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Research Article

Binomial Operator as a Hausdorff Operator of the Euler Type

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ABSTRACT. In this paper, we prove that the binomial operator is a Hausdorff operator of the Euler type and consequently, the binomial matrix domain associated with this operator is nothing new except an Euler sequence space. Therefore, all the results of published papers on the binomial sequence spaces like [4], can be extracted easily from [1] and the relation between the binomial and Euler operators that we introduce. Moreover, we compute the norm and the lower bound of the binomial operator on some sequence spaces.

Keywords: Binomial operator, Euler operator, norm, lower bound, Hausdorff matrix, sequence spaces.

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1. INTRODUCTION

Let $p \geq 1$ and ω denote the set of all real-valued sequences. The space ℓ_p is the set of all real sequences $x = (x_k) \in \omega$ such that

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$

Definition 1.1. *The Hausdorff matrix $H^\mu = (h_{jk})_{j,k=0}^{\infty}$ is defined by*

$$h_{j,k} := \begin{cases} \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta) & , \quad 0 \leq k \leq j, \\ 0 & , \quad k > j \end{cases}$$

for all $j, k \in \mathbb{N}_0$, where μ is a probability measure on $[0,1]$.

Theorem 1.1 (Hardy's formula, [9, Theorem 216]). *The Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty$ and*

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) \qquad (1 < p < \infty). \tag{1.1}$$

Hausdorff operator has the following norm property.

Theorem 1.2 ([3, Theorem 9]). *Let $p \geq 1$ and H^μ, H^φ and H^ν be Hausdorff matrices such that $H^\mu = H^\varphi H^\nu$. Then, H^μ is bounded on ℓ_p if and only if both H^φ and H^ν are bounded on ℓ_p. Moreover, we have*

$$\|H^\mu\|_{\ell_p} = \|H^\varphi\|_{\ell_p} \|H^\nu\|_{\ell_p}.$$

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Definition 1.2. *For $0 < r < 1$ and $d\mu(\theta) =$ point evaluation at $\theta = r$, the associated Hausdorff matrix is the Euler matrix of order r, $E^r = (e^r_{j,k})$, who has the entries*

$$e^r_{j,k} = \begin{cases} \binom{j}{k}(1-r)^{j-k}r^k & , 0 \le k \le j \\ 0 & , \text{otherwise} \end{cases}$$

and the ℓ_p-norm $\|E^r\|_{\ell_p} = r^{\frac{-1}{p}}$.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_n) \in \omega \ : \ Ax \in \lambda\}. \tag{1.2}$$

It is easy to see that for invertible matrix A and normed space λ, the matrix domain λ_A is a normed space with $\|x\|_{\lambda_A} := \|Ax\|_\lambda$. Let $1 \le p < \infty$. The matrix domains $e^r(p)$ and $e^r(\infty)$ associated by the Euler matrix E^r are

$$e^r(p) = \left\{ x = (x_n) \in w \ : \ \sum_j \left| \sum_{k=0}^{j} \binom{j}{k} (1-r)^{j-k} r^k \right|^p < \infty \right\}$$

and

$$e^r(\infty) = \left\{ x = (x_n) \in w \ : \ \sup_j \left| \sum_{k=0}^{j} \binom{j}{k} (1-r)^{j-k} r^k \right| < \infty \right\}.$$

By the notation of (1.2), the Euler sequence spaces $e^r(p)$ and $e^r(\infty)$ can be redefined by the matrix domain of

$$e^r(p) = (\ell_p)_{E^r} \qquad \text{and} \qquad e^r(\infty) = (\ell_\infty)_{E^r}.$$

Let r, s be two non-negative numbers that $r + s \neq 0$. The binomial matrix $B^{r,s} = (b^{r,s}_{j,k})$ is defined by

$$b^{r,s}_{j,k} = \begin{cases} \frac{1}{(r+s)^j}\binom{j}{k} s^{j-k} r^k & , 0 \le k \le j \\ 0 & , \text{otherwise} \end{cases}.$$

If $r + s = 1$, one can easily see that $B^{r,s} = E^r$. For $1 \le p < \infty$, the binomial sequence spaces $b^{r,s}(p)$ and $b^{r,s}(\infty)$ generated by $B^{r,s}$ are defined by

$$b^{r,s}(p) = \left\{ x = (x_n) \in w \ : \ \sum_j \left| \sum_{k=0}^{j} \frac{1}{(r+s)^j} \binom{j}{k} s^{j-k} r^k \right|^p < \infty \right\}$$

and

$$b^{r,s}(\infty) = \left\{ x = (x_n) \in w \ : \ \sup_j \left| \sum_{k=0}^{j} \frac{1}{(r+s)^j} \binom{j}{k} s^{j-k} r^k \right| < \infty \right\}.$$

The binomial sequence spaces $b^{r,s}(p)$ and $b^{r,s}(\infty)$ can be represented by the matrix domain of

$$b^{r,s}(p) = (\ell_p)_{B^{r,s}} \qquad \text{and} \qquad b^{r,s}(\infty) = (\ell_\infty)_{B^{r,s}}.$$

