Turk. J. Math. Comput. Sci.
13(2)(2021) 226–233
© MatDer
DOI : 10.47000/tjmcs.780474

On Pell and Pell-Lucas Generalized Octonions

Ümit Tokeşer[1], Tuğba Mert[2,*], Zafer Ünal[1], Göksal Bilgici[3]

[1]*Department of Mathematics, Faculty of Arts and Sciences, Kastamonu University, 37100, Kastamonu, Turkey.*
[2]*Department of Mathematics, Faculty of Arts and Sciences, Cumhuriyet University, 58100, Sivas, Turkey.*
[3]*Department of Mathematics Education, Education Faculty, Kastamonu University, 37100, Kastamonu, Turkey.*

Received: 14-08-2020 • Accepted: 15-07-2021

Abstract. In this study, we gave a generalization on Pell and Pell-Lucas octonions over the algebra $\mathbb{O}(a, b, c)$ where a, b and c are real numbers. For these number sequences, we obtain Binet formulas and gave some well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity.

2010 AMS Classification: 11B39, 11R52

Keywords: Generalized octonion, Pell octonion, Pell-Lucas octonion.

1. Introduction

Let a and b be real constants, the generalized quaternion algebra is $\mathbb{H}(a, b)$ with the basis $\{1, e_1, e_2, e_3\}$. And the multiplication table for this basis of $\mathbb{H}(a, b)$ can be given as follows:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-a$	e_3	$-ae_2$
e_2	e_2	$-e_3$	$-b$	be_1
e_3	e_3	ae_2	$-be_1$	$-ab$

For $a = b = 1$, $\mathbb{H}(1, 1)$ is the quaternion division algebra, for $a = 1, b = -1$, $\mathbb{H}(1, -1)$ is the algebra of split-quaternions or also called coquaternions, para-quaternions, anti-quaternions, pseudo-quaternions or hyperbolic quaternions.

The octonions constitute the largest normed division algebra over the real numbers and with notation $\mathbb{O}$. The octonions have eight dimensions and they are alternative, flexible, power-associative, non-commutative and non-associative.

**Corresponding Author*
Email addresses: utokeser@kastamonu.edu.tr (Ü. Tokeşer), tmert@cumhuriyet.edu.tr (T. Mert), zunal@kastamonu.edu.tr (Z. Ünal), gbilgici@kastamonu.edu.tr (G. Bilgici)

Let $\mathbb{O}(a,b,c)$ be the generalized octonion algebra over the $\mathbb{R}$ with the basis $\{e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$. And its the multiplication table as follows

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-a$	e_3	$-ae_2$	e_5	$-ae_4$	$-e_7$	ae_6
e_2	e_2	$-e_3$	$-b$	be_1	e_6	e_7	$-be_4$	$-be_5$
e_3	e_3	ae_2	$-be_1$	$-ab$	e_7	$-ae_6$	be_5	$-abe_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-c$	ce_1	ce_2	ce_3
e_5	e_5	ae_4	$-e_7$	ae_6	$-ce_1$	$-ac$	$-ce_3$	ace_2
e_6	e_6	e_7	be_4	$-be_5$	$-ce_2$	ce_3	$-bc$	$-bce_1$
e_7	e_7	$-ae_6$	be_5	abe_4	$-ce_3$	$-ace_2$	bce_1	$-abc$

If $\gamma \in \mathbb{O}(a,b,c)$, then we can write $\gamma = \gamma_0 + \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4 + \gamma_5 e_5 + \gamma_6 e_6 + \gamma_7 e_7$. The conjugate of γ is $\overline{\gamma} = \gamma_0 - \gamma_1 e_1 - \gamma_2 e_2 - \gamma_3 e_3 - \gamma_4 e_4 - \gamma_5 e_5 - \gamma_6 e_6 - \gamma_7 e_7$. The trace and the norm of γ are, respectively

$$t(\gamma) = \gamma + \overline{\gamma} = 2\gamma_0$$

and

$$N(\gamma) = \gamma\overline{\gamma} = \gamma_0^2 + a\gamma_1^2 + b\gamma_2^2 + ab\gamma_3^2 + c\gamma_4^2 + ac\gamma_5^2 + bc\gamma_6^2 + abc\gamma_7^2$$

[5].

