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# BEST PROXIMITY POINT THEOREMS FOR PROXIMAL $b$-CYCLIC CONTRACTIONS ON $b$-METRIC SPACES

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Abstract. In this paper, we first introduce a new notion of the property $(M_C)$ to improve and generalize the property $(G_C)$. After that, we present two new concepts, proximal $b$-cyclic contraction of the first type and second type, on $b$-metric spaces. Then, we obtain two best proximity point theorems for such mappings in the framework of best proximally complete $b$-metric spaces by using the property $(M_C)$. Hence, we generalize some results existing in the literature. Finally, we present some illustrative and interesting examples.

## 1. Introduction and Preliminaries

Fixed point theory has great importance in dealing with various problems in differential equations, approximation theory, control systems, nonlinear analysis and game theory. Hence, many authors have studied to develop fixed point theory. In this sense, it was proved the Banach contraction principle [5] which is considered the start of the fixed point theory on metric spaces. Because of its importance, many generalizations of this principle have been made by many authors in the literature [3,7,11,14,15]. One of these generalizations is obtained by Kirk et al. [13]. They introduced a new notion of the cyclic mapping and obtained the following nice result.

**Theorem 1.** *Let $(\mho, \rho)$ be a complete metric space, $\wp, \Re$ be nonempty closed subsets of $\mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a mapping. Assume that $H$ is a cyclic mapping, that is, $H(\wp) \subseteq \Re$ and $H(\Re) \subseteq \wp$. If there exists $q$ in $[0, 1)$ such that*

$$\rho(H\xi, H\upsilon) \leq q\rho(\xi, \upsilon) \tag{1}$$

*for all $\xi \in \wp$ and $\upsilon \in \Re$, then $H$ has a fixed point in $\wp \cap \Re$.*

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Note that, unlike the Banach contraction principle, $H$ is not necessary to be continuous in Theorem 1. Due to its applicability, there are many fixed point results on this topic in the literature. However, when considering $\wp \cap \Re = \emptyset$ in Theorem 1, $H$ cannot have a fixed point. Taking into account this situation, a new concept of the cyclic contraction mapping which is a generalization of inequality (1) has been introduced by Eldred and Veeremani [8], and hence obtained the existence of a point $\xi$ such that $\rho(\xi, H\xi) = \rho(\wp, \Re)$ which is a best proximity point of $H$ and is also an optimal solution of the minimization problem $\min_{\xi\in\wp} \rho(\xi, H\xi)$. Since a best proximity point result is a natural generalization of fixed point result, many authors have studied to obtain best proximity point results [1, 2, 16, 17]. Now, we recall this notion related result.

Let $(\mho, \rho)$ be a metric space and $\emptyset \neq \wp, \Re \subseteq \mho$. We will use the subsets of $\wp$ and $\Re$, respectively:

$$\wp_0 = \{\xi \in \wp : \rho(\xi, \upsilon) = \rho(\wp, \Re) \text{ for some } \upsilon \in \Re\}$$

and

$$\Re_0 = \{\upsilon \in \Re : \rho(\xi, \upsilon) = \rho(\wp, \Re) \text{ for some } \xi \in \wp\}$$

where $\rho(\wp, \Re) = \inf\{\rho(\xi, \upsilon) : \xi \in \wp \text{ and } \upsilon \in \Re\}$

**Definition 2.** *Let $(\mho, \rho)$ be a metric space, $\emptyset \neq \wp, \Re \subseteq \mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. $H : \wp \cup \Re \to \wp \cup \Re$ is called a cyclic contraction if there exists $q$ in $[0, 1)$ such that*

$$\rho(H\xi, H\upsilon) \leq q\rho(\xi, \upsilon) + (1 - q)\rho(\wp, \Re) \tag{2}$$

*for all $\xi \in \wp$ and $\upsilon \in \Re$.*

Recall that, let $(\mho, \rho)$ be a metric space and $\emptyset \neq \wp \subseteq \mho$. A set $\wp$ is called a boundedly compact if every bounded sequence in $\wp$ has a convergent subsequence in $\wp$.

**Theorem 3.** *Let $(\mho, \rho)$ be a complete metric space, $\emptyset \neq \wp, \Re \subseteq \mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a cyclic contraction mapping. Assume that either $\wp$ or $\Re$ is boundedly compact. Then, there exists $\xi^*$ in $\wp \cup \Re$ such that $\rho(\xi^*, H\xi^*) = \rho(\wp, \Re)$.*

The following definitions and theorems about best proximity point theory are important for our results.

**Definition 4** ( [12])**.** *Let $(\mho, \rho)$ be a metric space, $\emptyset \neq \wp, \Re \subseteq \mho$ and $\{\xi_n\}$, $\{\upsilon_n\}$ be sequences in $\wp$ and $\Re$, respectively. If for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\rho(\xi_m, \upsilon_n) < \rho(\wp, \Re) + \varepsilon$$

*for all $m, n \geq n_0$, then the sequence $\{(\xi_n, \upsilon_n)\}$ in $(\wp, \Re)$ is said to be a cyclically Cauchy sequence.*

It can be easily seen that a sequence $\{\xi_n\}$ in $\wp$ is a Cauchy sequence iff the sequence $\{(\xi_n, \xi_n)\}$ is a cyclically Cauchy sequence in $(\wp, \wp)$.

**Definition 5** ( [9]). *Let $(\mho, \rho)$ be a metric space and $\emptyset \neq \wp, \Re \subseteq \mho$. Assume that it satisfies the following condition:*

$$\left.\begin{array}{l}\rho(\xi, \upsilon_1) = \rho(\wp, \Re)\\ \rho(\xi, \upsilon_2) = \rho(\wp, \Re)\end{array}\right\} \Longrightarrow \upsilon_1 = \upsilon_2$$

*for all $\xi$ in $\wp$, and $\upsilon_1, \upsilon_2$ in $\Re$. Then, the pair $(\wp, \Re)$ is called a semi-sharp proximinal pair.*

It can be seen that if $\wp \cap \Re \neq \emptyset$, then $(\wp, \Re)$ and $(\Re, \wp)$ are semi-sharp proximinal. In particular, for every subset $\wp$ of a metric space, $(\wp, \wp)$ is semi-sharp proximinal.