In this study, we investigate the norm and the lower bound of binomial operator $B^{r,s}$ from the sequence spaces A_p into the sequence spaces B_p and gain inequalities of the form

$$\|B^{r,s}x\|_{B_p} \le U\|x\|_{A_p} \quad \text{and} \quad \|B^{r,s}x\|_{B_p} \ge L\|x\|_{A_p}$$

for all sequences $x \in \ell_p$. The constants U and L are not depending on x, and the norm and the lower bound of T are the smallest and greatest possible value of U and L, respectively. The problem of finding the norm of matrix operators on the sequence space ℓ_p have been studied extensively by many mathematicians and abundant literature exists on the topic. Although topological properties and inclusion relations of $b^{r,s}(p)$ have largely been explored [4, 13, 12], computing the norm of binomial operators on sequence spaces has not been investigated to date. More recently, the author has computed the norm of operators on several sequence spaces, [7, 8, 21, 22, 14, 15, 16, 17, 18, 20, 19].

Several papers have published about the binomial sequence space, who is a matrix domain associated with the binomial operator or other spaces which obtain by this operator [4, 5, 11, 24, 23, 6]. Those are all have investigated the properties of this space such as inclusions, dual spaces, Schauder basis, compactness, matrix transformations etc. In this study, we reveal that this matrix is a Hausdorff one, of the Euler type, which is not worth wasting the mathematicians' time more. Moreover, the bounds of this operator has computed on some sequence spaces that has never done before.

2. Close relation of Binomial and Euler operators

In this section, we reveal the nature of binomial operator, its ℓ_p-norm and its relation with Euler operators. The following theorem is the main theorem of this study.

Theorem 2.3. *Suppose that r and s are two non-negative real numbers with $r+s\neq 0$. Then, $B^{r,s}$ is a Hausdorff matrix and*

- $B^{r,s}=E^{\frac{r}{r+s}}$, *where E^r is the Euler matrix of order r,*
- $E^r=B^{r,1-r}$,
- $\|B^{r,s}\|_{\ell_p}=\left(\frac{r+s}{r}\right)^{1/p}$,
- $B^{r,s}B^{t,u}=B^{rt,ru+st+su}$,
- $B^{r,s}$ *is invertible and its inverse is* $B^{1+\frac{s}{r},\frac{-s}{r}}$.

Proof. By letting $d\mu(\theta)=point\ evaluation\ at\ \theta=\frac{r}{r+s}$, the associated Hausdorff matrix is the binomial operator which accorollaryding to the Hardy's formula has the ℓ_p-norm $\left(\frac{r+s}{r}\right)^{1/p}$. This proves the first and the third parts. The second part is obvious. By applying the identity $E^rE^s=E^{rs}$ and part one, we have

$$\begin{aligned} B^{r,s}B^{t,u} &= E^{\frac{r}{r+s}}E^{\frac{t}{t+u}}=E^{\frac{rt}{(r+s)(t+u)}} \\ &= E^{\frac{rt}{rt+ru+st+su}}=B^{rt,ru+st+su} \end{aligned}$$

which results the fourth item. For obtaining the last part, since $(E^r)^{-1}=E^{\frac{1}{r}}$, applying the second part results in

$$(B^{r,s})^{-1}=(E^{\frac{r}{r+s}})^{-1}=E^{\frac{r+s}{r}}=B^{1+\frac{s}{r},\frac{-s}{r}}.$$

□

Remark 2.1. *One can verify the first result of the Theorem 2.3 directly by*

$$\begin{aligned} b^{r,s}_{j,k} &= \frac{1}{(r+s)^j}\binom{j}{k}s^{j-k}r^k=\binom{j}{k}\left(\frac{s}{r+s}\right)^{j-k}\left(\frac{r}{r+s}\right)^k \\ &= \binom{j}{k}\left(1-\frac{r}{r+s}\right)^{j-k}\left(\frac{r}{r+s}\right)^k \\ &= e^{\frac{r}{r+s}}_{j,k}. \end{aligned}$$

Remark 2.2. *One can also check the second part of Theorem 2.3 straightly by*

$$
\begin{aligned}
(B^{r,s}B^{t,u})_{j,k} &= \frac{1}{(r+s)^j}\sum_{i=k}^{j}\binom{j}{i}\binom{i}{k}s^{j-i}r^i\frac{1}{(t+u)^i}t^{i-k}u^k \\
&= \frac{1}{(r+s)^j}(\frac{u}{t})^k s^j\frac{j!}{k!}\sum_{i=k}^{j}\frac{1}{(j-i)!(i-k)!}\left(\frac{rt}{s}\right)^i\frac{1}{(t+u)^i} \\
&= \frac{1}{(r+s)^j}(\frac{u}{t})^k s^j\frac{j!}{k!}\sum_{i=0}^{j-k}\frac{1}{(j-i-k)!i!}\left(\frac{rt}{s(t+u)}\right)^{i+k} \\
&= (\frac{s}{r+s})^j(\frac{ru}{st+su})^k\binom{j}{k}\sum_{i=0}^{j-k}\binom{j-k}{i}\left(\frac{rt}{s(t+u)}\right)^i \\
&= (\frac{s}{r+s})^j(\frac{ru}{st+su})^k\binom{j}{k}\left(1+\frac{rt}{s(t+u)}\right)^{j-k} \\
&= (\frac{1}{r+s})^j(\frac{ru}{t+u})^k\binom{j}{k}\left(s+\frac{rt}{t+u}\right)^{j-k} \\
&= \frac{1}{(r+s)^j}\frac{1}{(t+u)^j}(ru)^k\binom{j}{k}(st+su+rt)^{j-k} \\
&= B^{rt,ru+st+su}_{j,k}.
\end{aligned}
$$