The Pell sequence $\{P_n\}_{n=0}^{\infty}$ is well known sequence among integer sequences which satisfies the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}$$

with the initial conditions $P_0 = 0$ and $P_1 = 1$. Similarly, Pell-Lucas sequence $\{PL_n\}_{n=0}^{\infty}$ satisfy the recurrence relation

$$PL_n = 2PL_{n-1} + PL_{n-2}$$

with Pell sequence except the initial conditions $PL_0 = 1$ and $PL_1 = 1$. In this case, the Pell-Lucas sequence is called modified Pell sequence.

The generating functions for the Pell sequence and Pell-Lucas sequence are as follows

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2}$$

and

$$\sum_{n=0}^{\infty} PL_n x^n = \frac{2-x}{1-2x-x^2}$$

respectively. Moreover, the Binet formulas for these sequences are defined as

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

and

$$PL_n = \frac{\gamma^n + \delta^n}{2}$$

respectively, where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are solutions of the characteristic equation of $x^2 - 2x - 1 = 0$. The positive root γ is called silver ratio (see for details [13]).

Pell and Pell-Lucas numbers appear in many subjects of mathematics. They appear as solutions of the Pell equation $x^2 - 2y^2 = (-1)^n$. The solutions of this equation are (PL_n, P_n).

In 1963, Horadam [7] defined Fibonacci and Lucas quaternions for detail [6, 8, 10, 12]). And also generalizations of Fibonacci and Lucas quaternions are ([2, 9, 14, 16]).

Fibonacci and Lucas octonions are deffined by Kecioglu and Akkus as follows

$$Q_n = \sum_{s=0}^{7} F_{n+s} e_s \text{ and } T_n = \sum_{s=0}^{7} L_{n+s} e_s$$

where F_n and L_n are nth Fibonacci and Lucas numbers [11]. They gave generating function, Binet formulas and some identities for the Fibonacci and Lucas octonions. Also, they defined Split Fibonacci and Lucas octonions similarly in [1]. Savin [15] gave generalized Fibonacci and Lucas octonions over the octonion algebras $\mathbb{O}_{\mathbb{R}}(a+1, 2a+1, 3a+1)$ where for real number a.

Cimen and İpek [4] defined Pell and Pell-Lucas quaternions as

$$QP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$

and

$$QPL_{k,n} = PL_{k,n} + iPL_{k,n+1} + jPL_{k,n+2} + kPL_{k,n+3}$$

where P_n and PL_n are the nth Pell and Pell-Lucas numbers. They obtained many properties of these quaternions such as Binet formulas and Cassini's identity. Szynal-Liana and Wloch [17] introduced the Pell quaternions, the Pell octonions and gave some properties of them.

Catarino [3] studied on modified Pell and modified k-Pell quaternions which are defined by

$$MP_n = q_n + iq_{n+1} + jq_{n+2} + kq_{n+3}$$

and

$$MP_{k,n} = q_{k,n} + iq_{k,n+1} + jq_{k,n+2} + kq_{k,n+3}$$

where q_n and q_k, n are the nth modified Pell and modified k-Pell numbers, respectively.

In this paper, we study the Pell and Pell-Lucas octonions over the octonion algebra $\mathbb{O}(a, b, c)$. At first, we show the Pell and Pell-Lucas octonions over the octonion algebra $\mathbb{O}(a, b, c)$ by the following notations:

$$PO_n = \sum_{s=0}^{7} P_{n+s}e_s \text{ and } RO_n = \sum_{s=0}^{7} PL_{n+s}e_s$$

For generalized case with negative indices is given by

$$PO_{-n} = \sum_{s=0}^{7}(-1)^{n+s+1}P_{n-s}e_s \text{ and } RO_{-n} = \sum_{s=0}^{7}(-1)^{n+s}PL_{n-s}e_s$$

respectively. We also obtained some properties of these octonions including Binet formulas, Cassini's, Catalan's and D'Ocagne's identities.

2. Binet Formulas and Generalizations for Some Identities

Binet formulas for the Pell and Pell-Lucas generalized octonions are given by the following theorem.

Theorem 2.1. *For any integer n, nth Pell generalized octonion is*

$$PO_n = \frac{\gamma^*\gamma^n - \delta^*\delta^n}{\gamma - \delta} \tag{2.1}$$

and nth Pell-Lucas generalized octonion is

$$RO_n = \frac{\gamma^*\gamma^n + \delta^*\delta^n}{2} \tag{2.2}$$

where $\gamma = 1 + \sqrt{2}, \delta = 1 - \sqrt{2}, \gamma^* = \sum\limits_{s=0}^{7} \gamma^s e_s$ *and* $\delta^* = \sum\limits_{s=0}^{7} \delta^s e_s$.

