(q,t;\alpha,n)|\varphi'(a)|^q + \zeta_4(q,t;\alpha,n)|\varphi'(b)|^q\right)^{\frac{1}{q}}\right].$$

4. Conclusion

In this paper, by using a new identity of Simpson-like type for conformable fractional integral for harmonic convex functions, we obtained some new Simpson type conformable fractional integral inequalities. Furthermore, some interesting conclusions were obtained for some special values of α.

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