A Note On Simplicial Groupoids

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Abstract: Showing the equivalence of the two categories is very important in terms of investigating the properties. This equivalence shows that the properties existing in the category exist in the other category without the need to examination. From this point of view, since the category of groupoids is a category better known than the category of simplicial groupoids, thanks to equivalence, all the properties in the category of groupoids are valid for the category of simplicial groupoids. Main objective of this paper is to give simplicial object in category of groupoids and to prove the equivalence between the category of simplicial groupoids and the category of crossed modules of groupoids.

Simplisel Grupoidler Üzerine Bir Not

Anahtar Kelimeler
Grupoid,
Simlisel Obje,
Çaprazlanmış Modül

Öz: İki kategorinin denkliğini göstermek özelliklerin incelenmesi bakımında oldukça önemlidir. Bu denklik, kategoride var olan özelliklerin, incelemeye gerek kalmadan diğer kategoride de var olduğunu gösterir. Bu açıdan bakıldığında grupoidler kategorisi, simplisel grupoidler kategorisine göre daha iyi bilinen bir kategori olduğundan denklik sayesinde grupoidler kategorisindeki özelliklerin tamamı simplisel grupoidler kategorisi için de geçerlidir. Bu makalenin temel amacı grupoidler kategorisinde simplisel objeyi vermek ve simplisel grupoidler kategorisiyle grupoidler üzerinde çaprazlanmış modüller kategorisinin denkliğini göstermektir.

1. Introduction

Crossed modules over groups originally were defined by Whitehead [20]. He mentioned crossed modules in this study on which algebraic structures of relative homotopy groups. Since then, notion of crossed module has taken an important place in other areas. Some of studies are [4,5,10,13,14,15] about this topic. Simplicial groups were initially introduced by Kan [17]. After that, different authors defined concept of simplicial such as simplicial Lie algebras, simplicial Leibniz algebras and equivalences in related categories were given in some of the studies [1,2,6,14].

Groupoids were defined by Brandt [7] after a long working process. Afterwards, many studies have been done by researchers. According to Brown [8] groupoids can be considered as the generalization of groups. In this context, it seems possible to transfer the work done on groups to groupoids. See [3,9,18,19] for details about groupoids.

2. Material and Method

2.1. Groupoids and Crossed Modules

We recall some definitions from [9].
A groupoid is a small category in which every morphism is an isomorphism, i.e invertible.

That is, there exists morphism α^{-1} for any morphism α such that

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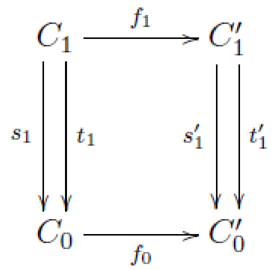
$$\alpha \circ \alpha^{-1} = e_{t(\alpha)} \text{ and } \alpha^{-1} \circ \alpha = e_{s(\alpha)},$$

where $e : C_0 \to C_1$ converts the identity morphism at an object and $s, t : C_1 \to C_0$ are group morphisms. We write a groupoid as $C = (C_0, C_1)$, where C_0 is the set of objects and C_1 is the set of morphisms. A groupoid C is called totally disconnected, if $C_1(x, y)$ is empty, for all $x, y \in C_0$ with $x \neq y$, i.e $s = t$. There exists a morphism of groupoids

$$f = (f_0, f_1) : (C_0, C_1) \to (C_0', C_1')$$

such that the diagram



is commutative. Thus, it can be formed category of groupoids denoted by **Gpd**.

Examples:

- **i.** Every group can be considered as a groupoid with single object.
- **ii.** A category whose objects are all sets and morphisms are bijective functions is a groupoid.
- **iii.** A category whose objects are topological spaces and morphisms are homeomorphisms is a groupoid.
- **iv.** Let X be a set and (x, y) be a morphism from x to y, for all $x, y \in X$. If the composition is defined as
$$(y, z) \circ (x, y) = (x, z)$$
for all $x, y, z \in X$, then $(X, X \times X)$ becomes a groupoid.
- **v.** Let X be a set, G be a group and triple (x, g, y) be a morphism from x to y, for all $x, y \in X$ and $g \in G$. If the composition is defined as
$$(y, h, z) \circ (x, g, y) = (x, hg, z),$$
for all $g, h \in G$, then $(X, X \times G \times X)$ is a groupoid which is called trivial groupoid.

Remark 2.1. Let $f = (f_0, f_1) : C \to C'$ be a morphism of groupoids. Kernel of the morphism is the set $\ker f = \{a \in C : \text{ there exists an } x \in C_0 \text{ such that } f(a) = e_x\}$.