**Definition 6.** *The pair $(\wp, \Re)$ is said to be a best proximally complete space if and only if for every cyclically Cauchy sequence $\{(\xi_n, \upsilon_n)\}$ in $(\wp, \Re)$ the sequences $\{\xi_n\}$ and $\{\upsilon_n\}$ are convergent in $\wp$ and $\Re$, respectively.*

We give some facts about best proximally complete space. Let $(\mho, \rho)$ be a metric space and $\emptyset \neq \wp, \Re \subseteq \mho$.

- If $(\wp, \Re)$ is a best proximally complete space, then $(\wp, \Re)$ and $(\Re, \wp)$ are semi-sharp proximinal.
- $\wp$ is a complete metric space iff $(\wp, \wp)$ is a best proximally complete space
- If $\wp, \Re$ are closed sets with $\rho(\wp, \Re) = 0$, then $(\wp, \Re)$ is a best proximally complete space.

Basha et al. [4] introduced a new and nice concept which is weaker than cyclic contraction mapping as follows:

**Definition 7.** *Let $(\mho, \rho)$ be a metric space, $\emptyset \neq \wp, \Re \subseteq \mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. If there exists $q$ in $[0, 1)$ such that*

$$\left.\begin{array}{l}\rho(\zeta, H\xi) = \rho(\wp, \Re)\\ \rho(v, H\upsilon) = \rho(\wp, \Re)\end{array}\right\} \Longrightarrow \rho(\zeta, v) \leq q\rho(\xi, \upsilon) + (1 - q)\rho(\wp, \Re)$$

*for all $\zeta, \xi$ in $\wp$ and $v, \upsilon$ in $\Re$, then $H$ is called proximal cyclic contraction of the first type.*

**Definition 8.** *Let $(\mho, \rho)$ be a metric space, $\emptyset \neq \wp, \Re \subseteq \mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. If there exists $q$ in $[0, 1)$ such that*

$$\left.\begin{array}{l}\rho(\zeta, H\xi) = \rho(\wp, \Re)\\ \rho(v, H\upsilon) = \rho(\wp, \Re)\end{array}\right\} \Longrightarrow \rho(H\zeta, Hv) \leq q\rho(H\xi, H\upsilon) + (1 - q)\rho(\wp, \Re)$$

*for all $\zeta, \xi$ in $\wp$ and $v, \upsilon$ in $\Re$, then $H$ is called proximal cyclic contraction of the second type.*

Then, Basha et al. [4] obtained best proximity point results for proximal cyclic contraction of the first and second type as follows:

**Theorem 9.** *Let $(\mho, \rho)$ be a metric space and $\emptyset \neq \wp, \Re \subseteq \mho$ such that $(\wp, \Re)$ is a best proximally complete space. Assume that $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ is a* *proximal cyclic contraction of the first type satisfying* $H(\wp_0) \subseteq \Re_0$ *and* $H(\Re_0) \subseteq \wp_0$. *Then, there exists a unique pair* $(\xi, \upsilon)$ *in* $\wp \times \Re$ *satisfying the conditions that*

$$\begin{aligned} \rho(\xi, H\xi) &= \rho(\wp, \Re), \\ \rho(\upsilon, H\upsilon) &= \rho(\wp, \Re), \\ \rho(\xi, \upsilon) &= \rho(\wp, \Re). \end{aligned}$$

**Theorem 10.** *Let* $(\mho, \rho)$ *be a metric space and* $\emptyset \neq \wp, \Re \subseteq \mho$ *such that* $(\wp, \Re)$ *is a best proximally complete space. Assume that* $\wp_0 \neq \emptyset$ *and* $H : \wp \cup \Re \to \wp \cup \Re$ *is a proximal relatively continuous cyclic contraction of the second type satisfying* $H(\wp_0) \subseteq \Re_0$ *and* $H(\Re_0) \subseteq \wp_0$. *Then, there exists a unique pair* $(\xi, \upsilon)$ *in* $\wp \times \Re$ *satisfying the conditions that*

$$\begin{aligned} \rho(\xi, H\xi) &= \rho(\wp, \Re), \\ \rho(\upsilon, H\upsilon) &= \rho(\wp, \Re), \\ \rho(\xi, \upsilon) &= \rho(\wp, \Re). \end{aligned}$$

On the other hand, by introducing a new concept which is b-metric, Czerwik [6] obtained a generalization of the Banach contraction principle in a different way from used in the literature.

**Definition 11.** *Let* $\mho \neq \emptyset$ *and* $\rho : \mho \times \mho \to [0, \infty)$ *be a function such that for all* $\xi, \upsilon, z \in \mho$,

b1) $\rho(\xi, \upsilon) = 0$ *if and only if* $\xi = \upsilon$,
b2) $\rho(\xi, \upsilon) = \rho(\upsilon, \xi)$,
b3) $\rho(\xi, z) \leq s[\rho(\xi, \upsilon) + \rho(\upsilon, z)]$ *where* $s \geq 1$.

*Then,* $\rho$ *is called b-metric on* $\mho$ *and* $(\mho, \rho)$ *is called b-metric space.*

Note that every metric space is a $b$-metric space, but the converse may not be true. Indeed, the following well-known example of $b$-metric spaces shows this fact. Let $\mho = \mathbb{R}$ and $\rho : \mho \times \mho \to [0, \infty)$ be a function defined as $\rho(\xi, \upsilon) = (\xi - \upsilon)^2$ for all $\xi, \upsilon \in \mho$. Then $(\mho, \rho)$ is a $b$-metric space with $s = 2$. If we take $\xi = 10$, $\upsilon = 5$ and $z = 1$, then

$$\rho(10, 1) = 81 > \rho(10, 5) + \rho(5, 1).$$

Hence, it is not a metric space .