2.1. **Factorization of the Binomial operators and its applications.** In this part of study, we find some factorization for the binomial operator and obtain several inequalities and inclusions who are all the straightforward result of the Theorem 2.3.

Corollary 2.1. *Let r,s,t,u be positive numbers that $\frac{r}{s}<\frac{t}{u}$. The binomial operator $B^{r,s}$ has a factorization of the form*

$$B^{r,s}=E^{\frac{r(t+u)}{t(r+s)}}B^{t,u}.$$

In particular,

- $E^r=E^{\frac{r(t+u)}{t}}B^{t,u}$, $\quad r<\frac{t}{t+u}$,
- $B^{r,s}=E^{\frac{r}{t(r+s)}}E^t$, $\quad \frac{r}{r+s}<t$,
- $E^r=E^{\frac{r}{t}}E^t$, $\quad r<t$.

As a result of the above factorization, we have the following inequalities.

Corollary 2.2. *Let r,s,t,u be positive numbers that $\frac{r}{s}<\frac{t}{u}$ and $x\in\ell_p$. Then,*

$$\sum_{k=0}^{\infty}\left|\frac{1}{(r+s)^j}\binom{j}{k}s^{j-k}r^kx_k\right|^p\le\frac{t(r+s)}{r(t+u)}\sum_{k=0}^{\infty}\left|\frac{1}{(t+u)^j}\binom{j}{k}u^{j-k}t^kx_k\right|^p.$$

In particular,

$$\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-r)^{j-k}r^kx_k\right|^p\le\frac{t}{r(t+u)}\sum_{k=0}^{\infty}\left|\frac{1}{(t+u)^j}\binom{j}{k}u^{j-k}t^kx_k\right|^p \qquad ,r<\frac{t}{t+u},$$

$$\sum_{k=0}^{\infty}\left|\frac{1}{(r+s)^j}\binom{j}{k}s^{j-k}r^kx_k\right|^p\le\frac{t(r+s)}{r}\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-t)^{j-k}t^kx_k\right|^p \qquad ,\frac{r}{r+s}<t$$

and

$$\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-r)^{j-k}r^k x_k\right|^p \leq \frac{t}{r}\sum_{k=0}^{\infty}\left|\binom{j}{k}(1-t)^{j-k}t^k x_k\right|^p \qquad , r<t.$$

Proof. The proof is obvious by Corollary 2.1. □

Theorem 2.4. *Let r, s, t, u be positive numbers that $\frac{r}{s} < \frac{t}{u}$. Then, $b^{t,u}(p) \subset b^{r,s}(p)$.*
In particular,

- $b^{t,u}(p) \subset e^r(p), \quad r < \frac{t}{t+u}$,
- $e^t(p) \subset b^{r,s}(p), \quad \frac{r}{r+s} < t$,
- $e^t(p) \subset e^r(p), \quad r < t$.

Proof. This is the straightforward result of Corollary 2.2. □

Remark 2.3. *The last part of previous theorem is the Theorem 3.4 of* [1].

Remark 2.4. *Since $\frac{r}{s} < \frac{t}{u}$, hence $\frac{r}{r+s} < \frac{t}{t+u}$. Now, accorollaryding to the Theorem 3.4 of* [1], $b^{t,u}(p) = e^{\frac{t}{t+u}}(p) \subset e^{\frac{r}{r+s}}(p) = b^{r,s}(p)$.

2.2. **The α-, β- and γ-dual of $b^{r,s}(p)$.** In this example, we show that Theorem 4.2 of [4] can be easily gained from Theorem 4.4 of [1]. Therefore, let us bring that theorem first

Theorem 2.5 ([1, Theorem 4.4])**.** *Define the sets A^r_q and A^r_∞ as follows. For $1 \leq p < \infty$,*

$$A^r_q = \left\{ a=(a_k)\in w \ : \ \sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}(r-1)^{n-k}r^{-n}a_n\right|^q < \infty \right\}$$

and

$$A^r_\infty = \left\{ a=(a_k)\in w \ : \ \sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}(r-1)^{n-k}r^{-n}a_n\right| < \infty \right\}.$$

Then, $(E^r_1)^\alpha = A^r_\infty$ and $(e^r(p))^\alpha = A^r_q$, where $1 < p \leq \infty$.