Definition 2.2. Let G and C be groupoids over same object set and C be totally disconnected. Then an action of G on C is defined by a function

$$G_s \times_t C \to C$$
$$(g, c) \mapsto {}^g c$$

which satisfies

a) ${}^g c$ is defined if and only if $t(c) = s(g)$ and then $t({}^g c) = t(g)$,

b) ${}^g(c_1 \circ c_2) = {}^g c_1 \circ {}^g c_2$,

c) ${}^{g \circ h} c_1 = {}^g({}^h c_1)$ and ${}^{e_x} c_1 = c_1$,

for all $c_1, c_2 \in C(x, x)$ and $g \in G(x, y)$, $h \in G(y, z)$.

Definition 2.3. Let G and C be groupoids providing the above conditions. If the morphism $\partial : C \to G$ which satisfies

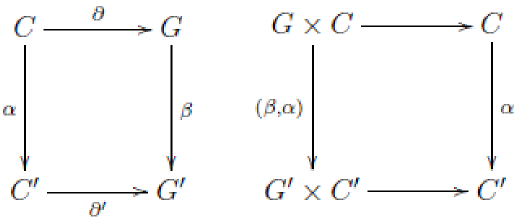
CM1) $\partial({}^g c) = g^{-1} \circ \partial(c) \circ g$

CM2) ${}^{\partial(c_1)} c = c_1^{-1} \circ c \circ c_1$

for all $c_1, c \in C(x,x)$, $g \in G(x,y)$, and $x, y \in C_0$ then triple (C, G, ∂) is called a crossed module of groupoids.
A morphism of crossed modules of groupoids from (C, G, ∂) to (C', G', ∂') is a pair of groupoids morphisms

$$\alpha : C \to C' \text{ and } \beta : G \to G'$$

such that the diagrams



are commutative. Thus, it can be get category of crossed modules of groupoids denoted by **XMod(Gpd)**.

3. Results

We recall some definitions from [12,16].
Let Δ be the category of finite ordinals. A simplicial object C in **Gpd**, or briefly a simp. gpd, is the functor

$$C : \Delta^{op} \to \textbf{Gpd}.$$

In other words, let $C = \{C_0, C_1, ..., C_n, ...\}$ be a family of groupoids and

$$d_i^n : C_n \to C_{n-1} \text{ and } s_i^n : C_n \to C_{n+1}$$

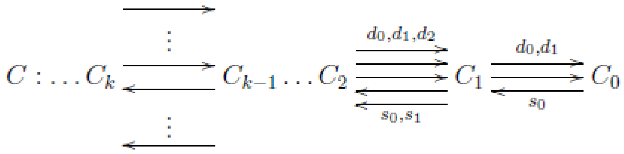
groupoid morphisms for $0 \le i \le n$. The triple (C, d_i, s_j) is called simplicial groupoid if d_i and s_j satisfy the following axioms which called simplicial identities

$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ id & \text{for } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \quad \text{for } i \le j,$$

for $0 \le i \le n$, where d_i and s_j is called face and degeneracy map respectively. Note that face and dejeneracy morphisms of C are identity on objects. Simplicial groupoids can be pictured as follows;



A simplicial groupoid morphism $f : C \to C'$ with d_i and s_j maps is a family of groupoid homomorphisms $f_n : C_n \to C_n'$ such that

$$d_i f_n = f_{n-1} d_i \text{ and } f_n s_i = s_i f_{n-1}$$

for all i and n. So, it can be obtained the category of simplicial groupoids denoted by **Simp(Gpd)**.

3.1. The Moore Complex and Truncated Objects

The Moore complex NC of a simplicial object C in **Simp(Gpd)** is the sequence

$$NC : \cdots \longrightarrow NC_n \xrightarrow{\partial_n} NC_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} NC_1 \xrightarrow{\partial_1} NC_0$$

where $NC_0 = C_0$, $NC_n = \bigcap_{i=0}^{n-1} \ker d_i$ and ∂_n is the restriction of d_n to NC_n. It is called that NC is Moore complex of simplicial groupoid C and is of length k if $NC_n = 0$, for all $n \geq k+1$. Thus, one can define a category whose objects are simplicial groupoids with Moore complex of length k and the morphisms are families of homomorphisms compatible with face and degeneracy maps. We denote that category by **Simp** $_{\leq k}$ **(Gpd)**.

The following terminology for groupoid case is adapted from [11]. Let C_i be a groupoid for $0 \leq i \leq k$. A simplicial groupoid defined by

$$\cdots \longrightarrow 0 \longrightarrow C_k \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow}\; C_{k-1} \cdots C_2 \;\underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow}}}\; C_1 \;\underset{s_0}{\overset{d_0, d_1}{\substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow}}}\; C_0$$

is called k-truncated simplicial groupoid and denoted by **Tr$_k$Simp(Gpd)**. This structure has truncated functor

$$tr_k : \textbf{Simp(Gpd)} \rightarrow \textbf{Tr}_\textbf{k}\textbf{Simp(Gpd)}$$

has left adjoint st_k called k-skeleton and right adjoint $cost_k$ called k-coskeleton. These can be shown as follows;

$$\textbf{Tr}_\textbf{k}\textbf{Simp}(\textbf{Gpd}) \underset{tr_k}{\overset{\cos tr_k}{\rightleftarrows}} \textbf{Simp}(\textbf{Gpd}) \underset{tr_k}{\overset{st_k}{\rightleftarrows}} \textbf{Tr}_\textbf{k}\textbf{Simp}(\textbf{Gpd}).$$

See [11] for details.