Let $(\mho, \rho)$ be a $b$-metric space with $s \geq 1$. We denote the family of all open subsets of $\mho$ by $\tau_\rho$ which has as a base the family

$$\{B(\xi, r) : \xi \in \mho \text{ and } r > 0\}$$

where

$$B(\xi, r) = \{\upsilon \in \mho : \rho(\xi, \upsilon) < r\}.$$

The sequence $\{\xi_n\}$ in $\mho$ is said to be a Cauchy sequence if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho(\xi_n, \xi_m) < \varepsilon$ for all $n, m \geq n_0$. $(\mho, \rho)$ is said to be a complete $b$-metric space if every Cauchy sequence in $\mho$ converges to some $\xi \in \mho$ with respect to $\tau_\rho$.

Note that, unlike ordinary metric, $b$-metric may not be continuous. To overcome this disadvantage, Felhi and Aydi [10] introduced the following definition.

**Definition 12.** *Let $(\mho,\rho)$ be a $b$-metric space with $s \geq 1$. We say that $(\mho,\rho)$ satisfies the property $(G_C)$ if for all sequences $\{\xi_n\}$, $\{\upsilon_n\}$ in $\mho$ and $\xi,\upsilon \in \mho$, we have*

$$\lim_{n\to\infty}\rho(\xi_n,\xi) = \lim_{n\to\infty}\rho(\upsilon_n,\upsilon) = 0 \implies \lim_{n\to\infty}\rho(\xi_n,\upsilon_n) = \rho(\xi,\upsilon).$$

In this paper, we present two new concepts, proximal $b$-cyclic contraction of the first type and second type, on $b$-metric spaces. Then, we obtain two best proximity point theorems for such mappings in the framework of best proximally complete $b$-metric spaces. To do this, we introduce a new notion of the property $(M_C)$ to improve and generalize the property $(G_C)$. Hence, we extend some results used in the literature such as main result of [4]. Finally, we present some illustrative and interesting examples.

## 2. Proximal $b$-cyclic contraction of the first type

In this section, we first state the following definitions of the property $(M_C)$.

**Definition 13.** *Let $(\mho,\rho)$ be a $b$-metric space with $s \geq 1$ and $\emptyset \neq \wp,\Re \subseteq \mho$. The pair $(\wp,\Re)$ satisfies the property $(M_C)$ if for all sequences $\{\xi_n\}$ in $\wp_0$, $\{\upsilon_n\}$ in $\Re_0$ and $\xi \in \wp$, $\upsilon \in \Re$, we have*

$$\lim_{n\to\infty}\rho(\xi_n,\xi) = \lim_{n\to\infty}\rho(\upsilon_n,\upsilon) = 0 \implies \lim_{n\to\infty}\rho(\xi_n,\upsilon_n) = \rho(\xi,\upsilon).$$

It can be seen that if $(\mho,\rho)$ satisfies the property $(G_C)$, then $(\wp,\Re)$ satisfies the property $(M_C)$, but the converse may not be true. The following example is important to show this fact.

**Example 14.** *Let $\mho = \left\{0, \frac{1}{4}, \frac{1}{8}, \cdots, \frac{1}{4n}, \cdots\right\}$ and the function $\rho : \mho \times \mho \to \mathbb{R}$ defined by*

$$\rho(\xi,\upsilon) = \begin{cases} 0 & , \quad \xi = \upsilon \\ 2 & , \quad \xi \neq \upsilon \in \left\{0, \frac{1}{4}\right\} \\ |\xi - \upsilon| & , \quad \xi \neq \upsilon \in \{0\} \cup \left\{\frac{1}{4n} : n \geq 2\right\} \\ 5 & , \quad \textit{otherwise} \end{cases}$$

*for all $\xi,\upsilon \in \mho$. Then, it is clear that $\rho$ is a $b$-metric space with $s = \frac{5}{2}$. Further, $(\mho,\rho)$ does not satisfy the property $(G_C)$. Indeed, if we take $\xi_n = \frac{1}{4}$ and $\upsilon_n = \frac{1}{4n}$ for all $n \geq \dot{2}$, it can be seen that $\xi_n \to \frac{1}{4}$ and $\upsilon_n \to 0$. However,*

$$\lim_{n\to\infty}\rho\left(\frac{1}{4n},\frac{1}{4}\right) = 5 \neq 2 = \rho\left(0,\frac{1}{4}\right).$$

*Let $\wp = \left\{\frac{1}{4n} : n \geq 3\right\}$ and $\Re = \left\{\frac{1}{4},\frac{1}{8}\right\}$. Then, it is clear that the pair $(\wp,\Re)$ satisfies the property $(M_C)$.*

Now, we introduce definition of a proximal $b$-cyclic contraction of the first type.

