Communications in Advanced Mathematical Sciences
Vol. III, No. 3, 155-161, 2020
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.780899

Norm of Operators on the Generalized Cesàro Matrix Domain

Maryam Sinaei [1*]

Abstract
Roopaei in [13] has introduced some factorization for the infinite Hilbert matrix and the Cesàro matrix of order n based on the generalized Cesàro matrix. In this research, we investigate the norm of these two operators on the generalized Cesàro matrix domain. Moreover we introduce some factorizations for the Hilbert matrix. Hence the present study is a complement of Roopaei's research.

Keywords: Hilbert matrix, Cesàro matrix, Norm, Sequence space.
2010 AMS: 26D15, 40C05, 40G05, 47B37.

[1] *Department of mathematics, Azad university of Shiraz, Shiraz branch, Shiraz, Iran*
***Corresponding author**: marysinaei@yahoo.com
Received: 15 August 2020, **Accepted:** 22 September 2020, **Available online:** 29 September 2020

1. Introduction

Let ω be the space of all real-valued sequences. The space ℓ_p consists all real sequences $x = (x_k)_{k=0}^{\infty} \in \omega$ such that $\sum_{k=0}^{\infty} |x_k|^p < \infty$ which a Banach space with the norm

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$.

Let T is a matrix with non-negative entries, assumed to map ℓ_p into itself and satisfies the inequality

$$\|Tx\|_{\ell_p} \leq K\|x\|_{\ell_p},$$

where K is a constant which is not depending on x for every $x \in \ell_p$. The constant K is called an upper bound for operator T and the smallest possible value of K is called the norm of T.

For an infinite matrix A and sequence space X, we define the matrix domain $A(X)$ as the set

$$A(X) = \{x \in \omega : Ax \in X\}$$

which is also a sequence space. In this study, we use the notation A_p for the matrix domain associated with the matrix A on the space $X = \ell_p$. For an invertible matrix A, the matrix domain A_p is a normed space with $\|x\|_{A_p} := \|Ax\|_{\ell_p}$. There are several new Banach spaces who have introduced and studied by using matrix domains of special lower triangular matrices. For more references we encourage the readers to some papers [1, 3, 17, 18] and textbook [2]. Recently, several mathematicians have computed the bounds of operators on some matrix domains in [9, 11, 12, 15, 16, 17, 18, 19].

Cesàro matrix. The infinite Cesàro operator is defined by

$$c_{j,k} = \left\{ \begin{array}{ll} \frac{1}{j+1} & 0 \le k \le j \\ 0 & otherwise, \end{array} \right\}$$

for all $j,k \in \mathbb{N}$. It can be represented by its arrays as

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix has the ℓ_p-norm $\|C\|_{\ell_p} = \frac{p}{p-1}$. The inequality

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{k=0}^{\infty} |x_k|^p,$$

which is called Hardy's inequality is resulted from the boundedness of Cesàro operator.

The matrix domain associated with the Cesàro matrix is the set

$$C_p \quad = \quad \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+1} \right|^p < \infty \right\},$$

which is a Banach space with norm

$$\|x\|_{C_p} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+1} \right|^p \right)^{\frac{1}{p}}.$$

The Cesàro sequence space C_p is studied in [10, 20]. Recently, Roopaei et al. [16] have investigated the general case C_p^n, its inclusion relations, dual spaces, matrix transformations as well as computing the norm of operators on this matrix domain in the case $1 \le p < \infty$.

Generalized Cesàro matrix. Let $N \ge 1$ be a real number, the generalized Cesàro matrix, $C^N = (c_{j,k}^N)$, is defined by

$$c_{j,k}^N = \left\{ \begin{array}{ll} \frac{1}{j+N} & 0 \le k \le j \\ 0 & otherwise, \end{array} \right.$$

and has the ℓ_p-norm $\|C^N\|_{\ell_p} = \frac{p}{p-1}$ ([6], Lemma 2.3). That is

$$C^N = \begin{pmatrix} \frac{1}{N} & 0 & 0 & \cdots \\ \frac{1}{1+N} & \frac{1}{1+N} & 0 & \cdots \\ \frac{1}{2+N} & \frac{1}{2+N} & \frac{1}{2+N} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that, C^1 is the well-known Cesàro matrix C. For more examples

$$C^2 = \begin{pmatrix} 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad and \quad C^3 = \begin{pmatrix} 1/3 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The sequence space associated with the generalized Cesàro matrix is the set

$$C(N,p) \quad = \quad \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+N} \right|^p < \infty \right\},$$

who has the norm

$$\|x\|_{C(N,p)}=\left(\sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\frac{x_k}{j+N}\right|^p\right)^{\frac{1}{p}}.$$

Note that for $N=1$ we use the notation C_p instead of $C(1,p)$.