Proof. Let us consider the following for Eq. (2.1) and $PO_n = \sum_{s=0}^{7} P_{n+s}e_s$:

$$\gamma PO_n + PO_{n-1} = \sum_{s=0}^{7} (\gamma P_{n+s} + P_{n+s-1})\, e_s.$$

By the help of the identity $\gamma^n = \gamma P_n + P_{n-1}$, we get

$$\gamma PO_n + PO_{n-1} = \gamma^*\gamma^n. \tag{2.3}$$

Similarlay, using the identity $\delta^n = \delta P_n + P_{n-1}$, we have

$$\delta PO_n + PO_{n-1} = \delta^*\delta^n. \tag{2.4}$$

From the equations (2.3) and (2.4), we obtain

$$PO_n = \frac{\gamma^* \gamma^n - \delta^* \delta^n}{\gamma - \delta}.$$

Eq. (2.2) can we obtain smilarly. □

Now, we give some useful identities. These identities play very important roles throughout the paper for calculations.

Lemma 2.2. *We have the followings identities*

$$\begin{aligned}
\gamma^{*2} &= \xi_1 + 2RO_0 + 2\sqrt{2}\left(\xi_2 + PO_0\right), && (2.5)\\
\delta^{*2} &= \xi_1 + 2RO_0 - 2\sqrt{2}\left(\xi_2 + PO_0\right), &&\\
\gamma^* \delta^* &= \tau + 2RO_0 + 2\sqrt{2}\sigma, && (2.6)\\
\delta^* \gamma^* &= \tau + 2RO_0 - 2\sqrt{2}\sigma && (2.7)
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= 1 - 114243abc - 3363ac - 19601bc - 99ab - 3a - 17b - 577c,\\
\xi_2 &= -40391abc - 6930bc - 1189ac - 35ab - 204c - 6b - a,\\
\tau &= abc + ab + ac - bc + a - b - c - 1,\\
\sigma &= (bc - b - c)\, e_1 + (2ac - 2a - 2c)\, e_2 + (-6c + 1)\, e_3\\
&\quad +(-12ab - 12a + 12b)e_4 + (-34b + 5)e_5 + (68a - 2)e_6 - 33e_7.
\end{aligned}$$

Proof. Using the multiplication table for the basis of $\mathbb{O}(a, b, c)$, we have

$$\begin{aligned}
\gamma^{*2} &= \left(\textstyle\sum_{s=0}^{7} \gamma^s e_s\right)\left(\textstyle\sum_{s=0}^{7} \gamma^s e_s\right)\\
&= 1 - 114243abc - 3363ac - 19601bc - 99ab - 3a - 17b - 577c\\
&\quad +2RO_0\\
&\quad +2\sqrt{2}\,(-40391abc - 6930bc - 1189ac - 35ab - 204c - 6b - a\\
&\quad +PO_0)\\
&= \xi_1 + 2RO_0 + 2\sqrt{2}\left(\xi_2 + PO_0\right).
\end{aligned}$$

The last equation is the Eq. (2.5). Smilarly

$$\begin{aligned}
\gamma^* \delta^* &= \left(\textstyle\sum_{s=0}^{7} \gamma^s e_s\right)\left(\textstyle\sum_{s=0}^{7} \delta^s e_s\right)\\
&= abc + ab + ac - bc + a - b - c - 1\\
&\quad +2RO_0 + 2\sqrt{2}\,[(bc - b - c)\, e_1\\
&\quad + (2ac - 2a - 2c)\, e_2 + (-6c + 1)\, e_3 + (-12ab - 12a + 12b)e_4\\
&\quad (-34b + 5)e_5 + (68a - 2)e_6 - 33e_7]\\
&= \tau + 2RO_0 + 2\sqrt{2}\sigma.
\end{aligned}$$

The last equation is the Eq. (2.6). The others can be proved similarly. □

After having Binet formulas, we can obtain some identities for the Pell and Pell-Lucas generalized octonions. The following theorem gives us Catalan's identities for the Pell and Pell-Lucas generalized octonions as follows.