**Definition 15.** *Let $(\mho,\rho)$ be a b-metric space with $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. $H$ is said to be a proximal b-cyclic contraction of the first type if there exists $q$ in $\left(0, \frac{1}{s^3}\right)$ such that*

$$\left.\begin{array}{l}\rho(\zeta_1, H\xi_1) = \rho(\wp, \Re)\\ \rho(\zeta_2, H\xi_2) = \rho(\wp, \Re)\end{array}\right\} \Longrightarrow \rho(\zeta_1, \zeta_2) \leq q\rho(\xi_1, \xi_2) + (1-q)\rho(\wp, \Re) \tag{3}$$

*for all $\zeta_1, \xi_1 \in \wp$ and $\zeta_2, \xi_2 \in \Re$.*

**Proposition 16.** *Let $(\mho,\rho)$ be a b-metric space with $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \mho$ with $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a proximal b-cyclic contraction of the first type satisfying $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. Then, there exist two sequences $\{\xi_n\}$ and $\{\upsilon_n\}$ with initial point $\xi_0$ in $\wp$ and $\upsilon_0$ in $\Re$ such that*

$$\rho(\xi_n, H\xi_{n-1}) = \rho(\wp, \Re)$$

*and*

$$\rho(\upsilon_n, H\upsilon_{n-1}) = \rho(\wp, \Re)$$

*for all $n \geq 1$. Moreover, we have $\rho(\xi_n, \upsilon_n) \to \rho(\wp, \Re)$ as $n \to \infty$.*

*Proof.* Let $\xi_0 \in \wp_0$ be an arbitrary point. Since $H\xi_0 \in H(\wp_0) \subseteq \Re_0$, there exists $\xi_1 \in \wp_0$ such that

$$\rho(\xi_1, H\xi_0) = \rho(\wp, \Re).$$

Similarly, since $H\xi_1 \in H(\wp_0) \subseteq \Re_0$, there exists $\xi_2 \in \wp_0$ such that

$$\rho(\xi_2, H\xi_1) = \rho(\wp, \Re).$$

Repeating this process, we can construct a sequence $\{\xi_n\}$ in $\wp_0$ such that

$$\rho(\xi_n, H\xi_{n-1}) = \rho(\wp, \Re).$$

for all $n \geq 1$. Similarly, we can find a sequence $\{\upsilon_n\}$ in $\Re_0$ with initial point $\upsilon_0 \in \Re_0$ such that

$$\rho(\upsilon_n, H\upsilon_{n-1}) = \rho(\wp, \Re).$$

for all $n \geq 1$. Using the implication (3), we have

$$\rho(\xi_n, \upsilon_n) \leq q\rho(\xi_{n-1}, \upsilon_{n-1}) + (1-q)\rho(\wp, \Re)$$

for all $n \geq 1$. Hence, we get

$$\begin{aligned}
\rho(\xi_n, \upsilon_n) &\leq q\rho(\xi_{n-1}, \upsilon_{n-1}) + (1-q)\rho(\wp, \Re)\\
&\leq q\left(q\rho(\xi_{n-2}, \upsilon_{n-2}) + (1-q)\rho(\wp, \Re)\right) + (1-q)\rho(\wp, \Re)\\
&= q^2\rho(\xi_{n-2}, \upsilon_{n-2}) + (1-q)(1+q)\rho(\wp, \Re)\\
&\ \ \vdots\\
&\leq q^n\rho(\xi_0, \upsilon_0) + (1-q)\left(1 + q + \cdots + q^{n-1}\right)\rho(\wp, \Re)\\
&= q^n\rho(\xi_0, \upsilon_0) + (1-q^n)\rho(\wp, \Re)
\end{aligned}$$

for all $n \geq 1$ and so we have $\rho(\xi_n, \upsilon_n) \to \rho(\wp, \Re)$ as $n \to \infty$. □

**Remark 17.** *If we consider the sequences $\{\xi_n\}$ and $\{\upsilon_n\}$ mentioned Proposition 16, we have $\rho(\xi_n, \upsilon_m) \leq \rho(\xi_{n-1}, \upsilon_{m-1})$ for all $n, m \in \mathbb{N}$.*

**Proposition 18.** *Let $(\mho, \rho)$ be a b-metric space with $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \mho$ with $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a proximal b-cyclic contraction of the first type satisfying $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. Then, the sequences $\{\xi_n\}$ and $\{\upsilon_n\}$ which are constructed as in Proposition 16 are bounded.*

*Proof.* From Proposition 16, we have $\rho(\xi_n, \upsilon_n) \to \rho(\wp, \Re)$ as $n \to \infty$, and so it is enough to prove that $\{\xi_n\}$ is a bounded sequence.

$$\begin{aligned} \rho(\xi_n, \upsilon_0) &\leq s\rho(\xi_n, \upsilon_{n+1}) + s^2\rho(\upsilon_{n+1}, \xi_{n+1}) + s^3\rho(\xi_{n+1}, \upsilon_1) + s^3\rho(\upsilon_1, \upsilon_0) \\ &\leq s\rho(\xi_0, \upsilon_1) + s^2\rho(\upsilon_0, \xi_0) \\ &\quad + s^3 \{q\rho(\xi_n, \upsilon_0) + (1-q)\rho(\wp, \Re)\} + s^3\rho(\upsilon_1, \upsilon_0). \end{aligned}$$

Hence, we have

$$(1 - s^3 q)\rho(\xi_n, \upsilon_0) \leq s^3 \left(\rho(\xi_0, \upsilon_1) + \rho(\upsilon_0, \xi_0) + \rho(\upsilon_1, \upsilon_0) + \rho(\wp, \Re)\right)$$

which implies

$$\rho(\xi_n, \upsilon_0) \leq \frac{1}{\left(\frac{1}{s^3} - q\right)} \left(\rho(\xi_0, \upsilon_1) + \rho(\upsilon_0, \xi_0) + \rho(\upsilon_1, \upsilon_0) + \rho(\wp, \Re)\right)$$

Hence, $\{\xi_n\}$ is a bounded sequence in $\wp$. Then, $\{\upsilon_n\}$ is a bounded sequence in $\Re$. □

**Theorem 19.** *Let $\wp, \Re$ be nonempty subsets of a b-metric space $(\mho, \rho)$ with $s \geq 1$ such that $(\wp, \Re)$ is a best proximally complete space and $H : \wp \cup \Re \to \wp \cup \Re$ be a proximal b-cyclic contraction of the first type. Assume that $\wp_0 \neq \emptyset$, $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$ and the pair $(\wp, \Re)$ satisfies the property $(M_C)$. Then, there exists unique pair $(\xi^*, \upsilon^*)$ in $\wp \times \Re$ such that*

$$\begin{aligned} \rho(\xi^*, H\xi^*) &= \rho(\wp, \Re), \\ \rho(\upsilon^*, H\upsilon^*) &= \rho(\wp, \Re), \\ \rho(\xi^*, \upsilon^*) &= \rho(\wp, \Re). \end{aligned}$$

