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A NEW FORMULATION FOR THE BOUNDARY ELEMENT ANALYSIS OF HEAT CONDUCTION PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract: An effective numerical method based on the boundary element formulation is presented to solve heat conduction equations which are governed by the Fourier equation, with nonlinear boundary conditions on one or more sections of the prescribed boundary. The solution involves the manipulation of the system matrices of the boundary element method and obtaining a smaller ranked matrix equation in which the unknown is only the temperature difference over the nonlinear boundary condition region. This way, the iterations to deal with the nonlinear conditions are performed faster. After finding the solution over the nonlinear boundary condition region, the solution over the entire boundary is obtained as a post-process through a prescribed relation. An example with a proven exact solution is employed to assess the results.
Keywords: Boundary element method, heat conduction, nonlinear boundary conditions.

DOĞRUSAL OLMAYAN SINIR KOŞULLARINA SAHİP ISI TRANSFERİ PROBLEMLERİNİN SINIR ELEMAN YÖNTEMİ İLE ANALİZİNE YÖNELİK YENİ BİR FORMÜLASYON

Özet: Bu çalışmada Fourier denklemi ile ifade edilen ısı transferi problemlerinin bir ya da daha fazla bölgesinde tanımlı doğrusal olmayan sınır koşulları altında çözümüne yönelik sınır eleman yöntemi tabanlı etkili bir sayısal çözüm sunulmaktadır. Çözüm, sınır eleman yöntemi sistem matrislerinin üzerinde yapılan matematiksel işlemler ile bilinmeyenleri sadece doğrusal olmayan sınır bölgesindeki sıcaklık farkı olan indirgenmiş matris denklemleri elde etmektedir. Bu sayede doğrusal olmayan sınır koşullarına dayalı iterasyonlar daha hızlı gerçekleştirilebilmektedir. Doğrusal olmayan sınır koşullarının tanımlı olduğu bölgelerde çözüm elde edildikten sonra tüm sınır çözümü tanımlı bir son-işlem ile gerçekleştirilebilmektedir. Gerçek çözümü elde edilmiş bir örnek kullanılarak elde edilen sonuçlar değerlendirilmiştir.
Anahtar Kelimler: Sınır eleman yöntemi, ısı transferi, lineer (doğrusal) olmayan sınır koşulları

INTRODUCTION

It is an easy task to solve heat conduction equations with Dirichlet, Neumann or linear Robin type of boundary conditions (BC) using the boundary element method (BEM) (Becker, 1992). Thanks to its boundary-only nature, the BEM discretizes only the boundary of the domain, and due to its semi-analytical nature, it easily and accurately solves the heat conduction problems, for which the governing equation is a second order linear differential equation in one variable. It has been shown that for the same level of discretization, the BEM gives more accurate results when compared with FEM (Mukherjee and Morjaria, 1984).

One of the main advantages of the BEM is the direct application of the boundary conditions through manipulations over the system matrices. In case of Neumann BC no action is required and for Dirichlet BC only a swapping of the respective columns would suffice to impose the boundary condition. In case of linear Robin type BC, a linear operation is employed combined with swapping, which leads to direct imposition of the corresponding BC (Beer et al., 2008). Since no approximation or penalty is in place, the boundary conditions are exact through the boundary of the domain. Yet, this is true for only linear boundary conditions. In case of the non-linear boundary conditions, iterations should be performed to obtain the solution (Bialecki and Nowak, 1981, Wrobel and Brebbia, 1992). In heat transfer problems, the non-linear boundary conditions may appear, for example, in case of radiative or combined convective-radiative heat transfer. Also, inverse problems assume nonlinear boundary conditions in which the coefficients are obtained through inverse analysis (Slodicka et al., 2010). Since the application of such non-linear algorithm would involve iterations and since the system matrices of the BEM are highly ill-conditioned, there is a high possibility that the results would diverge or be not accurate. In cases where iterations converge to a value, the rate of convergence may be very low.

It should also be stated here that, the conventional iteration methods would require an initial solution - which is mostly very problematic in case of BEM since after the application of the boundary conditions, the known vector consists of two different field variables - the temperature difference and the heat flux - which are very different in order of magnitude (noting that flux is the gradient of temperature difference). Aside from assigning an initial value, also, finding a proper norm for the iterations is not an easy task, since the norm should involve the field variable, temperature difference along with its derivative quantity and the normal flux, which are, most probably, not in the same order of magnitude.