Theorem 2.3. *Let n and r be integers, we have*

$$PO_{n+r}PO_{n-r} - PO_n^2 = (-1)^{n-r+1}\left[\frac{1}{4}(\tau + 2RO_0)\left(PL_{2r} + (-1)^{r-1}\right) + \sigma P_{2r}\right]$$

and

$$RO_{n+r}RO_{n-r} - RO_n^2 = (-1)^{n-r}\left[\frac{1}{2}(\tau + 2RO_0)(PL_{2r} - (-1)^r) + 2\sigma P_{2r}\right].$$

Proof. By using the Binet formulas for the Pell generelized octonions, we have

$$\begin{aligned}
PO_{n+r}PO_{n-r} - PO_n^2 \quad &= \frac{1}{8}\left[(\gamma^*\gamma^{n+r} - \delta^*\delta^{n+r})(\gamma^*\gamma^{n-r} - \delta^*\delta^{n-r})\right.\\
&\left. - (\gamma^*\gamma^n - \delta^*\delta^n)^2\right]\\
&= \frac{1}{8}[-\gamma^*\delta^*\gamma^{n+r}\delta^{n-r} - \delta^*\gamma^*\delta^{n+r}\gamma^{n-r}\\
&+\gamma^*\delta^*\gamma^n\delta^n + \delta^*\gamma^*\delta^n\gamma^n]\\
&= \frac{1}{8}\left[-\gamma^{n-r}\delta^{n-r}\left(\gamma^*\delta^*\gamma^{2r} + \delta^*\gamma^*\delta^{2r}\right) + 2(-1)^n(\tau + 2RO_0)\right]\\
&= \frac{1}{8}\left[(-1)^{n-r+1}\left((\tau + 2RO_0)\left(\gamma^{2r} + \delta^{2r}\right)\right.\right.\\
&\left.\left. 2\sqrt{2}\sigma\left(\gamma^{2r} - \delta^{2r}\right)\right) + 2(-1)^n(\tau + 2RO_0)\right]\\
&= \frac{1}{8}\left[2(-1)^{n-r+1}(\tau + 2RO_0)\left(PL_{2r} + (-1)^{r-1}\right)\right.\\
&= \left. +8(-1)^{n-r+1}\sigma P_{2r}\right]\\
&= (-1)^{n-r+1}\left[\tfrac{1}{4}(\tau + 2RO_0)\left(PL_{2r} + (-1)^{r-1}\right) + \sigma P_{2r}\right].
\end{aligned}$$

The second identity in theorem can be obtained similarly. □

For $r = 1$, Theorem 3 gives Cassini's identities for the Pell and Pell-Lucas generalized octonions as folllows.

Corollary 2.4. *For any integer n, we have*

$$PO_{n+1}PO_{n-1} - PO_n^2 = (-1)^n\left[\frac{1}{2}(\tau + 2RO_0) + \sigma\right]$$

and

$$RO_{n+1}RO_{n-1} - RO_n^2 = (-1)^{n-1}[(\tau + 2RO_0) + 2\sigma].$$

D'Ocagne's identities for the Pell and Pell-Lucas generalized octonions are given in the following theorem.

Theorem 2.5. *Let n and m be integers, we have*

$$PO_mPO_{n+1} - PO_{m+1}PO_n = (-1)^n[(\tau + 2RO_0)P_{m-n} + 2\sigma PL_{m-n}]$$

and

$$RO_mRO_{n+1} - RO_{m+1}RO_n = 2(-1)^{n+1}[(\tau + 2RO_0)P_{m-n} + 2\sigma PL_{m-n}].$$

Proof. By using the Binet formulas for the Pell generalized octonions, we get

$$\begin{aligned} PO_mPO_{n+1} - PO_{m+1}PO_n &= \frac{1}{8}\left(\gamma^*\gamma^m - \delta^*\delta^m\right)\left(\gamma^*\gamma^{n+1} - \delta^*\delta^{n+1}\right) \\ &-\frac{1}{8}\left(\gamma^*\gamma^{m+1} - \delta^*\delta^{m+1}\right)\left(\gamma^*\gamma^n - \delta^*\delta^n\right) \\ &= \frac{\sqrt{2}}{4}\left[(-1)^n\left(\gamma^*\delta^*\gamma^{m-n} - \delta^*\gamma^*\delta^{m-n}\right)\right]. \end{aligned}$$

If we substitute equations (2.6) and (2.7) into the last equation, then we have

$$\begin{aligned} PO_mPO_{n+1} - PO_{m+1}PO_n &= \frac{\sqrt{2}}{4}(-1)^n\left[2\sqrt{2}\left(\tau + 2RO_0\right)P_{m-n} + 4\sqrt{2}\sigma PL_{m-n}\right] \\ &= (-1)^n\left[\left(\tau + 2RO_0\right)P_{m-n} + 2\sigma PL_{m-n}\right]. \end{aligned}$$

We can obtain the second identity, similarly. □

3. Some Results for The Pell and Pell-Lucas Generalized Octonions

In this section, we give some identities which can be obtained from Binet formulas for the Pell and Pell-Lucas generalized octonions.