*Proof.* Let $\{\xi_n\}$ and $\{\upsilon_n\}$ be two sequences constructed as in Proposition 16. Hence, we get

$$\begin{aligned} \rho(\xi_n, H\xi_{n-1}) &= \rho(\wp, \Re) \qquad (4) \\ \rho(\upsilon_n, H\upsilon_{n-1}) &= \rho(\wp, \Re) \end{aligned}$$

for all $n \geq 1$. From Proposition 18, we have $\{\xi_n\}$ and $\{\upsilon_n\}$ are bounded sequences. Now, we shall show that $\{(\xi_n, \upsilon_n)\}$ is a cyclically Cauchy sequence. Without lost of generality, assume that $n > m$. Hence, we get

$$\begin{aligned} \rho(\xi_m, \upsilon_n) &\leq q\rho(\xi_{m-1}, \upsilon_{n-1}) + (1-q)\rho(\wp, \Re) \\ &\leq q\left(q\rho(\xi_{m-2}, \upsilon_{n-2}) + (1-q)\rho(\wp, \Re)\right) + (1-q)\rho(\wp, \Re) \end{aligned}$$

$$\begin{aligned}
&= q^2\rho(\xi_{m-2}, \upsilon_{n-2}) + (1-q)(1+q)\rho(\wp, \Re) \\
&\ \vdots \\
&\le q^m\rho(\xi_0, \upsilon_{n-m}) + (1-q)\left(1+q+\cdots+q^{m-1}\right)\rho(\wp, \Re) \\
&= q^m\rho(\xi_0, \upsilon_{n-m}) + (1-q^m)\rho(\wp, \Re)
\end{aligned}$$

Hence, using boundedness of $\{\xi_n\}$ and $\{\upsilon_n\}$, $\{(\xi_n, \upsilon_n)\}$ is a cyclically Cauchy sequence. Since $(\wp, \Re)$ is a best proximally complete space, there exist $\xi^* \in \wp$ and $\upsilon^* \in \Re$ such that

$$\lim_{n\to\infty} \xi_n = \xi^*$$

and

$$\lim_{n\to\infty} \upsilon_n = \upsilon^*$$

Further, since the pair $(\wp, \Re)$ satisfies the property $(M_C)$, from Proposition 16, we have

$$\rho(\xi^*, \upsilon^*) = \lim_{n\to\infty} \rho(\xi_n, \upsilon_n) = \rho(\wp, \Re).$$

Hence, we get $\xi^* \in \wp_0$ and $\upsilon^* \in \Re_0$. Since $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$, there exist $a \in \wp_0$ and $b \in \Re_0$ such that

$$\begin{aligned}
\rho(a, H\xi^*) &= \rho(\wp, \Re) \\
\rho(b, H\upsilon^*) &= \rho(\wp, \Re). \qquad (5)
\end{aligned}$$

Using the implication (3), we have

$$\rho(a, b) \le q\rho(\xi^*, \upsilon^*) + (1-q)\rho(\wp, \Re).$$

Therefore, it can be seen that

$$\rho(a, b) = \rho(\wp, \Re).$$

Using the implication (3), from (4) and (5), we have

$$\rho(\xi_{n+1}, b) \le q\rho(\xi_n, \upsilon^*) + (1-q)\rho(\wp, \Re). \qquad (6)$$

Taking limit $n \to \infty$ in the last inequality, we obtain

$$\rho(\xi^*, b) = \rho(\wp, \Re).$$

Since $(\wp, \Re)$ is a best proximally complete space, the pair $(\wp, \Re)$ is semi-sharp proximinal. Hence, we have $\xi^* = a$. Therefore, we can conclude that

$$\rho(\xi^*, H\xi^*) = \rho(\wp, \Re).$$

Similarly, it can be seen that

$$\rho(\upsilon^*, H\upsilon^*) = \rho(\wp, \Re).$$

Moreover, since $(\wp, \Re)$ and $(\Re, \wp)$ are semi-sharp proximinal, we have $\xi^* = H\upsilon^*$ and $\upsilon^* = H\xi^*$.

Now, we shall show the uniqueness. Let's consider another pair $(\zeta, \upsilon)$ in $\wp \times \Re$ satisfying

$$\begin{aligned} \rho(\zeta, H\zeta) &= \rho(\wp, \Re) \\ \rho(\upsilon, H\upsilon) &= \rho(\wp, \Re) \\ \rho(\zeta, \upsilon) &= \rho(\wp, \Re). \end{aligned}$$

Since the pair $(\wp, \Re)$ is semi-sharp proximinal, we have $H\zeta = \upsilon$ and $H\upsilon = \zeta$. Using the implication (3), we have

$$\rho(\xi^*, \upsilon) \leq q\rho(\xi^*, \upsilon) + (1-q)\rho(\wp, \Re),$$

and so we get $\rho(\xi^*, \upsilon) = \rho(\wp, \Re)$. The pair $(\wp, \Re)$ is semi-sharp proximinal and $\rho(\xi^*, \upsilon) = \rho(\wp, \Re) = \rho(\zeta, \upsilon)$, so we can conclude that $\xi^* = \zeta$. Similarly, it can be seen that $\upsilon^* = \upsilon$. □