A practical solution to this problem is given by (Chan, 1993) where a local iteration scheme is proposed. The method first starts with an initial guess of the temperature field within the nonlinear portion of the domain (in the given study, this initial guess is assumed to be 1 over the nonlinear BC boundary) and solves the system equations of the BEM assuming temperature difference is prescribed over the nonlinear BC boundary. After obtaining the solution, using an exact derivative of the BEM equations over the domain which gives the partial derivative of the fluxes with respect to the potential (in this case, temperature difference), a local iteration is performed to obtain an updated guess on the temperature difference field over the nonlinear domain. This updated solution is re-inserted into the initial BEM equations and solved until a convergence is obtained.

It should be noted that, for large systems, (Chan, 1993) proposes a two-domain solution. Large systems occur when the domain is discretized into a large number of elements, therefore, to obtain a lower number of elements and yet have a proper solution (Xu and Kamiya, 1997) proposed an adaptive mesh refinement over the nonlinear domain. In this study, the number of elements on the linear BC boundaries are kept constant whereas the element mesh is refined on the nonlinear BC boundaries. This way, convergence gets faster.

Note also that, if the non-linear BC is affecting a small section of all the solution domain, as it can be in many 3D complex geometries with a small part exposed to radiative heat transfer, an iterative solution that considers all the unknowns over the boundary would most probably have a very low convergence rate - or no convergence at all. A recent study in cathodic protection proposes a domain decomposition technique for such problems in 3D (Santos et al., 2018).

The procedure becomes more complex, if, for example, there exists more than one region with non-linear boundary conditions, each with different functions. Assume, for example, different regions of the body receiving radiative heat transfer from different sources and different emissivities. In this case, the iterations should be made for two -or more- regions with different parameters and nonlinearity, which makes the problem more complex. Also, problems with combined nonlinear conditions (such that, there exists linear BC resulting from convective heat transfer and a fourth order nonlinear BC from radiative heat transfer) the modeling requires special treatment (Dehghani et al., 2011, Mosayebidorcheh et al., 2014). Also, in case of inverse analysis, repeated runs to obtain the nonlinear coefficients are needed (Lesnic et al., 2009, Onyago et al., 2009).

In this study, a novel method is proposed to impose nonlinear boundary conditions in the BEM. The method involves operations over the system matrices, in a similar way to the impedance method (Mengi and Argeso, 2006, Yalçın and Mengi, 2013, Karakaya et al., 2015). The resulting system is a smaller rank system of equations that involves only the field variable (and not its derivative) over the nonlinear boundary as boundary unknowns. Yet, in the impedance method the system of equations is reduced to an impedance-like relation in which solving the equation reveals a desired dependence (e.g., solving a unit-load or unit-displacement problem in interaction problems or the particle velocities in a particle-tracking problem). In the proposed method, however, the matrix manipulations are so arranged that the reduced system of equations are used to solve the nonlinear BC unknowns, which then reveals the total solution again using algebraic matrix equations. It is shown in this study that with such manipulations the resulting system is stable and the iterations are robust. The iterations are performed over only one field variable as unknowns, therefore a simple norm can easily be applied. The solution is obtained for the boundary nodes with the nonlinear BC only - and obtaining the solution at the other boundary nodes and also the internal solution is treated as a post-processing. One other major advantage of the formulation is the treatment of all non-linear boundary conditions in a unified manner, so that, the iterations are performed over the same set of unknowns.

The formulation is presented in steady-state, but is general in nature and can be easily applied to time-dependent problems.

To assess the formulation, two problems in 2D are considered. It has been found that the new algorithm proves well for the examples presented.

FORMULATION

The governing equation for the isotropic steady-state heat conduction is given by the Fourier's equation (Nowacki, 1967):

$$k\nabla^2 u = 0 \tag{1}$$

where k is the heat conduction coefficient, ∇ is the Laplace operator with $\nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i}$, the double index implying summation over the range of the variable, and u is the field variable, in this case the temperature difference from the reference temperature. The flux associated with the field variable is given by

$$q_i = k\frac{\partial u}{\partial x_i} \quad (2)$$

with the normal flux being

$$q = q_i n_i = k\frac{\partial u}{\partial n} \quad (3)$$

Here, n denotes the outward unit normal direction, where n_i represents the components of the unit vector along this direction. This would seem contradictory to the heat equations, defining the flux in reverse direction, yet mathematically practical and the application to other problems with Laplace operators are straightforward.