Theorem 3.1 The category **XMod(Gpd)** is naturally equivalent to the category **Simp** $_{\leq 1}$ **(Gpd)**.

Proof: Let C be a simplicial groupoid with Moore complex of length 1 and C_0, C_1 be two groupoids. Take $G = \ker d_0$ and ∂ is the restriction of d_1 to G. Define the action of C_0 on G by

$$C_0 \times G \rightarrow G$$
$$(c,g) \mapsto {}^c g = s_0(c)^{-1} \circ g \circ s_0(c)$$

for all $c \in C_0$ and $g \in G$. By using the action given above $\partial : G \rightarrow C_0$ becomes a crossed module. Because

CM1)

$$\begin{aligned}
\partial\left({}^c g\right) &= \partial\left(s_0(c)^{-1} \circ g \circ s_0(c)\right) \\
&= \partial s_0(c)^{-1} \circ \partial(g) \circ \partial s_0(c) \\
&= d_1 s_0(c)^{-1} \circ \partial(g) \circ d_1 s_0(c) \\
&= c^{-1} \circ \partial(g) \circ c
\end{aligned}$$

CM2)

$$\begin{aligned}
{}^{\partial(g)} g' &= s_0 \partial(g)^{-1} \circ g' \circ s_0 \partial(g) \\
&= s_0 d_1(g)^{-1} \circ g' \circ s_0 d_1(g) \\
&= s_0 d_1(g)^{-1} \circ g' \circ s_0 d_1(g) \circ \left[\left(g^{-1} \circ (g')^{-1} \circ g\right) \circ \left(g^{-1} \circ (g') \circ g\right)\right] \\
&= s_0 d_1(g)^{-1} \circ g' \circ s_0 d_1(g) \circ \left(g^{-1} \circ (g')^{-1} \circ g\right) \circ \left(g^{-1} \circ (g') \circ g\right) \\
&= d_2 s_0(g)^{-1} \circ d_2 s_1(g') \circ d_2 s_0(g) \circ \; d_2 s_1(g^{-1}) \circ d_2 s_1(g')^{-1} \circ d_2 s_1(g) \circ \left(g^{-1} \circ (g') \circ g\right) \\
&= d_2\left(s_0(g)^{-1} \circ s_1(g') \circ s_0(g)\right) \circ \; d_2\left(s_1(g^{-1}) \circ s_1(g')^{-1} \circ s_1(g)\right) \circ \left(g^{-1} \circ (g') \circ g\right) \\
&= \left(g^{-1} \circ (g') \circ g\right)
\end{aligned}$$

for all $c \in C_0$ and $g, g' \in G$. So we obtain the functor

$$U:\mathbf{Simp}_{\leq 1}\,\mathbf{(Gpd)} \longrightarrow \mathbf{XMod(Gpd)}.$$

On the other hand, let $\partial : G \to C_0$ be a crossed module. With the action of C_0 on G one can obtain the semi-direct product

$$G\tilde{a}\; C_0 = \{(g,c) : g \in G \text{ and } c \in C_0\}$$

where multiplication of elements is

$$(g_1,c_1)(g_2,c_2) = \left(g_1 \circ {}^{c_1}g_2, c_1 \circ c_2\right)$$

for all $(g_1,c_1),(g_2,c_2) \in G\tilde{a}\; C_0$. Also, we have the morphism

$d_0 :\quad G\tilde{a}\; C_0 \to C_0$

$\qquad (g,c) \mapsto c$

$d_1 :\quad G\tilde{a}\; C_0 \to C_0$

$\qquad (g,c) \mapsto \partial(g) \circ c$

$s_0 :\quad C_0 \to G\tilde{a}\; C_0$

$\qquad c \mapsto (0,c)$

which satisfy the simplicial identities. Consequently,

$$C_1 \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} C_0$$

is a 1-truncated simplicial groupoid. As a result, we obtain the functor

$$V:\mathbf{XMod(Gpd)} \longrightarrow \mathbf{Simp}_{\leq 1}\,\mathbf{(Gpd)}.$$

Thus, these functors can be showed as

$$\mathbf{Simp}_{\leq 1}\,\mathbf{(Gpd)} \underset{V}{\overset{U}{\rightleftarrows}} \mathbf{XMod(Gpd)}.$$

4. Discussion and Conclusions

In this study, notion of simplicial object was given in the category of groupoids and it was shown that the category of simplicial groupoids is equivalent to the category of crossed modules of groupoids. Thanks to this equivalence, it was understood that these two categories have common properties.

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