**Example 20.** *Let $\mho = \mathbb{N} \cup \{\infty\}$ and the function $\rho : \mho \times \mho \to [0, \infty)$ defined by*

$$\rho(u,v) = \begin{cases} 0 & , \quad u = v \\ 1 & , \quad u \neq v \in \{1,2\} \\ \left|\frac{1}{u} - \frac{1}{v}\right| & , \quad \text{if one of } u \text{ and } v \text{ is odd and the other is odd or } \infty, \\ 4 & , \quad \text{if one of } u \text{ and } v \text{ is even and the other is even or } \infty \\ 3 & , \quad \text{otherwise} \end{cases}$$

*Then, $(\mho, \rho)$ is a b-metric space with $s = \frac{4}{3}$. Consider the sets $\wp = \{2u-1 : u \geq 1\}$ and $\Re = \{2u : u \geq 1\}$. Hence, $\rho(\wp, \Re) = 1$, $\wp_0 = \{1\}$ and $\Re_0 = \{2\}$. It can be seen that $(\wp, \Re)$ satisfies the property $(M_C)$. Also the pair $(\wp, \Re)$ is a best proximally complete space. Indeed, let $(\xi_n, \upsilon_n)$ be a cyclically Cauchy sequence in $(\wp, \Re)$. Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\rho(\xi_m, \upsilon_n) < \rho(\wp, \Re) + \varepsilon$$

*for all $m, n \geq n_0$. Then, there exists $n_0 \in \mathbb{N}$ such that $\xi_m = 1$ and $\upsilon_n = 2$ for all $n, m \geq n_0$. Therefore, $\{\xi_n\}$ and $\{\upsilon_n\}$ are convergent sequences in $\wp$ and $\Re$, respectively. Let $H : \wp \cup \Re \to \wp \cup \Re$ be a mapping as*

$$H\xi = \begin{cases} 2u & , \quad \xi = 2u-1 \\ 2u-1 & , \quad \xi = 2u \end{cases}$$

*for all $u \geq 1$. Hence, it is clear that $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. Further, $H$ satisfies the implication (3). Thus, all hypotheses of Theorem 19 hold. Hence, $u = 1$ is a best proximity point of $H$ in $\wp \cup \Re$. Note that, the pair $(\wp, \Re)$ does not have the property $(G_C)$.*

If we take $s = 1$ in Theorem 19, we obtain the following result which is the main result of [4].

**Corollary 21.** *Let $\wp, \Re$ be nonempty subsets of a metric space $(\mho, \rho)$ such that $(\wp, \Re)$ is a best proximally complete space and $H : \wp \cup \Re \to \wp \cup \Re$ be a proximal cyclic* *contraction of the first type. Assume that $\wp_0 \neq \emptyset$, $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. Then, there exists unique pair $(\xi^*, \upsilon^*)$ in $\wp \times \Re$ such that*

$$\begin{aligned} \rho(\xi^*, H\xi^*) &= \rho(\wp, \Re), \\ \rho(\upsilon^*, H\upsilon^*) &= \rho(\wp, \Re), \\ \rho(\xi^*, \upsilon^*) &= \rho(\wp, \Re). \end{aligned}$$

## 3. Proximal $b$-cyclic contraction of the second type

Now, we introduce the following two definitons on $b$-metric spaces.

**Definition 22.** *Given nonempty subsets $\wp$ and $\Re$ of a b-metric space with $s \geq 1$, $\Re$ is said to have s-uniform approximation in $\wp$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left. \begin{array}{c} \rho(\xi_1, \upsilon_1) = \rho(\wp, \Re) \\ \rho(\xi_2, \upsilon_2) = \rho(\wp, \Re) \\ \rho(\upsilon_1, \upsilon_2) < \delta \end{array} \right\} \Longrightarrow \rho(\xi_1, \xi_2) < \varepsilon \tag{7}$$

*for all $\xi_1, \xi_2$ in $\wp$ and $\upsilon_1, \upsilon_2$ in $\Re$.*

**Definition 23.** *Let $(\mho, \rho)$ be a b-metric space with $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \mho$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. $H$ is said to be proximal b-cyclic contraction of the second type if there exists $q \in \left(0, \frac{1}{s^3}\right)$ such that*

$$\left. \begin{array}{c} \rho(\zeta_1, H\xi_1) = \rho(\wp, \Re) \\ \rho(\zeta_2, H\xi_2) = \rho(\wp, \Re) \end{array} \right\} \Longrightarrow \rho(H\zeta_1, H\zeta_2) \leq q\rho(H\xi_1, H\xi_2) + (1-q)\rho(\wp, \Re) \tag{8}$$

*for all $\zeta_1, \xi_1 \in \wp$ and $\zeta_2, \xi_2 \in \Re$.*

**Proposition 24.** *Let $(\mho, \rho)$ be a b-metric space with $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \mho$ with $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a proximal b-cyclic contraction of the second type satisfying $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. Then, there exist two sequences $\{\xi_n\}$ and $\{\upsilon_n\}$ such that*

$$\rho(\xi_n, H\xi_{n-1}) = \rho(\wp, \Re)$$

*and*

$$\rho(\upsilon_n, H\upsilon_{n-1}) = \rho(\wp, \Re)$$

*for all $n \geq 1$. Moreover, we have $\rho(H\xi_n, H\upsilon_n) \to \rho(\wp, \Re)$ as $n \to \infty$.*