Now, we obtain the α-dual of $b^{r,s}(p)$ and $b^{r,s}(1)$. By applying the above theorem and the identity $B^{r,s} = E^{\frac{r}{r+s}}$ of Theorem 2.3,

$$\begin{aligned}
(b^{r,s}(p))^\alpha &= (e^{\frac{r}{r+s}}(p))^\alpha = A_q^{\frac{r}{r+s}}\\
&= \left\{a=(a_k)\in w \ : \ \sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}\left(\frac{r}{r+s}-1\right)^{n-k}\left(\frac{r}{r+s}\right)^{-n}a_n\right|^q<\infty\right\}\\
&= \left\{a=(a_k)\in w \ : \ \sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n\right|^q<\infty\right\} = V_1^{r,s}
\end{aligned}$$

and

$$\begin{aligned}
(b^{r,s}(1))^\alpha &= (e^{\frac{r}{r+s}}(1))^\alpha = A_\infty^{\frac{r}{r+s}}\\
&= \left\{a=(a_k)\in w \ : \ \sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}\left(\frac{r}{r+s}-1\right)^{n-k}\left(\frac{r}{r+s}\right)^{-n}a_n\right|<\infty\right\}\\
&= \left\{a=(a_k)\in w \ : \ \sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n\right|<\infty\right\} = V_2^{r,s},
\end{aligned}$$

where $V_1^{r,s}$ and $V_2^{r,s}$ are the α-duals of $b^{r,s}(p)$ and $b^{r,s}(1)$ respectively, as the author of [4] has proved in Theorem 4.2. Note that for obtaining the β- and γ-duals of $b^{r,s}(p)$ and $b^{r,s}(1)$ ([4], Theorem 4.3), we only need changing r to $\frac{r}{r+s}$ in Theorems 4.5 and 4.6 of [1].

3. Bounds of binomial operator on some sequence spaces

In this section, we investigate the bounds of binomial operator on some sequence spaces. In so doing, the following lemma is needed.

Lemma 3.1 ([15, Lemma 2.1]). *Let U is a bounded operator on ℓ_p, and A_p and B_p be two matrix domains such that $A_p \simeq \ell_p$. Then, the following statements hold:*

(i) If BT is a bounded operator on ℓ_p, then T is a bounded operator from ℓ_p into B_p and

$$\|T\|_{\ell_p,B_p} = \|T\|_{\ell_p} \quad \text{and} \quad L(T)_{\ell_p,B_p} = L(BT).$$

(ii) If T has a factorization of the form $T = UA$, then T is a bounded operator from the matrix domain A_p into ℓ_p and

$$\|T\|_{A_p,\ell_p} = \|U\|_{\ell_p} \quad \text{and} \quad L(T)_{A_p,\ell_p} = L(U).$$

(iii) If $BT = UA$, then T is a bounded operator from the matrix domain A_p into B_p and

$$\|T\|_{A_p,B_p} = \|U\|_{\ell_p} \quad \text{and} \quad L(T)_{A_p,B_p} = L(U).$$

In particular, if $AT = UA$, then T is a bounded operator from the matrix domain A_p into itself and $\|T\|_{A_p} = \|U\|_{\ell_p}$ and $L(T)_{A_p} = L(U)$. Also, if T and A commute, then $\|T\|_{A_p} = \|T\|_{\ell_p}$ and $L(T)_{A_p} = L(T)$.

Throughout this section, we use the notations $L(\cdot)$ for the lower bound of operators on ℓ_p and $L(\cdot)_{X,Y}$ for the lower bound of operators from the sequence space X into the sequence space Y.

3.1. Norm of binomial operator on difference sequence space.

The backward difference matrix $\Delta = (\delta_{j,k})$ is defined by

$$\delta_{j,k} = \begin{cases} 1 & , k = j \\ -1 & , k = j-1 \\ 0 & , otherwise \end{cases}$$

and the difference sequence space associated with this matrix is called bv_p

$$bv_p = \left\{ x = (x_n) \ : \ \sum_{n=1}^{\infty} |x_n - x_{n-1}|^p < \infty \right\}, \quad 1 \le p < \infty,$$

which has the norm $\|x\|_{bv_p} = \left(\sum_{n=1}^{\infty} |x_n - x_{n-1}|^p\right)^{1/p}$. The idea of difference sequence spaces was introduced by Kizmaz [10]. Recently, Roopaei in [14] has computed the norm of Hausdorff operators on bv_p sequence space.

Theorem 3.6 ([14, Theorem 2.4]). *The Hausdorff operator H^μ is a bounded operator on bv_p and*

$$\|H^\mu\|_{bv_p} = 1.$$

We have proved that the binomial operator is a Hausdorff operator of Euler type, hence

Corollary 3.3. *The binomial operator $B^{r,s}$ is a bounded operator on bv_p and $\|B^{r,s}\|_{bv_p} = 1$.*

3.2. **Bounds of binomial operator on the Hausdorff sequence space.** The Hausdorff matrix contains the famous classes of matrices. For $\alpha > 0$, some of these classes are as follows:

- The choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ gives the Cesàro matrix of order α,
- The choice $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ gives the Gamma matrix of order α,
- The choice $d\mu(\theta) = \frac{|\log\theta|^{\alpha-1}}{\Gamma(\alpha)}d\theta$ gives the Hölder matrix of order α.