Obtaining the boundary element equation (BEE) for the Laplace's equation is straight-forward, resulting in the integral equation given by

$$C(A)\cdot u(A) + \int_S H(A,P)\cdot u(P)\cdot \mathrm{d}A = \int_S H(A,P)\cdot u(P)\cdot \mathrm{d}A \quad (4)$$

where A is the source, P is the field point, $G(A,P)$ and $H(A,P)$ are the first and second fundamental solutions of the Laplace's equation. In Eq.4, $C(A)$ is the free term that takes values depending on the location of the point A, eg., if A is on a smooth boundary C(A) = 1/2, if A is within the solution domain C(A) = 1 and if A is outside the solution domain, C(A) = 0. When discretized using N constant boundary elements, Eq.4 can be expressed in matrix form as:

$$\mathbf{H}\cdot\mathbf{u} = \mathbf{G}\cdot\mathbf{q} \quad (5)$$

Here, H is an $N\times N$ matrix involving the integrals from the second fundamental solution, augmented with 1/2 across the diagonal, G is $N\times N$ matrix involving the integrals evaluated with the first fundamental solution, and u and q are $N\times 1$ vectors.
Let us divide the boundary of the solution region, S into two regions:

$$\begin{aligned} S &= S^a \cup S^b \\ S^a \cap S^b &= \emptyset \end{aligned} \quad (6)$$

where

$$\begin{aligned} S^a &= S^d \cup S^n \\ S^d \cap S^n &= \emptyset \end{aligned} \quad (7)$$

Here, S^d and S^n represent the sections of the boundary with Dirichlet and Neumann type boundary condition respectively. Over S^b , on the other hand, the boundary conditions are defined as:

$$q = f(u) \quad (8)$$

where the single valued function f can be linear or non-linear. If, f is linear, the solution is straightforward. In this study, we will assume that the function f takes a non-linear form.

We first subdivide the matrix equation given in Eq.5 as

$$\begin{bmatrix}\mathbf{H}_{aa} & \mathbf{H}_{ab}\\ \mathbf{H}_{ba} & \mathbf{H}_{bb}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{u}_a\\ \mathbf{u}_b\end{bmatrix} = \begin{bmatrix}\mathbf{G}_{aa} & \mathbf{G}_{ab}\\ \mathbf{G}_{ba} & \mathbf{G}_{bb}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{q}_a\\ \mathbf{q}_b\end{bmatrix} \quad (9)$$

where the index a refers to the elements belonging to the S^a boundary and b refers to those belonging to the S^b boundary. If the number of nodes on S^a boundary is N_a, and the number of nodes on S^b boundary is N_b, $\mathbf{H}_{aa}$ is $N_a\times N_a$, $\mathbf{H}_{ab}$ is $N_a\times N_b$, and etc.

Note that, over S^a the boundary conditions will be applicable through proper column changes; thus Eq.9 can be re-written after the application of Dirichlet and Neumann boundary conditions as:

$$\begin{bmatrix}\mathbf{K}_{aa} & \mathbf{H}_{ab}\\ \mathbf{K}_{ba} & \mathbf{H}_{bb}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}_a\\ \mathbf{u}_b\end{bmatrix} = \begin{bmatrix}\mathbf{M}_{aa} & \mathbf{G}_{ab}\\ \mathbf{M}_{ba} & \mathbf{G}_{bb}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{b}_a\\ \mathbf{q}_b\end{bmatrix} \quad (10)$$

where **K** and **M** represents the left- and right- coefficient matrices that would appear after column changes respectively, **b** is the column vector containing the boundary conditions and **x** is the column vector with the unknowns. Note that, since there is a one-to-one relation between the flux and the temperature difference as in Eq.8, for all nodes on S^b boundary, one can write:

$$\mathbf{q}_b = \mathbf{D}_{bb}\cdot\mathbf{u}_b \quad (11)$$

where $\mathbf{D}_{bb}$ is a diagonal square matrix of dimensions $N_b\times N_b$ where the components are determined by a non-linear relation with $\mathbf{u}_b$. The evaluation of the matrix $\mathbf{D}_{bb}(\mathbf{u}_b)$ is simple: Assume, for example, that the nonlinear function is given in the form:

$$q = m\times u^n \quad (12)$$

T
hen, it is obvious that at all points on the S^b boundary, one would have

$$q_i = m\times u_i^n = m\times u_i^{n-1}\times u_i \quad (13)$$

which results in the matrix equation:

$$\begin{bmatrix} q_1\\ q_2\\ \vdots\\ q_{N_b}\end{bmatrix} = \begin{bmatrix} m\times u_1^{n-1} & 0 & 0 & 0\\ 0 & m\times u_1^{n-1} & 0 & 0\\ 0 & 0 & m\times u_1^{n-1} & 0\\ 0 & 0 & 0 & m\times u_1^{n-1}\end{bmatrix}\begin{bmatrix} u_1\\ u_2\\ \vdots\\ u_{N_b}\end{bmatrix} \quad (14)$$

Similar expressions can be derived with all forms of explicit functions, noting that

$$q = f(u) = \frac{f(u)}{u}\times u \quad (15)$$

where, in such a case, the diagonal components of the $\mathbf{D}_{bb}$ matrix are determined through the augmented function $f(u)/u$.