*Proof.* Proceeding as in the proof of Proposition 16, we can construct a sequence $\{\xi_n\}$ in $\wp_0$ and $\{\upsilon_n\}$ in $\Re_0$ such that

$$\rho(\xi_n, H\xi_{n-1}) = \rho(\wp, \Re).$$

and

$$\rho(\upsilon_n, H\upsilon_{n-1}) = \rho(\wp, \Re).$$

for all $n \geq 1$. Using the implication (8), we have

$$\rho(H\xi_n, H\upsilon_n) \leq q\rho(H\xi_{n-1}, H\upsilon_{n-1}) + (1-q)\rho(\wp, \Re)$$

for all $n \geq 1$. Hence, we get

$$\begin{aligned}
\rho(H\xi_n, Hv_n) &\leq q\rho(H\xi_{n-1}, Hv_{n-1}) + (1-q)\rho(\wp, \Re) \\
&\leq q\left(q\rho(H\xi_{n-2}, Hv_{n-2}) + (1-q)\rho(\wp, \Re)\right) + (1-q)\rho(\wp, \Re) \\
&= q^2\rho(H\xi_{n-2}, Hv_{n-2}) + (1-q)(1+q)\rho(\wp, \Re) \\
&\vdots \\
&\leq q^n\rho(H\xi_0, Hv_0) + (1-q)\left(1 + q + \cdots + q^{n-1}\right)\rho(\wp, \Re) \\
&= q^n\rho(H\xi_0, Hv_0) + (1-q^n)\rho(\wp, \Re)
\end{aligned}$$

for all $n \geq 1$ and so we have $\rho(H\xi_n, Hv_n) \to \rho(\wp, \Re)$ as $n \to \infty$. □

**Remark 25.** *If we consider the sequences $\{\xi_n\}$ and $\{v_n\}$ mentioned Proposition 24, we have $\rho(H\xi_n, Hv_m) \leq \rho(H\xi_{n-1}, Hv_{m-1})$ for all $n, m \in \mathbb{N}$.*

**Proposition 26.** *Let $(\mho, \rho)$ be a b-metric space with $s \geq 1$, $\emptyset \neq \wp, \Re \subseteq \mho$ with $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a proximal b-cyclic contraction of the second type satisfying $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. Then, for the sequence which is constructed as in Proposition 24, the sequences $\{H\xi_n\}$ and $\{Hv_n\}$ are bounded.*

*Proof.* From Proposition 24, we have $\rho(H\xi_n, Hv_n) \to \rho(\wp, \Re)$ as $n \to \infty$, and so it is enough to prove that $\{H\xi_n\}$ is a bounded sequence.

$$\begin{aligned}
\rho(H\xi_n, Hv_0) &\leq s\rho(H\xi_n, Hv_{n+1}) + s^2\rho(Hv_{n+1}, H\xi_{n+1}) + s^3\rho(H\xi_{n+1}, Hv_1) \\
&\quad + s^3\rho(Hv_1, Hv_0) \\
&\leq s\rho(H\xi_0, Hv_1) + s^2\rho(Hv_0, H\xi_0) \\
&\quad + s^3\left\{q\rho(H\xi_n, Hv_0) + (1-q)\rho(\wp, \Re)\right\} + s^3\rho(Hv_1, Hv_0).
\end{aligned}$$

Hence, we have

$$(1 - s^3 q)\rho(H\xi_n, Hv_0) \leq s^3\left(\rho(H\xi_0, Hv_1) + \rho(Hv_0, H\xi_0) + \rho(Hv_1, Hv_0) + \rho(\wp, \Re)\right)$$

which implies

$$\rho(H\xi_n, Hv_0) \leq \frac{1}{\left(\frac{1}{s^3} - q\right)}\left(\rho(H\xi_0, Hv_1) + \rho(Hv_0, H\xi_0) + \rho(Hv_1, Hv_0) + \rho(\wp, \Re)\right)$$

Hence, $\{H\xi_n\}$ is a bounded sequence in $\wp$. Then, $\{Hv_n\}$ is a bounded sequence in $\Re$ □

**Theorem 27.** *Let $\wp, \Re$ be nonempty subsets of a b-metric space $(\mho, \rho)$ with $s \geq 1$ such that $(\wp, \Re)$ is a best proximally complete space, $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a continuous proximal b-cyclic contraction of the second type satisfying $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. If the pair $(\wp, \Re)$ satisfies the property $(M_C)$ and $\Re$ has a s-uniform approximation in $\wp$, then there exists unique pair $(\xi^*, v^*)$ in $\wp \times \Re$ such that*

$$\rho(\xi^*, H\xi^*) = \rho(\wp, \Re),$$

$$\begin{aligned}\rho(\upsilon^*, H\upsilon^*) &= \rho(\wp, \Re),\\ \rho(\xi^*, \upsilon^*) &= \rho(\wp, \Re).\end{aligned}$$

*Proof.* Let $\{\xi_n\}$ and $\{\upsilon_n\}$ be two sequences satisfying

$$\begin{aligned}\rho(\xi_{n+1}, H\xi_n) &= \rho(\wp, \Re)\\ \rho(\upsilon_{n+1}, H\upsilon_n) &= \rho(\wp, \Re)\end{aligned} \tag{9}$$

for all $n \geq 1$, as in Proposition 24. Then, Proposition 26, $\{H\xi_n\}$ and $\{H\upsilon_n\}$ are bounded sequences. Without lost of generality, we can take $n > m$. Then, we have

$$\begin{aligned}\rho(H\xi_m, H\upsilon_n) &\leq q\rho(H\xi_{m-1}, H\upsilon_{n-1}) + (1-q)\rho(\wp, \Re)\\ &\leq q\left(q\rho(H\xi_{m-2}, H\upsilon_{n-2}) + (1-q)\rho(\wp, \Re)\right) + (1-q)\rho(\wp, \Re)\\ &= q^2\rho(H\xi_{m-2}, H\upsilon_{n-2}) + (1-q)(1+q)\rho(\wp, \Re)\\ &\vdots\\ &\leq q^m\rho(H\xi_0, H\upsilon_{n-m}) + (1-q)\left(1+q+\cdots+q^{m-1}\right)\rho(\wp, \Re)\\ &= q^m\rho(H\xi_0, H\upsilon_{n-m}) + (1-q^m)\rho(\wp, \Re)\end{aligned}$$