Theorem 3.7 ([3, Theorem 1]). *Let $p \geq 1$, and let H^μ is a bounded Hausdorff matrix on ℓ_p. Then,*

$$\|H^\mu x\|_{\ell_p} \geq L\|x\|_{\ell_p}$$

for every decreasing sequence x of non-negative terms, where

$$L^p = \sum_{k=0}^{\infty}\left(\int_0^1 (1-\theta)^k d\mu(\theta)\right)^p. \tag{3.3}$$

The constant in (3.3) is the best possible, and there is equality only when $x = 0$ or $p = 1$ or $d\mu(\theta)$ is the point mass at 1.

As an example of Theorem 3.7, we compute the lower bound of the Cesàro, Gamma and Euler operators by choosing their associated $d\mu(\theta)$.

- $L(C^\alpha) = \left\{\sum_{k=0}^{\infty}\left(\frac{\alpha}{\alpha+k}\right)^p\right\}^{1/p}$,
- $L(\Gamma^\alpha) = \left\{\sum_{k=0}^{\infty}\binom{\alpha+k}{k}^{-p}\right\}^{1/p}$,
- $L(E^\alpha) = \frac{1}{[1-(1-\alpha)^p]^{1/p}}, \quad 0 < \alpha < 1,$
- $L(B^{r,s}) = \frac{1}{[1-(\frac{s}{r+s})^p]^{1/p}}$ (by Theorem 2.3).

We use the notation $hau(p)$ as the set of all sequences whose H^μ-transforms are in the space ℓ_p, that is

$$hau(p) = \left\{x = (x_j) \in \omega : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\int_0^1 \binom{j}{k}\theta^k(1-\theta)^{j-k}d\mu(\theta)x_k\right|^p < \infty\right\},$$

where μ is a fixed probability measure on $[0,1]$.

Theorem 3.8. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $hau(p)$ and*

$$\|B^{r,s}\|_{\ell_p,hau(p)} = \left(\frac{r+s}{r}\right)^{1/p}\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta)$$

and

$$L(B^{r,s})_{\ell_p,hau(p)} \geq \left\{\frac{\sum_{k=0}^{\infty}\left(\int_0^1(1-\theta)^k d\mu(\theta)\right)^p}{1-(\frac{s}{r+s})^p}\right\}^{1/p}.$$

In particular, for $r+s=1$, the Euler operator E^r is a bounded operator from ℓ_p into $hau(p)$ and

$$\|E^r\|_{\ell_p,hau(p)} = r^{\frac{-1}{p}}\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta) \quad \textit{and} \quad L(E^r)_{\ell_p,hau(p)} \geq \left\{\frac{\sum_{k=0}^{\infty}\left(\int_0^1(1-\theta)^k d\mu(\theta)\right)^p}{1-(1-r)^p}\right\}^{1/p}.$$

Proof. Applying Lemma 3.1 part (i) and Theorems 1.2 and 2.3, result that

$$\|B^{r,s}\|_{\ell_p,hau(p)} = \|H^\mu B^{r,s}\|_{\ell_p} = \|H^\mu\|_{\ell_p}\|B^{r,s}\|_{\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta).$$

Also, the identity $L(AB) \geq L(A)L(B)$ results in

$$L(B^{r,s})_{\ell_p,hau(p)} = L(B^{r,s}H^\mu) \geq \left\{\frac{1}{1-(\frac{s}{r+s})^p}\right\}^{1/p}\left\{\sum_{k=0}^{\infty}\left(\int_0^1 (1-\theta)^k d\mu(\theta)\right)^p\right\}^{1/p}.$$

□

By letting $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix $C^\alpha = (C^\alpha_{jk})$ of order α is defined as follows

$$C^\alpha_{j,k} = \begin{cases} \dfrac{\binom{\alpha+j-k-1}{j-k}}{\binom{\alpha+j}{j}} & ,0 \leq k \leq j \\ 0 & ,\text{otherwise} \end{cases},$$

which accorollaryding to the Hardy's formula has the ℓ_p-norm

$$\|C^\alpha\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+1/p^*)}.$$

Note that, $C^0 = I$, where I is the identity matrix and C^1 is the well-known Cesàro matrix C which has the ℓ_p-norm $\|C\|_{\ell_p} = p^*$ and the lower bound $L(C) = \zeta(p)^{1/p}$. We use the notation $ces(\alpha,p)$ as the set of all sequences whose C^α-transforms are in the space ℓ_p, that is

$$ces(\alpha,p) = \left\{x = (x_j) \in \omega : \sum_{j=0}^{\infty}\left|\frac{1}{\binom{\alpha+j}{j}}\sum_{k=0}^{j}\binom{\alpha+j-k-1}{j-k}x_k\right|^p < \infty\right\}.$$

The space $ces(\alpha,p)$ is a Banach space which has the norm

$$\|x\|_{ces(\alpha,p)} = \left(\sum_{j=0}^{\infty}\left|\frac{1}{\binom{\alpha+j}{j}}\sum_{k=0}^{j}\binom{\alpha+j-k-1}{j-k}x_k\right|^p\right)^{1/p}.$$

We use the notation $ces(p)$ instead of $ces(1,p)$ as the sequence space associated with the well-known Cesàro matrix C. For more information about Cesàro matrix, the readers can refer to [20, 19].