Recall the infinite Hilbert matrix which is defined by $H=(h_{j,k})=\frac{1}{j+k+1}$ for all non-negative integers j and k and has the matrix representation

$$H=\begin{pmatrix}1 & 1/2 & 1/3 & \cdots\\ 1/2 & 1/3 & 1/4 & \cdots\\ 1/3 & 1/4 & 1/5 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

According to [8] Theorem 323, the Hilbert matrix is a bounded operator on ℓ_p with

$$\|H\|_{\ell_p}=\frac{\pi}{\sin(\pi/p)}.$$

It has proved by Bennett [5] that the Hilbert operator can be factorized of the form $H=BC$, where C is the Cesàro matrix and $B=(b_{j,k})$ is defined by

$$b_{j,k}=\frac{k+1}{(j+k+1)(j+k+2)}\qquad (j,k=0,1,\ldots). \tag{1.1}$$

The matrix B is also a bounded operator on ℓ_p, ([5], Proposition 2), and $\|B\|_{\ell_p}=\frac{\pi}{p^*\sin(\pi/p)}$, where p^* is the conjugate of p i.e. $1/p+1/p^*=1$.

More recently, Roopaei in [13, 14] has generalized Bennett's factorization to introduce several factorization for the Hilbert matrix. He has showed that H can be presented of the form $H=B^NC^N$, where C^N is the generalized Cesàro matrix of the form:

Theorem 1.1 ([13], Theorem 2.2)**.** *The Hilbert matrix H, admits a factorization of the form $H=B^NC^N$, where $B^N=(b^N_{j,k})$ has the entries*

$$b^N_{j,k}=\frac{k+N}{(j+k+1)(j+k+2)}\qquad (j,k=0,1,\ldots). \tag{1.2}$$

and is a bounded operator on ℓ_p with bounds

$$\frac{\pi}{p^*\sin(\pi/p)}\le\|B^N\|_{\ell_p}\le\frac{N\pi}{p^*\sin(\pi/p)}.$$

In particular, for $N=1$, $H=BC$ and $\|B\|_{\ell_p}=\frac{\pi}{p^*\sin(\pi/p)}$.

2. Norm of Hilbert operator on generalized Cesàro space

The main purpose of this section is computing the norm of Hilbert operator on the generalized Cesàro space. Meanwhile, we introduce some factorization for the Hilbert matrix.

In sequel, we need the definition of another Hilbert matrix, H^1, who has the same norm as the Hilbert matrix and is defined by

$$h^1_{j,k}=\frac{1}{j+k+2}\qquad (j,k=0,1,\ldots), \tag{2.1}$$

or

$$H^1=\begin{pmatrix}1/2 & 1/3 & 1/4 & \cdots\\ 1/3 & 1/4 & 1/5 & \cdots\\ 1/4 & 1/5 & 1/6 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Theorem 2.1. *The Hilbert operator is a bounded operator from ℓ_p into the generalized Cesàro space $C(N,p)$ and*

$$\|H\|_{\ell_p,C(N,p)} \leq \frac{p^*\pi}{\sin(\pi/p)}.$$

Proof. We have

$$\begin{aligned}\|H\|_{\ell_p,C(N,p)} &= \sup_{x\in\ell_p}\frac{\|Hx\|_{C(N,p)}}{\|x\|_{\ell_p}} = \sup_{x\in\ell_p}\frac{\|C^N Hx\|_{\ell_p}}{\|x\|_{\ell_p}} \\ &= \|C^N H\|_{\ell_p} \leq \pi p^* \csc(\pi/p).\end{aligned}$$

□

Theorem 2.2. *The Hilbert operator is a bounded operator from the generalized Cesàro space $C(N,p)$ into ℓ_p and*

$$\|H\|_{C(N,p),\ell_p} \leq \frac{N\pi}{p^*\sin(\pi/p)}.$$

In particular, the Hilbert matrix is a bounded operator from the Cesàro sequence space into ℓ_p and

$$\|H\|_{C_p,\ell_p} = \frac{\pi}{p^*\sin(\pi/p)}.$$

Proof. According to Theorem 1.1, the Hilbert matrix can be written as $H = B^N C^N$, where B^N is a bounded operator on ℓ_p and

$$\frac{\pi}{p^*\sin(\pi/p)} \leq \|B^N\|_{\ell_p} \leq \frac{N\pi}{p^*\sin(\pi/p)}.$$

Since C_p^N and ℓ_p are isomorphic, hence

$$\begin{aligned}\|H\|_{C(N,p),\ell_p} &= \sup_{x\in C(N,p)}\frac{\|Hx\|_{\ell_p}}{\|x\|_{C(N,p)}} = \sup_{x\in C(N,p)}\frac{\|B^N C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} = \sup_{y\in\ell_p}\frac{\|B^N y\|_{\ell_p}}{\|y\|_{\ell_p}} \\ &= \|B^N\|_{\ell_p} \leq \frac{N\pi}{p^*\sin(\pi/p)}.\end{aligned}$$