Hence, using boundedness of $\{H\xi_n\}$ and $\{H\upsilon_n\}$, $\{(H\upsilon_n, H\xi_n)\}$ is a cyclically Cauchy sequence in $(\wp, \Re)$. Because $(\wp, \Re)$ is a best proximally complete space, there exist $\xi^* \in \wp$ and $\upsilon^* \in \Re$ such that

$$\lim_{n\to\infty} H\upsilon_n = \xi^*$$

and

$$\lim_{n\to\infty} H\xi_n = \upsilon^*.$$

Further, since $(\mho, \rho)$ satisfies the property $(M_C)$, we have, from Proposition 24,

$$\rho(\xi^*, \upsilon^*) = \lim_{n\to\infty} \rho(H\xi_n, H\upsilon_n) = \rho(\wp, \Re). \tag{10}$$

Let $\varepsilon > 0$. Since $\Re$ has a $s$-uniform approximation in $\wp$, there exists $\delta > 0$ satisfying the implication (7). Because of the fact that $H\xi_n \to \upsilon^*$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho(H\xi_n, \upsilon^*) < \delta$$

for all $n \geq n_0$. Hence, from (9) and (10), we have $\rho(\xi_{n+1}, \xi^*) < \varepsilon$ for all $n \geq n_0$ and so we get $\xi_n \to \xi^*$ as $n \to \infty$. Similarly, we can obtain that $\upsilon_n \to \upsilon^*$ as $n \to \infty$. On the other hand, since $H$ is continuous, we have $\rho(H\xi_n, H\xi^*) = 0$ and $\rho(H\upsilon_n, H\upsilon^*) = 0$. Because of uniqueness of limit, we have $\xi^* = H\upsilon^*$ and $\upsilon^* = H\xi^*$. From (10), we have

$$\begin{aligned}\rho(\xi^*, H\xi^*) &= \rho(\wp, \Re)\\ \rho(\upsilon^*, H\upsilon^*) &= \rho(\wp, \Re)\end{aligned}$$

Now, we shall show the uniqueness. Let's consider another pair $(\zeta, \upsilon)$ in $\wp \times \Re$ satisfying

$$\begin{aligned} \rho(\zeta, H\zeta) &= \rho(\wp, \Re) \\ \rho(\upsilon, H\upsilon) &= \rho(\wp, \Re) \\ \rho(\zeta, \upsilon) &= \rho(\wp, \Re). \end{aligned}$$

Since the pair $(\wp, \Re)$ is semi-sharp proximinal, we have $H\zeta = \upsilon$ and $H\upsilon = \zeta$. Using the implication (8), we have

$$\rho(\xi^*, \upsilon) \leq q\rho(\xi^*, \upsilon) + (1-q)\rho(\wp, \Re)$$

and so we get $\rho(\xi^*, \upsilon) = \rho(\wp, \Re)$. The pair $(\wp, \Re)$ is semi-sharp proximinal and $\rho(\xi^*, \upsilon) = \rho(\wp, \Re) = \rho(\zeta, \upsilon)$, so we can conclude that $\xi^* = \zeta$. Similarly, it can be seen that $\upsilon^* = \upsilon$. □

**Example 28.** *Let $(\mho,\rho)$ be a b-metric space as in Example 14. If we take $\wp = \left\{\frac{1}{4n} : n \geq 3\right\}$ and $\Re = \left\{0, \frac{1}{4}, \frac{1}{8}\right\}$, then we have $\wp_0 = \left\{\frac{1}{12}\right\}$, $\Re_0 = \left\{\frac{1}{8}\right\}$ and $\rho(\wp, \Re) = \frac{1}{24}$. Now, we shall show that the pair $(\wp, \Re)$ is a best proximally complete space. Indeed, let $\{(\xi_n, \upsilon_n)\}$ be a cyclically Cauchy sequence in $(\wp, \Re)$. Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\rho(\xi_m, \upsilon_n) < \rho(\wp, \Re) + \varepsilon \tag{11}$$

*for all $n, m \geq n_0$. In this case, from (11), it can be seen that $\xi_m = \frac{1}{12}$ and $\upsilon_n = \frac{1}{8}$ for all $n, m \geq n_0$. Hence, $\{\xi_n\}$ and $\{\upsilon_n\}$ are convergent sequences. Let a mapping $H : \wp \cup \Re \to \wp \cup \Re$ defined by*

$$H\xi = \begin{cases} \frac{1}{8} & , \quad \xi = \frac{1}{12} \\ 0 & , \quad \xi = \frac{1}{4n},\ n \geq 4 \\ \frac{1}{12} & , \quad \xi \in \Re \end{cases}$$

*Hence, $H$ satisfies the implication (8). Further, we get $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. It is clear that the pair $(\wp, \Re)$ satisfies the property $(M_C)$. Therefore, all hypotheses of Theorem 27 hold. Hence, $\xi = \frac{1}{12}$ is a best proximity point of $H$ in $\wp \cup \Re$.*

If we take $s = 1$ in Theorem 27, we obtain the following result.

**Corollary 29.** *Let $\wp, \Re$ be nonempty subsets of a metric space $(\mho, \rho)$ such that $(\wp, \Re)$ is a best proximally complete space, $\wp_0 \neq \emptyset$ and $H : \wp \cup \Re \to \wp \cup \Re$ be a continuous proximal cyclic contraction of the second type satisfying $H(\wp_0) \subseteq \Re_0$ and $H(\Re_0) \subseteq \wp_0$. If $\Re$ has a uniform approximation in $\wp$, then there exists unique pair $(\xi^*, \upsilon^*)$ in $\wp \times \Re$ such that*

$$\begin{aligned} \rho(\xi^*, H\xi^*) &= \rho(\wp, \Re), \\ \rho(\upsilon^*, H\upsilon^*) &= \rho(\wp, \Re), \\ \rho(\xi^*, \upsilon^*) &= \rho(\wp, \Re). \end{aligned}$$

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