Corollary 3.4. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $ces(\alpha,p)$ and*

$$\|B^{r,s}\|_{\ell_p,ces(\alpha,p)} = \frac{\left(\frac{r+s}{r}\right)^{1/p}\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+1/p^*)}$$

and

$$L(B^{r,s})_{\ell_p,ces(\alpha,p)} \geq \left\{\frac{\sum_{k=0}^{\infty}\left(\frac{\alpha}{\alpha+k}\right)^p}{1-(\frac{s}{r+s})^p}\right\}^{1/p}.$$

In particular, for $r+s=1$ and $\alpha=1$, the Euler operator E^r is a bounded operator from ℓ_p into $ces(p)$ and $\|E^r\|_{\ell_p,ces(p)} = \frac{r^{-1/p}p}{p-1}$ and $L(E^r)_{\ell_p,ces(p)} \geq \left\{\frac{\zeta(p)}{1-(1-r)^p}\right\}^{1/p}$.

By letting $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ in the definition of the Hausdorff matrix, the Gamma matrix of order α, $\Gamma^\alpha = (\gamma^\alpha_{j,k})$, is

$$\gamma^\alpha_{j,k} = \begin{cases} \frac{\binom{\alpha+k-1}{k}}{\binom{\alpha+j}{j}} & , \ 0 \le k \le j \\ 0 & , \ otherwise \end{cases},$$

which accorollaryding to the Hardy's formula has the ℓ_p-norm $\|\Gamma^\alpha\|_{\ell_p} = \frac{\alpha p}{\alpha p-1}$. Note that, Γ^1 is the well-known Cesàro matrix. The Gamma space of order α, $gam(\alpha, p)$, is

$$gam(\alpha,p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{\alpha+j}{j}} \sum_{k=0}^{j} \binom{\alpha+k-1}{k} x_k \right|^p < \infty \right\},$$

which is a Banach spaces with the norm

$$\|x\|_{gam(\alpha,p)} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{\alpha+j}{j}} \sum_{k=0}^{j} \binom{\alpha+k-1}{k} x_k \right|^p \right)^{\frac{1}{p}}.$$

Note that $gam(1,p) = ces(p)$.

Corollary 3.5. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $gam(\alpha,p)$ and*

$$\|B^{r,s}\|_{\ell_p,gam(\alpha,p)} = \frac{\left(\frac{r+s}{r}\right)^{1/p} \alpha p}{\alpha p - 1}$$

and

$$L(B^{r,s})_{\ell_p,gam(\alpha,p)} \ge \left\{ \frac{\sum_{k=0}^{\infty} \binom{\alpha+k}{k}^{-p}}{1 - (\frac{s}{r+s})^p} \right\}^{1/p}.$$

In particular, for $r+s=1$ and $\alpha=1$, the Euler operator E^r is a bounded operator from ℓ_p into $ces(p)$ and $\|E^r\|_{\ell_p,ces(p)} = \frac{r^{-1/p}p}{p-1}$ and $L(E^r)_{\ell_p,ces(p)} \ge \left\{\frac{\zeta(p)}{1-(1-r)^p}\right\}^{1/p}$.

Corollary 3.6. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $e^\alpha(p)$ and*

$$\|B^{r,s}\|_{\ell_p,e^\alpha(p)} = \left(\frac{r+s}{r\alpha}\right)^{1/p} \quad and \quad L(B^{r,s})_{\ell_p,e^\alpha(p)} \ge \frac{1}{[1-(1-\alpha)^p]^{1/p}[1-(\frac{s}{r+s})^p]^{1/p}}.$$

In particular, for $r+s=1$, the Euler operator E^r is a bounded operator from ℓ_p into $e^\alpha(p)$ and $\|E^r\|_{\ell_p,e^\alpha(p)} = (r\alpha)^{-1/p}$ and $L(E^r)_{\ell_p,e^\alpha(p)} \ge \frac{1}{[1-(1-\alpha)^p]^{1/p}[1-(1-r)^p]^{1/p}}$.

Corollary 3.7. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $hol(\alpha,p)$ and*

$$\|B^{r,s}\|_{\ell_p,hol(\alpha,p)} = \left(\frac{r+s}{r}\right)^{1/p} \left(\frac{p}{p-1}\right)^\alpha.$$

In particular, for $r+s=1$ and $\alpha=1$, the Euler operator E^r is a bounded operator from ℓ_p into $ces(p)$ and $\|E^r\|_{\ell_p,ces(p)} = \frac{r^{-1/p}p}{p-1}$.