Theorem 3.1. *Pell and Pell-Lucas generalized octonions satisfy the following identities;*

$$RO_n^2 + PO_n^2 = \frac{3}{8}\left[2\left(\xi_1 + 2RO_0\right)PL_{2n} + 8\left(\xi_2 + PO_0\right)P_{2n}\right] + \frac{1}{4}(-1)^n\left(\tau + 2RO_0\right),$$

$$RO_n^2 - PO_n^2 = \frac{1}{8}\left[2\left(\xi_1 + 2RO_0\right)PL_{2n} + 8\left(\xi_2 + PO_0\right)P_{2n}\right] + \frac{3}{4}(-1)^n\left(\tau + 2RO_0\right),$$

$$RO_{n+r}PO_{n+s} - RO_{n+s}PO_{n+r} = (-1)^{n+r}\left(\tau + 2RO_0\right)P_{s-r},$$

$$PO_{m+n} + (-1)^n PO_{m-n} = 2PL_nPO_m,$$

$$PO_mRO_n - PO_nRO_m = (-1)^{m+1}\left(\tau + 2RO_0\right)P_{n-m},$$

$$PO_mRO_n - RO_mPO_n = (-1)^{m+1}\left[\left(\tau + 2RO_0\right)P_{n-m} - 2\sigma PL_{n-m}\right].$$

Proof. We prove the first and fourth identities. We need the Binet formulas for the Pell and Pell-Lucas generalized octonions.

$$\begin{aligned} RO_n^2 + PO_n^2 &= \frac{1}{4}\left(\gamma^*\gamma^n + \delta^*\delta^n\right)\left(\gamma^*\gamma^n + \delta^*\delta^n\right) \\ &+\frac{1}{8}\left(\gamma^*\gamma^n - \delta^*\delta^n\right)\left(\gamma^*\gamma^n - \delta^*\delta^n\right) \\ &= \frac{3}{8}\left[\gamma^{*2}\gamma^{2n} + \delta^{*2}\delta^{2n}\right] + \frac{1}{8}(-1)^n\left[\gamma^*\delta^* + \delta^*\gamma^*\right] \\ &= \frac{3}{8}\left[\left(\xi_1 + 2RO_0\right)\left(\gamma^{2n} + \delta^{2n}\right) + 2\sqrt{2}\left(\xi_2 + PO_0\right)\left(\gamma^{2n} - \delta^{2n}\right)\right] \\ &+\frac{1}{8}(-1)^n\left(\tau + 2RO_0\right) \\ &= \frac{3}{8}\left[2\left(\xi_1 + 2RO_0\right)PL_{2n} + 8\left(\xi_2 + PO_0\right)P_{2n}\right] + \frac{1}{4}(-1)^n\left(\tau + 2RO_0\right). \end{aligned}$$

Similarly, using Binet formulas again, we get

$$\begin{aligned}
PO_{m+n} + (-1)^n PO_{m-n} &= \frac{1}{2\sqrt{2}} \left[\gamma^* \gamma^{m+n} - \delta^* \delta^{m+n} + (-1)^n \left(\gamma^* \gamma^{m-n} - \delta^* \delta^{m-n}\right)\right] \\
&= \frac{1}{2\sqrt{2}} \left[\gamma^* \gamma^m \left(\gamma^n + \delta^n\right) - \delta^* \delta^m \left(\delta^n + \gamma^n\right)\right] \\
&= \frac{1}{\sqrt{2}} PL_n \left(\gamma^* \gamma^m - \delta^* \delta^m\right) \\
&= 2PL_n PO_m.
\end{aligned}$$

The other four identities in this theorem can be obtained similarly. □

Since the algebra $\mathbb{O}(a, b, c)$ is non-commutative, then we have the following theorem.