The first row of Eq.10 reads:

$$\mathbf{K}_{aa} \cdot \mathbf{x}_a + \mathbf{H}_{ab} \cdot \mathbf{u}_b = \mathbf{M}_{aa} \cdot \mathbf{b}_a + \mathbf{G}_{ab} \cdot \mathbf{q}_b \quad (16)$$

from which, one can write, for $\mathbf{x}_a$:

$$\mathbf{x}_a = \mathbf{K}_{aa}^{-1} \cdot (\mathbf{M}_{aa} \cdot \mathbf{b}_a + \mathbf{G}_{ab} \cdot \mathbf{q}_b - \mathbf{H}_{ab} \cdot \mathbf{u}_b) \quad (17)$$

Inserting this to the second row of Eq. 10 we have:

$$\begin{aligned} &(\mathbf{H}_{bb} - \mathbf{K}_{ba} \cdot \mathbf{K}_{aa}^{-1} \cdot \mathbf{H}_{ab}) \cdot \mathbf{u}_b - \\ &(\mathbf{G}_{bb} - \mathbf{K}_{ba} \cdot \mathbf{K}_{aa}^{-1} \cdot \mathbf{G}_{ab}) \cdot \mathbf{q}_b = \\ &(\mathbf{M}_{ba} - \mathbf{K}_{ba} \cdot \mathbf{K}_{aa}^{-1} \cdot \mathbf{M}_{aa}) \cdot \mathbf{b}_a \end{aligned} \quad (18)$$

With Eq. 11, Eq.18 becomes:

$$(\mathbf{Q}_{bb} + \mathbf{R}_{bb} \cdot \mathbf{D}_{bb}) \cdot \mathbf{u}_b = \mathbf{b}_b \quad (19)$$

Where

$$\mathbf{b}_b = (\mathbf{M}_{ba} - \mathbf{K}_{ba} \cdot \mathbf{K}_{aa}^{-1} \cdot \mathbf{M}_{aa}) \cdot \mathbf{b}_a \quad (20)$$

is a known load vector which is constant throughout the iteration process, and similarly,

$$\begin{aligned} \mathbf{Q}_{bb} &= \mathbf{H}_{bb} - \mathbf{K}_{ba} \cdot \mathbf{K}_{aa}^{-1} \cdot \mathbf{H}_{ab} \\ \mathbf{R}_{bb} &= \mathbf{G}_{bb} - \mathbf{K}_{ba} \cdot \mathbf{K}_{aa}^{-1} \cdot \mathbf{G}_{ab} \end{aligned} \quad (21)$$

are constant matrices. Thus, the solution of the problem can be obtained through an iterative process on Eq. 19. Once the solution is obtained for $\mathbf{u}_b$, the rest of the solution can be easily obtained using Eq.17 through matrix multiplications.

After model input, the solution algorithm involves an iterative nature. In this study, the simplest iteration method, namely the fixed-point iteration, is employed. The general procedure can be itemized as below:

1. Evaluate the system matrices **G** and **H**
2. Apply Dirichlet BCs through column changes
3. Subdivide the modified matrices **G** and **H** to obtain $\mathbf{K}_{aa}$, $\mathbf{K}_{ba}$, $\mathbf{H}_{ab}$, $\mathbf{H}_{bb}$, $\mathbf{M}_{aa}$, $\mathbf{M}_{ba}$, $\mathbf{G}_{ab}$ and $\mathbf{G}_{bb}$
4. Evaluate the constant system matrices $\mathbf{Q}_{bb}$, $\mathbf{R}_{bb}$ and the load vector $\mathbf{b}_b$
5. Assign an initial value to the vector $\mathbf{u}_b$
6. While a selected error norm, *e*, is less than a prescribed value, $\bar{e}$,
    a. Evaluate $\mathbf{D}_{bb}$ (Eq.11)
    b. Solve Eq.19 for the new value of $\mathbf{u}_b^n$
    c. Obtain the error norm *e* using $\mathbf{u}_b^n$ and $\mathbf{u}_b$
    d. Set $\mathbf{u}_b = \mathbf{u}_b^n$
7. Obtain the boundary solution $\mathbf{x}_a$ from Eq.17
8. If required, obtain the internal solution as post-processing

NUMERICAL EXAMPLES

The formulation presented in the previous section is general for 3D and 2D problems. For simplicity, without losing generality, the numerical examples will be given in 2D and a MATLAB program is coded for evaluating BEM results. The numerical examples will be discussed through a thick-walled infinite tube example. The geometry is shown in Figure 1.