In particular, for the symbol $N=1$, $C^N = C$ and $B^N = B$, where B is the factor in the Bennett's factorization of the Hilbert operator. Now, we have the desired result. □

Theorem 2.3. *The Hilbert operator is a bounded operator on the generalized Cesàro space and*

$$\|H\|_{C(N,p)} \leq \frac{N\pi}{\sin(\pi/p)}.$$

In special case, the Hilbert operator is a bounded operator on the Cesàro matrix domain and

$$\|H\|_{C_p} = \frac{\pi}{\sin(\pi/p)}.$$

Proof. Let $D^N = (d_{j,k}^N)$ be $C^N B^N$, where B^N was defined by the relation (1.2). Then

$$\begin{aligned}d_{i,k}^N &= \sum_{j=0}^{i}\frac{1}{i+N}\frac{k+N}{(j+k+1)(j+k+2)} \\ &= \left(\frac{k+N}{k+1}\right)\left(\frac{i+1}{i+N}\right)\frac{1}{i+k+2}.\end{aligned}$$

But, $\frac{k+N}{k+1} \leq N$ and $\frac{i+1}{i+N} \leq 1$, for all non-negative integers j,k. Hence, $d_{j,k}^N \leq N h_{j,k}^1$ which results in

$$\|D^N\|_{\ell_p} \leq N\|H^1\|_{\ell_p} = N\frac{\pi}{\sin(\pi/p)}.$$

The map $x \to C^N x$ shows that the two sequence spaces $C(N,p)$ and ℓ_p are isomorphic, hence

$$\begin{aligned}\|H\|_{C(N,p)} &= \sup_{x\in C(N,p)} \frac{\|Hx\|_{C(N,p)}}{\|x\|_{C(N,p)}} = \sup_{x\in C(N,p)} \frac{\|C^N Hx\|_{\ell_p}}{\|C^N x\|_{\ell_p}} \\ &= \sup_{x\in C(N,p)} \frac{\|D^N C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} = \sup_{y\in \ell_p} \frac{\|D^N y\|_{\ell_p}}{\|y\|_{\ell_p}} \\ &= \|D^N\|_{\ell_p} \leq \frac{N\pi}{\sin(\pi/p)}.\end{aligned}$$

In particular, for $N=1$, $C^N = C$ and $D^N = H^1$ which results the desired result. □

Corollary 2.4. *The Hilbert operator is a bounded operator from the generalized Cesàro space $C(N,p)$ into Cesàro sequence space C_p and*

$$\|H\|_{C(N,p),C_p} \leq \frac{N\pi}{\sin(\pi/p)}.$$

In particular, the Hilbert matrix is a bounded operator on the Cesàro matrix domain and

$$\|H\|_{C_p} = \frac{\pi}{\sin(\pi/p)}.$$

Proof. Let $P^N = (p^N_{j,k})$ be CB^N, where B^N was defined by the relation (1.2). Then

$$\begin{aligned}p^N_{i,k} &= \sum_{j=0}^{i} \frac{1}{i+1} \frac{k+N}{(j+k+1)(j+k+2)} \\ &= \left(\frac{k+N}{k+1}\right) \frac{1}{i+k+2}.\end{aligned}$$

But, $\frac{k+N}{k+1} \leq N$ for all non-negative integer k. Hence, $p^N_{j,k} \leq N h^1_{j,k}$ which results in

$$\|P^N\|_{\ell_p} \leq N \|H^1\|_{\ell_p} = N \frac{\pi}{\sin(\pi/p)}.$$

Since C^N_p and ℓ_p are isomorphic, hence

$$\begin{aligned}\|H\|_{C(N,p),C_p} &= \sup_{x\in C(N,p)} \frac{\|Hx\|_{C_p}}{\|x\|_{C(N,p)}} = \sup_{x\in C(N,p)} \frac{\|CB^N C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} \\ &= \sup_{y\in \ell_p} \frac{\|P^N y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|P^N\|_{\ell_p} \leq \frac{N\pi}{\sin(\pi/p)}.\end{aligned}$$

In particular, for the symbol $N=1$, $C^N = C$ and $B^N = B$, where B is the factor in the Bennett's factorization of the Hilbert operator. Now, we have the desired result. □

Similar to the above corollary we have the following result.