Theorem 3.2. *Let m and n be integers, then we have*

$$PO_n PO_m - PO_m PO_n = \sqrt{2} (-1)^{m+1} \sigma PL_{n-m}$$

and

$$RO_n RO_m - RO_m RO_n = 4 (-1)^m \sigma P_{n-m}. \tag{3.1}$$

Proof. Using Binet formulas for the Pell generalized octonions gives

$$\begin{aligned}
PO_n PO_m - PO_m PO_n &= \frac{1}{8} \left(\gamma^* \gamma^n - \delta^* \delta^n\right) \left(\gamma^* \gamma^m - \delta^* \delta^m\right) \\
&\quad -\frac{1}{8} \left(\gamma^* \gamma^m - \delta^* \delta^m\right) \left(\gamma^* \gamma^n - \delta^* \delta^n\right) \\
&= \frac{1}{8} \left[-\gamma^* \delta^* \gamma^n \delta^m - \delta^* \gamma^* \delta^n \gamma^m \right. \\
&\quad \left. +\gamma^* \delta^* \gamma^m \delta^n + \delta^* \gamma^* \delta^m \gamma^n\right] \\
&= \frac{1}{8} \left[-4\sqrt{2} \sigma \left(\gamma^n \delta^m + \delta^n \gamma^m\right)\right] \\
&= \frac{-1}{\sqrt{2}} \sigma \left[\gamma^m \delta^m \left(\gamma^{n-m} + \delta^{n-m}\right)\right] \\
&= \sqrt{2} (-1)^{m+1} \sigma PL_{n-m}.
\end{aligned}$$

Eq. (3.1) can be proved similarly. □

4. Results and Suggestions

In this paper, we study on the Pell and Pell-Lucas generalized octonions. We derive some new and interesting properties for the Pell and Pell-Lucas generalized octonions. After this study and results, Binet formulas, Catalan's, Cassini's and D'Ocagne's identities and some properties can be obtained on the k-Pell and k-Pell-Lucas generalized octonions.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] Akkus, I., Kecilioglu, O., *Split Fibonacci and Lucas Octonions*, Adv. Appl. Clifford Algebras, **25**(2015), 517–525.

[2] Akyiğit, M., Kösal, H.H., Tosun, M., *Fibonacci Generalized Quaternions*, Adv. Appl. Clifford Algebras, **24**(2014), 631–641.

[3] Catarino, P., *The Modified Pell and The Modified k-Pell Quaternions and Octonions*, Adv. Appl. Clifford Algebras, **26**(2)(2016), 577–590.

[4] Cimen, C.B., İpek, A., *On Pell Quaternions and Pell-Lucas Quaternions*, Adv. Appl. Clifford Algebras, **26**(2016), 39–51.

[5] Flaut, C., Stefanescu, M., *Some Equations over Generalized Quaternion and Octonion Division Algebras*, Bull. Math. Soc. Sci. Math. Roum., **52**(100)(4)(2009), 427–439.

[6] Halici, S., *On Fibonacci Quaternions*, Adv. Appl. Clifford Algebras, **22**(2012), 321–327.

[7] Horadam, A.F., *Complex Fibonacci Numbers and Fibonacci Quaternions*, Amer. Math. Monthly, **70**(1963), 289–291.

[8] Horadam, A.F., *Quaternion Recurrence Relations*, Ulam Quarterly, **2**(1993), 23–33.

[9] Iakin, A.L., *Generalized Quaternions of Higher Order*, Fibonacci Quart, **15**(1977), 343–346.

[10] Iyer, M.R., *A Note on Fibonacci Quaternions*, Fibonacci Quart, **3**(1969), 225–229.

[11] Keçilioğlu, O., Akkus, I., *The Fibonacci Octonions*, Adv. Appl. Clifford Algebras, **25**(2015), 151–158.

[12] Koshy, T., Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, Canada, 2001.

[13] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[14] Ramirez, J.L., *Some Combinatorial Properties of the k-Fibonacci and the k-Lucas Quaternions*, An. St. Univ. Ovidius Constanta, **23**(2)(2015), 201–212.

[15] Savin, D., *Some Properties of Fibonacci numbers, Fibonacci Octonions, and generalized Fibonacci-Lucas Octonions*, Adv. Difference Equ., **2015.1**(2015), 298.

[16] Swamy, M.N.S., *On Generalized Fibonacci Quaternions*, The Fib. Quarterly, **5**(1973), 547–550.

[17] Szynal-Liana, A., Wloch, I., *The Pell Quaternions and The Pell Octonions*, Adv. Appl. Clifford Algebras, **26**(2016), 435–440.