Figure 1. The infinite cylinder problem

If axially symmetric boundary conditions are imposed, a planar quarter model would suffice to solve the problem (Figure 2). The inner radius, $r_i = a$ and the outer radius $r_o = b$ are selected as $a = 10$mm and $b = 20$mm. The boundaries at y=0 and x=0 lines are the symmetry planes, so the normal flux, q_n, is zero. To simplify the analysis, the inner surface temperature difference at r_i is given as Dirichlet BC with $u_{r=a} = T$. Note that, since the potential u is the temperature difference in the problems that are considered, its unit depends on the selection of the temperature scale (it can be Celcius, Kelvin or Fahrenheit or any other scale that is selected), thus, will not mentioned in the presented analyses explicitly. All other dependent quantities, like normal flux, *k*, etc., will have units consistent with the selected unit of the potential, u, and in the text will not be explicitly stated. The outer boundary will be exposed to nonlinear heat flux, which will be given in the form:

$$q_{r=b} = k\left(\frac{\partial u}{\partial n}\right)_{r=b} = N \cdot u^n \quad (22)$$

where *N* and *n* are constants. Note that the unit of *N* depends on the order *n*, therefore, in the following discussion, its unit will not be mentioned explicitly. Note also that, *n* = 0 results in Neumann BC and *n* = 1 results in Robin BC and higher values, negative values and non-integer values of *n* imposes nonlinearity in the BC. It is noted here that, for simplicity but without losing generality, constant elements are used in the analysis. The number of elements used on the symmetry sides (A and B) is N_s, on the temperature side is N_t and on the nonlinear flux side is N_n. This mesh results in $N_a = 2 \times N_s + N_t$ and $N_b = N_n$. The problem, since posed as axisymmetric, can be given in radial coordinates as (with constant *k*):

$$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=0 \tag{23}$$

which had the solution

$$u(r)=C_1\ln(r)+C_2 \tag{24}$$

with $u_{r=a}=T$

$$C_2=\mathrm{T}-\mathrm{C}_1\ln(a) \tag{25}$$

At this point we assume that on the inner boundary the temperature condition is so assigned as

$$\mathrm{T}=\mathrm{C}_1\ln(a) \tag{26}$$

which results in $C_2=0$. On the inner and outer surfaces, the boundary is circular, therefore, on the outer boundary

$$q=k\frac{\partial u}{\partial n}=k\left(\frac{\partial u}{\partial r}\right)_{r=b}=k\frac{C_1}{b} \tag{27}$$

The outer boundary condition results in, with above discussion,

$$k\frac{C_1}{b}=N\cdot[C_1\cdot\ln(b)]^n \tag{28}$$

Note that if *n = 1* this equality will not reveal a solution for C_1, thus this value of *n* is excluded from the presented solution. Cancelling the trivial solution of C_1=0:

$$C_1=\left[\frac{k}{b\cdot N\cdot[\ln(b)]^n}\right]^{\frac{1}{n-1}} \tag{29}$$

It follows from Eq.26 that the inner temperature is assigned as

$$T=\left[\frac{k}{b\cdot N\cdot[\ln(b)]^n}\right]^{\frac{1}{n-1}}\ln(a) \tag{30}$$

and the outer flux is defined as in Eq.22. The solution within the domain would be given by

$$u(r)=\left[\frac{k}{b\cdot N\cdot[\ln(b)]^n}\right]^{\frac{1}{n-1}}\ln(r) \tag{31}$$

for all $n\in\mathbb{R}-\{+1\}$.

Note that, a very simple form of solution is possible for all possible values of n if the value of k is assigned as

$$k=b\cdot N\cdot[\ln(b)]^n \tag{32}$$

which sets $C_1=1$ and $T=\ln(a)$. The solution will be given by

$$u(r)=\ln(r) \tag{33}$$

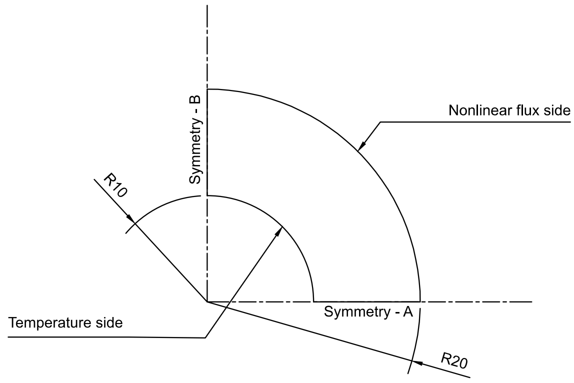

Figure 2. The quarter model for solution of the problem
The numerical results will be presented in three cases, with *n* = 2, *n* = −1 and *n* = 4.