Corollary 3.8. *The binomial operator $B^{r,s}$ is a bounded operator from ℓ_p into $b^{t,u}(p)$ and*

$$\|B^{r,s}\|_{\ell_p,b^{t,u}(p)} = \frac{(r+s)^{1/p}(t+u)^{1/p}}{(rt)^{1/p}} \quad and \quad L(B^{r,s})_{\ell_p,b^{t,u}(p)} \ge \frac{1}{[1-(\frac{u}{t+u})^p]^{1/p}[1-(\frac{s}{r+s})^p]^{1/p}}.$$

In particular,

- *for* $r+s=1$, *the Euler operator* E^r *is a bounded operator from* ℓ_p *into* $b^{t,u}(p)$ *and* $\|E^r\|_{\ell_p,b^{t,u}(p)} = \left(\frac{t+u}{rt}\right)^{1/p}$ *and* $L(E^r)_{\ell_p,b^{t,u}(p)} \geq \frac{1}{[1-(\frac{u}{t+u})^p]^{1/p}[1-(1-r)^p]^{1/p}}$,
- *for* $t+u=1$, *the binomial operator* $B^{r,s}$ *is a bounded operator from* ℓ_p *into* $e^t(p)$ *and* $\|B^{r,s}\|_{\ell_p,e^t(p)} = \left(\frac{r+s}{rt}\right)^{1/p}$ *and* $L(B^{r,s})_{\ell_p,e^t(p)} \geq \frac{1}{[1-(1-t)^p]^{1/p}[1-(\frac{s}{r+s})^p]^{1/p}}$,
- *for* $r+s=t+u=1$, *the Euler operator* E^r *is a bounded operator from* ℓ_p *into* $e^t(p)$ *and* $\|E^r\|_{\ell_p,e^t(p)} = (rt)^{-1/p}$ *and* $L(E^r)_{\ell_p,e^t(p)} \geq \frac{1}{[1-(1-t)^p]^{1/p}[1-(1-r)^p]^{1/p}}$.

We can also prove our results in the above corollary accorollaryding to Theorem 2.3.

Remark 3.5. *The binomial operator* $B^{r,s}$ *is a bounded operator from* ℓ_p *into* $b^{t,u}(p)$ *and*

$$\begin{aligned}\|B^{r,s}\|_{\ell_p,b^{t,u}(p)} &= \|B^{r,s}B^{t,u}\|_{\ell_p} = \|B^{rt,ru+st+su}\|_{\ell_p} \\ &= \left(\frac{rt+ru+st+su}{rt}\right)^{1/p} = \frac{(r+s)^{1/p}(t+u)^{1/p}}{(rt)^{1/p}}.\end{aligned}$$

In particular,

- *for* $r+s=1$, *the Euler operator* E^r *is a bounded operator from* ℓ_p *into* $b^{t,u}(p)$ *and*

$$\begin{aligned}\|E^r\|_{\ell_p,b^{t,u}(p)} &= \|E^rB^{t,u}\|_{\ell_p} = \|B^{r,1-r}B^{t,u}\|_{\ell_p} = \|B^{rt,ru+(1-r)t+(1-r)u}\|_{\ell_p} \\ &= \|B^{rt,u+t-rt}\|_{\ell_p} = \left(\frac{u+t}{rt}\right)^{1/p},\end{aligned}$$

- *for* $t+u=1$, *the binomial operator* $B^{r,s}$ *is a bounded operator from* ℓ_p *into* $e^t(p)$ *and*

$$\begin{aligned}\|B^{r,s}\|_{\ell_p,e^t(p)} &= \|E^tB^{r,s}\|_{\ell_p} = \|B^{t,1-t}B^{r,s}\|_{\ell_p} \\ &= \|B^{rt,r+s-rt}\|_{\ell_p} = \left(\frac{r+s}{rt}\right)^{1/p},\end{aligned}$$

- *for* $r+s=t+u=1$, *the Euler operator* E^r *is a bounded operator from* ℓ_p *into* $e^t(p)$ *and*

$$\|E^r\|_{\ell_p,e^t(p)} = \|E^rE^t\|_{\ell_p} = \|B^{r,1-r}B^{t,1-t}\|_{\ell_p} = \|B^{rt,1-rt}\|_{\ell_p} = (rt)^{-1/p}.$$

Accorollaryding to Lemma 3.1, for obtaining the bound of the operator T from the sequence space A_p into ℓ_p there is need that we have a factorization for T of the form $T=UA$. The existence of this factorization for the Hausdorff operators is a challenging problem.

Theorem 3.9. *If* $B^{r,s}$ *has a factorization of the form* $B^{r,s}=UH^\mu$, *then the binomial operator* $B^{r,s}$ *is a bounded operator from* $hau(p)$ *into* ℓ_p *and*

$$\|B^{r,s}\|_{hau(p),\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\left(\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta)\right)^{-1}.$$

In particular, for $r+s=1$, *the Euler operator* E^r *is a bounded operator from* $hau(p)$ *into* ℓ_p *and* $\|E^r\|_{hau(p),\ell_p} = r^{\frac{-1}{p}}\left(\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta)\right)^{-1}$.