Corollary 2.5. *The Hilbert operator is a bounded operator from the Cesàro sequence space C_p into the generalized Cesàro space $C(N,p)$ and*

$$\|H\|_{C_p,C(N,p)} \leq \frac{\pi}{\sin(\pi/p)}.$$

In particular, the Hilbert matrix is a bounded operator on the Cesàro sequence space and

$$\|H\|_{C_p} = \frac{\pi}{\sin(\pi/p)}.$$

Corollary 2.6. *The Hilbert matrix H, can be represented of the form $H = C^{-1}P^NC^N$, where $P^N = (p^N_{j,k})$ is defined by*

$$p^N_{j,k} = \frac{(k+N)}{(k+1)(j+k+2)} \qquad (j,k = 0,1,\ldots).$$

In particular, for $N = 1$, $\|P\|_{\ell_p} = \pi \csc(\pi/p)$.

Proof. By a simple calculation, $P^N = CB^N$. Therefore by applying Theorem 1.1, $C^{-1}P^NC^N = H$, which proves the factorization. Note that for $N = 1$, $P^1 = P = H^1$, where the Hilbert matrix H^1 is

$$h^1_{j,k} = \frac{1}{j+k+2} \qquad (j,k = 0,1,\ldots),$$

and has the norm $\|H^1\|_{\ell_p} = \frac{\pi}{\sin(\pi/p)}$. □

Theorem 2.7. *The Hilbert matrix H, has a factorization of the form $H = C^{-N}A^NC$, where $A^N = (a^N_{j,k})$ is defined by*

$$a^N_{j,k} = \frac{j+1}{(j+N)(j+k+2)} \qquad (j,k = 0,1,\ldots).$$

In particular, for $N = 1$, H has the factorization $H = C^{-1}AC$, where $\|A\|_{\ell_p} = \pi \csc(\pi/p)$.

Proof. It is not difficult to verify that $A^N = C^NB$, therefore by applying Theorem 1.1, $C^{-N}A^NC^N = H$, which proves the factorization. Note that for $N = 1$, $A^1 = A = H^1$ and has the norm $\|A\|_{\ell_p} = \|H^1\|_{\ell_p} = \frac{\pi}{\sin(\pi/p)}$. □

3. Norm of Cesàro operator on the generalized Cesàro space

In this section we intend to compute the norm of Cesàro operator of order n on the generalized Cesàro space.

For the probability measure μ on the interval $[0,1]$, the Hausdorff matrix $H^\mu = (h_{j,k})$, is defined by

$$h_{j,k} = \begin{cases} \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} d\mu(\theta) & 0 \le k \le j \\ 0 & otherwise, \end{cases}$$

For $1 \le p < \infty$, by Hardy's formula ([7], Theorem 216) one can obtain the norm of Hausdorff matrices. These operators are bounded iff $\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta).$$

By inserting $d\mu(\theta) = n(1-\theta)^{n-1}d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix of order n, $C^n = (c^n_{j,k})$ is

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & j \ge k \ge 0 \\ 0 & otherwise. \end{cases}$$

This matrix has the ℓ_p-norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)},$$

according to Hardy's formula. Note that, $C^1 = C$, where C is the well-known Cesàro matrix.

For computing the norm of Cesàro matrix of order n on the generalized Cesàro matrix domain we need the following theorem.

Theorem 3.1 ([13], Theorem 3.2)**.** *For $n \ge 1$, Cesàro matrix of order n, C^n, has a factorization of the form $C^n = R^{n,N}C^N$, where C^N is the generalized Cesàro matrix of order N and $R^{n,N}$ is a bounded operator on ℓ_p with*

$$\|R^{n,N}\|_{\ell_p} \le \frac{N\Gamma(n+1)\Gamma(1+1/p^*)}{\Gamma(n+1/p^*)}.$$

Corollary 3.2. *The Cesàro operator of order n is a bounded operator from the generalized Cesàro space $C(N,p)$ into sequence space ℓ_p and*

$$\|C^n\|_{C(N,p),\ell_p} \leq \frac{N\Gamma(n+1)\Gamma(1+1/p^*)}{\Gamma(n+1/p^*)}.$$

Proof. Since $C(N,p)$ and ℓ_p are isomorphic, hence according to the Theorem 3.1 we have

$$\begin{aligned}\|C^n\|_{C(N,p),\ell_p} &= \sup_{x\in C(N,p)} \frac{\|C^n x\|_{\ell_p}}{\|x\|_{C(N,p)}} = \sup_{x\in C(N,p)} \frac{\|R^{n,N}C^N x\|_{\ell_p}}{\|C^N x\|_{\ell_p}} \\ &= \sup_{y\in \ell_p} \frac{\|R^{n,N} y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|R^{n,N}\|_{\ell_p} \leq \frac{N\Gamma(n+1)\Gamma(1+1/p^*)}{\Gamma(n+1/p^*)}.\end{aligned}$$

Now, we have the desired result. □

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