To be general in the solutions, k given as in Eq.32 will not be considered; rather, explicit values of k will be imposed. Assuming N = 1 with proper units, for $n=-1$, k = 0.12 is selected which gives $T\approx 17.1742$, for n = 2, k = 2000 is selected which gives $T\approx$25.6565 and for n=4, k=3×106 is selected which gives $T\approx$28.3289.

The comparison between exact and BEM results are presented in Figure 3 along with the percent error in Figure 4. In these analyses, $N_s=50$, $N_t=70$ and $N_n=100$ which totals up to 270 elements. The results are presented with the absolute error norm of $\bar{e}=10^{-5}$. It can be seen that the maximum percent error does not exceed 0.045%.

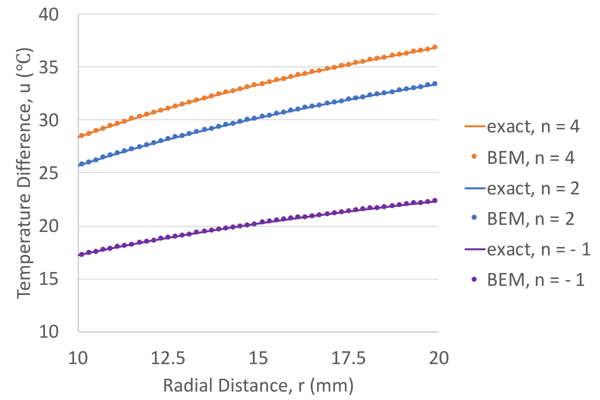

Figure 3. Comparison of exact and BEM results for $n=-1$, n = 2 and n = 4.

Another analysis follows for the number of iterations for obtaining the solution. Note that the simplest iterative algorithm is used in this study which has a constant convergence rate is used in the analysis. The number of iterations with respect to prescribed error norm is given in Figure 5. From the graph, the constant convergence nature of the iteration algorithm can easily be detected for all three n values. It is at this point necessary to note that the slowest solution time on a Macbook Pro computer with Intel i7 processor and 16 GB RAM for the presented problems is less than a second. Also note that, as an extreme case, $\bar{e}=10^{-10}$, the solution is obtained in 206 iterations for n = 4. Yet, the solution time is 0.71 seconds and the maximum percent error is 0.04%.

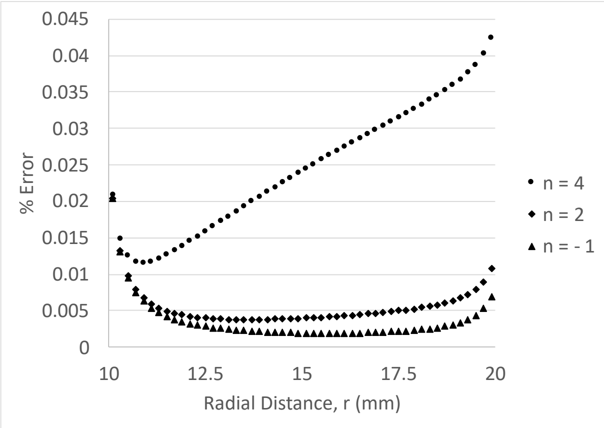

Figure 4. Comparison of exact and BEM results for $n = -1$, $n = 2$ and $n = 4$

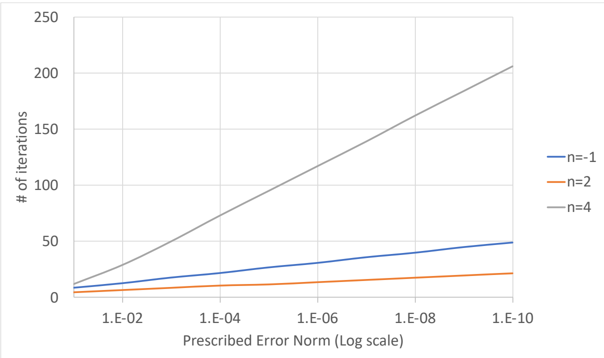

Figure 5. Number of iterations to obtain solution for $n = -1$, $n = 2$ and $n = 4$