Proof. Similar to Bennett ([2, p. 120]), if $B^{r,s}$ has a factorization of the form $B^{r,s}=H^\omega H^\mu$, where ω is a quotient measure, then Lemma 3.1 part (ii) and Theorem 1.2 result in

$$\|B^{r,s}\|_{hau(p),\ell_p} = \|H^\omega\|_{\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\left(\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta)\right)^{-1}.$$

□

Corollary 3.9. *The binomial operator $B^{r,s}$ is a bounded operator from $b^{t,u}(p)$ into ℓ_p and*

$$\|B^{r,s}\|_{b^{t,u}(p),\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}\left(\frac{t}{t+u}\right)^{1/p}.$$

In particular,

- *for $r+s=1$, the Euler operator E^r is a bounded operator from $b^{t,u}(p)$ into ℓ_p and $\|E^r\|_{b^{t,u}(p),\ell_p} = \left(\frac{t}{rt+ru}\right)^{1/p}$,*
- *for $t+u=1$, the binomial operator $B^{r,s}$ is a bounded operator from $e^t(p)$ into ℓ_p and $\|B^{r,s}\|_{e^t(p),\ell_p} = \left(\frac{rt+st}{r}\right)^{1/p}$,*
- *for $r+s=t+u=1$, the Euler operator E^r is a bounded operator from $e^t(p)$ into ℓ_p and $\|E^r\|_{e^t(p),\ell_p} = (\frac{t}{r})^{1/p}$.*

Proof. Let the binomial operator $B^{r,s}$ has a factorization of the form $B^{r,s} = UB^{t,u}$. Then, U is

$$U = B^{r,s}(B^{t,u})^{-1} = B^{r,s}B^{1+\frac{u}{t},\frac{-u}{t}} = B^{r+\frac{ru}{t},s-\frac{ru}{t}},$$

hence according to Lemma 3.1

$$\begin{aligned}\|B^{r,s}\|_{b^{t,u}(p),\ell_p} &= \|U\|_{\ell_p} = \|B^{r+\frac{ru}{t},s-\frac{ru}{t}}\|_{\ell_p} \\ &= \left(\frac{r+s}{r}\right)^{1/p}\left(\frac{t}{t+u}\right)^{1/p}.\end{aligned}$$

□

Corollary 3.10. *The binomial operator $B^{r,s}$ is a bounded operator on Hausdorff sequence space $hau(p)$ and*

$$\|B^{r,s}\|_{hau(p)} = \left(\frac{r+s}{r}\right)^{1/p} \quad \text{and} \quad L(B^{r,s})_{hau(p)} = \frac{1}{[1-(\frac{s}{r+s})^p]^{1/p}}.$$

In particular, for $r+s=1$, the Euler operator E^r is a bounded operator on $hau(p)$ and $\|E^r\|_{hau(p)} = r^{\frac{-1}{p}}$ and $L(E^r)_{hau(p)} = \frac{1}{[1-(1-r)^p]^{1/p}}$.

Proof. Since Hausdorff operators commute, hence by Lemma 3.1, we have

$$\|B^{r,s}\|_{hau(p)} = \|B^{r,s}\|_{\ell_p} = \left(\frac{r+s}{r}\right)^{1/p}$$

and

$$L(E^r)_{hau(p)} = L(E^r) = \frac{1}{[1-(1-r)^p]^{1/p}}.$$

□

4. Lower bound of the transposed binomial operator on the transposed Hausdorff matrix domains

In this section, we intend to compute the lower bound of the transposed binomial operator $(B^{r,s})^t$ on the transposed Hausdorff sequence space $hau^t(p)$ for $0 < p < 1$. For this reason, we need the following theorem which is an analogy of Hardy's formula.

Theorem 4.10 ([2, Theorem 7.18]). *Fix p, $0 < p < 1$, and let $H^{\mu t}$ be the transposed Hausdorff matrix. Then,*

$$\|H^{\mu t}x\|_{\ell_p} \geq \left(\int_0^1 \theta^{\frac{1-p}{p}}\, d\mu(\theta)\right) \|x\|_{\ell_p}$$

for every sequence x of non-negative terms. The constant is best possible, and there is equality only when $x = 0$ or $p = 1$ or $H = I$.

Theorem 4.11 ([2, Corollary 7.27]). *If $H^{\mu t}$ and $H^{\nu t}$ are two transposed Hausdorff matrices, then the lower bound (on $\ell_p, 0 < p < 1$) of their product is the product of their lower bounds.*

Theorem 4.12. *The transposed binomial operator is a bounded operator from ℓ_p into $hau^t(p)$ and*

$$L((B^{r,s})^t)_{\ell_p, hau^t(p)} = \left(\frac{r}{r+s}\right)^{1/p^*} \int_0^1 \theta^{\frac{1-p}{p}}\, d\mu(\theta).$$

In particular, for $r+s = 1$, the transposed Euler operator E^{rt} is a bounded operator from ℓ_p into $hau^t(p)$ and $L(E^{rt})_{\ell_p, hau^t(p)} = r^{1/p^*} \int_0^1 \theta^{\frac{1-p}{p}}\, d\mu(\theta)$.

Proof. The proof is obvious accorollaryding to the Lemma 3.1 and Theorems 4.11 and 4.10. □

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