A last analysis will be on the selection of the initial vector. The method is robust in the sense that selection of the initial vector does not affect convergence. A special note is placed for negative odd powers, since in this case C_1, in Eq.29, yields two solutions, a negative and a positive solution, and the results converge to the one that the initial vector is closer. To be comparable with all n selections previously stated, the results will be presented in terms of Δ where

$$\mathbf{u}_b = \left(T - \Delta\frac{T}{100}\right) \cdot \mathbf{i}_b \tag{34}$$

with T being the corresponding inner temperature condition from Eq.30 and $\mathbf{i}_b$ is the identity vector of dimension $N_b \times 1$. In the analysis, to see the effect of initial vector more clearly, $\bar{e} = 10^{-10}$ is selected. The results are presented in Figure 6 with scaling the % error by a factor of 1000. It can easily be seen that the method provides robust and stable solution independent of the selection of the initial vector. Note that $\Delta > 75$ is excluded from the figure for $n = -1$ since after that value the values converges to the second solution.

As a second benchmark problem, the square plate analysis from (Xu and Kamiya, 1997) is reproduced. The square domain with 5m sides is subjected to the boundary conditions as given in Figure 7. Since the thermal conduction coefficient used in the cited reference is not explicitly given, four different values are employed in the present study: k = 2.3, 3.0, 5.0 and 10.0 W/m°K. The lower values of k do not converge, which is most probably because a very simple iterative algorithm, the fixed-point iteration method, is employed in the present study. In presenting the results, as in the cited reference, a measured distance from the lower left corner is used as a reference distance value (denoted by s, in Figure 7).

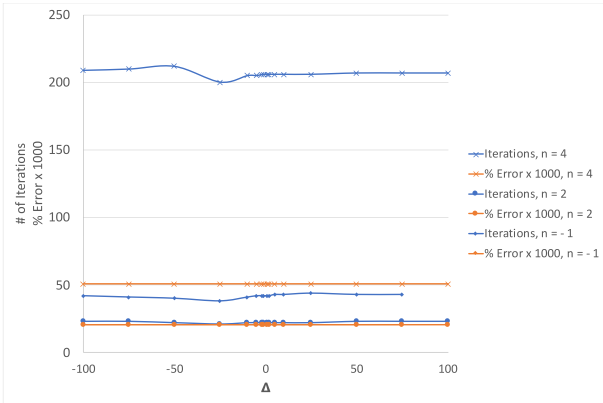

Figure 6. Sensitivity of the solution accuracy (% Error x 1000) and the number of iterations to obtain the solution with respect to Δ

The square plate is discretized using 50 elements per side which totals up to 200 nodes. A short sensitivity analysis shows that increasing the elements more does not result in cost-effective better solutions.

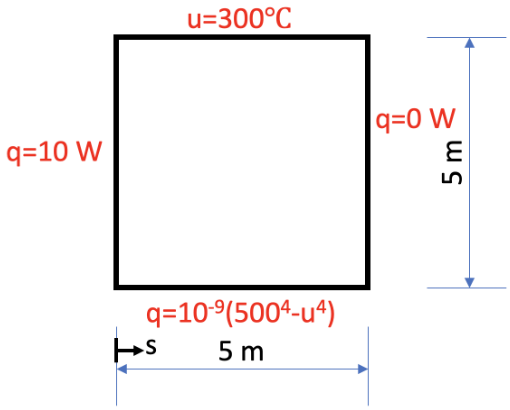

Figure 7. The square plate domain with dimensions and boundary conditions

The temperature difference distribution with respect to the measured distance, s, is given in Figure 8. The dependence of the temperature difference is the same as in the cited reference. Since the cited reference does not explicitly give the thermal conduction coefficient it is not possible to make a quantitative comparison.

It can be noted that, as the thermal conduction coefficient, k, increases, the average temperature in the plate decreases, as expected. This is mainly because, in steady state, the right side is kept at 300°C which enforces this boundary condition more as the thermal conduction coefficient increases.

In Figure 9 the dependence of normal flux with respect to the measured distance is displayed. Again, it can be concluded that the dependence is the same as in the cited reference.

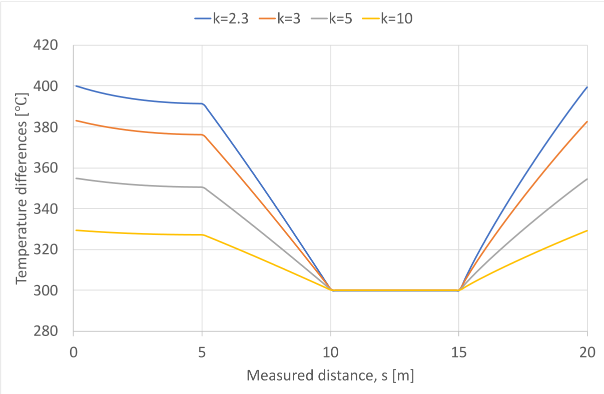

Figure 8. The temperature difference distribution with respect to the measured distance, s.

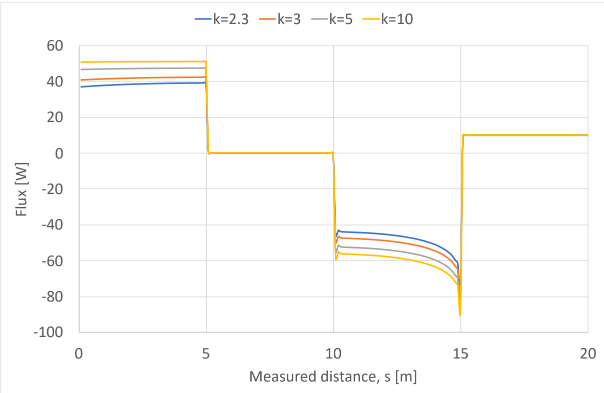

Figure 9. The normal flux with respect to the measured distance, s.

Figure 10 shows the temperature distribution of the square plate. Note that, the internal solution for such problems are obtained as a post processing.

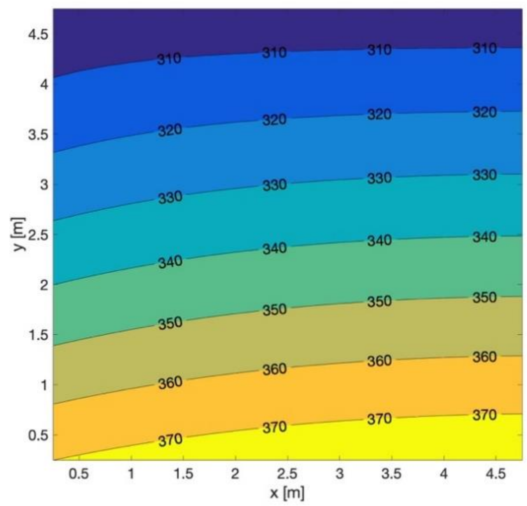

Figure 10. Temperature distribution for the square plate

CONCLUSION

A new method to solve steady-state heat transfer equations with nonlinear boundary conditions is proposed which is based on the Boundary Element Method. The method involves matrix manipulations over the system matrices of the BEM and reduces the total system of equations into a smaller ranked system where iterations involve only the temperature difference over the boundary that the nonlinear boundary condition is imposed. Iterative solution of the system reveals the temperature difference along that boundary which then can be used to obtain the total solution of the problem.

The method is applicable to 2D and 3D problems in heat transfer, as well as other problems that involves the Laplace's equation as the governing equation. It is a simple task to extend the capability of the method through time-dependent problems of thermal conduction since the time-dependent kernels are known in literature.

In this study, an example problem with a proven exact solution is employed to assess the efficiency of the proposed algorithm. The given exact solution is applicable to all polynomial order input functions and presented for the first time in literature. It has been seen that the method proves well both in accuracy and stability when compared with the presented exact solution.

To assess the formulation further, another example from a cited research is employed. It has been seen in this example also that the method gives comparable results with the reference. The only drawback of the study stems from the fact that in the code a very simple iterative algorithm, namely the fixed-point iteration, is used. This is the main reason why the results diverged for values of k less than 2.3.

The advantages of the proposed method lie in the following lines:

- Since the iterations are reduced to a system of equations that involve only the temperature difference along S^b, the method provides a robust and stable solution

- Assuming the geometry is fixed along with the discretization, the system matrices $\mathbf{Q}_{bb}$, $\mathbf{R}_{bb}$ and the load vector $\mathbf{b}_b$ is evaluated once and does not change within the iterations. This is a very advantageous property if repeated runs are required with different boundary conditions, which is in the case of an inverse analysis.

- The method is readily applicable for mesh refinement procedures over S_b. Note that, the main time-consuming step in the algorithm is obtaining the inverse of the sub-matrix $\mathbf{K}_{aa}$, especially if the number of elements in S_a is sufficiently large. But if the elements in this region are fixed, then it is not required to re-evaluate $\mathbf{K}_{aa}^{-1}$, instead other sub-matrices are re-evaluated with the new set of elements. Rest of the method is only matrix manipulations (multiplication and summation) which are sufficiently fast in nowadays computer architectures.

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