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Research Article

On the class of k-quasi-(n,m)-power normal operators

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Abstract

We introduce a family of operators called the family of k-quasi-(n,m)-power normal operators. Such family includes normal, n-normal and (n,m)-power normal operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi-(n,m)-power normal if it satisfies

$$T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0,$$

where k, n and m are natural numbers. Firstly, some basic structural properties of this family of operators are established with the help of special kind of operator matrix representation associated with such family of operators. Secondly, some properties of algebraically k-quasi-(n,m)-power normal operators are discussed. Thirdly, we consider the study of tensor products of k-quasi-(n,m)-power normal operators. A necessary and sufficient condition for $T \otimes S$ to be a k-quasi-(n,m)-power normal is given, when $T \neq 0$ and $S \neq 0$.

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators defined on $\mathcal{H}$. For every $T \in \mathcal{B}(\mathcal{H})$, denote the range, the nullspace and the adjoint of T by $T(\mathcal{H})$ $(= \mathcal{R}(T))$, $\mathcal{N}(T)(= T^{-1}(0))$ and T^*, respectively. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant if $T(\mathcal{M}) \subset \mathcal{M}$ and in addition, if $T^*(\mathcal{M}) \subseteq \mathcal{M}$, then $\mathcal{M}$ is called a reducing subspace for T. The closure of a subset F of $\mathcal{H}$ will be denoted by $\overline{F}$. For any arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we will denote the point spectrum, the approximate spectrum, the spectrum, the surjective spectrum, and the essential spectrum of T by $\sigma_p(T)$, $\sigma_a(T)$, $\sigma(T)$, $\sigma_s(T)$, and $\sigma_e(T)$, respectively.
For any $T \in \mathcal{B}(\mathcal{H})$, set $[T^*, T] = T^*T - TT^*$.
(1) T is normal if $[T,\ T^*] = 0$,
(2) T is paranormal if $\|Tu\|^2 \leq \|T^2u\|\|u\|$ for all $u \in \mathcal{H}$.
(3) T is normaloid if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T [8].

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(4) T belongs to class $\mathcal{A}$ if $|T^2| \geq |T|^2$ [13].
Recently, I.H. Jeon et al. in [11] have extended class $\mathcal{A}$ operators to quasi-class $\mathcal{A}$ operators. An operator $T \in \mathcal{B}(\mathcal{H})$ belongs to quasi-class $\mathcal{A}$ if

$$T^*\Big(|T^2| - |T|^2\Big)T \geq 0.$$

For further generalization, Tanahashi et al. [27] introduced the class of k-quasi-class $\mathcal{A}$ operators. T is said to be a k-quasi-class $\mathcal{A}$ operator if

$$T^{*k}\Big(|T^2| - |T|^2\Big)T^k \geq 0,$$

where k is a positive integer. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi-paranormal operator [24] if

$$\|S^{k+1}u\|^2 \leq \|S^{k+2}u\|\|S^k u\|,$$

for every $u \in \mathcal{H}$, k is a natural number.

A generalization of normal operators to the concept of n-normal operators has been introduced and studied by A.A. Jibril [12] and S.A. Alzuraiqi et al. [3]. An operator T is called n-normal if $T^nT^* = T^*T^n$. Very recently, several papers have appeared on n-normal operators. We refer to [4,5,18] for complete study.
An operator $T \in \mathcal{B}(\mathcal{H})$ is called (n,m)-power normal if $T^nT^{*m} - T^{*m}T^n = 0$ and it is said to be (n,m)-power quasi-normal if $\big(T^nT^{*m} - T^{*m}T^n\big)T = 0$ where n,m be two nonnegative integers. We refer the reader to [1], [2] and [5] for complete details on these families of operators.

Recall that an $T \in \mathcal{B}(\mathcal{H})$ have the single-valued extension property (SVEP) if for every open subset U of $\mathbb{C}$ and any analytic function $f : U \longrightarrow \mathcal{H}$ for which $(T - \lambda)f(\lambda) \equiv 0$ on U, we have $f(\lambda) \equiv 0$ on U. $T \in \mathcal{B}(\mathcal{H})$ has Bishop's property (β) if, for every open subset $\mathbb{D}$ of $\mathbb{C}$ and every sequence $g_n : \mathbb{D} \longrightarrow \mathcal{H}$ of analytic functions with $(T - \mu)g_n(\mu)$ converges uniformly to 0 in norm on compact subsets of $\mathbb{D}$, and $g_n(\mu)$ converges uniformly to 0 in norm on compact subsets of $\mathbb{D}$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called scalar of order m s.t. $0 \leq m \leq \infty$ if there exists a continuous unital homomorphism of topological algebra $\psi : C_0^m(\mathbb{C}) \longrightarrow \mathcal{B}(\mathcal{H})$ for which $\psi(z) = T$, where $C_0^m(\mathbb{C})$ is the Fréchet space of all continuously differentiable functions of order m with compact support. T is subscalar of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace. An operator $T \in \mathcal{B}(\mathcal{H})$ is called algebraic if there is a nonconstant polynomial $Q \in \mathbb{C}[z]$ for which $Q(T) = 0$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be isoloid [4] if every isolated point of $\sigma(T)$ belongs to the point spectrum of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is called polaroid [7] if $\pi(T) = \{\mu \in \operatorname{iso}\sigma(T)\}$, where $\operatorname{iso}\sigma(T)$ is the set of isolated points of the spectrum of T and $\pi(T)$ is the set of poles of the resolvent of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is quasinilpotent if $\sigma(T) = \{0\}$.

This paper is devoted to some class of operators on the Hilbert space which is a generalization of normal, n-normal and (n,m)-power normal operators. More precisely, we introduce a new class of operators which is called the class of k-quasi-(n,m)-power normal operators. It is proved in Example 2.4 that there is an operator which is k-quasi-(n,m)-power normal, but not (n,m)-power normal for some positive integers n,m and k, and thus, the proposed new class of operators contains the class of (n,m)-power normal operators as a proper subset. In Section 2 we characterize this class of operators in terms of (n,m)-power normal operators on the subspace $\overline{\mathcal{R}(T^k)}$ (Lemma 2.5). Other characterizations are given in Propositions 2.6, 2.7, 2.11 and Theorem 2.13. In Section 3, we study algebraically k-quasi-(n,m)-power normal operators. Using the operator matrix representation of k-quasi-(n,m)-power normal operators which is related to the (n,m)-power normal operators, we prove that in many cases the algebraically k-quasi-(n,m)-power normal operators are very close (even are equal in some cases) to power-scalar, nilpotent, polaroid, operator with Bishop's property (β) and the so-called SVEP operator. Section 4 is devoted to the tensor product for k-quasi-(n,m)-power normal operators, and some new results are obtained.

2. k-quasi-(n,m)-power normal operators

In this section, the family of k-quasi-(n,m)-power normal operators is introduced. In addition, we study several properties of members from this family of operators.

Definition 2.1. We say that $T \in \mathcal{B}(\mathcal{H})$ is k-quasi-(n,m)-power normal if

$$T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0. \tag{2.1}$$

for some positive integers k, n and m.

Remark 2.2. (1) A 1-quasi-(n,m)-power normal operator is a quasi-(n,m)-power normal operator.
(2) If $n = m = k = 1$, then (2.1) coincides with $\big(T^*T\big)^2 = T^{*2}T^2$, i.e., T is a class (Q) operator.
(3) Every k-quasi-(n,m)-power normal operator is a $(k+1)$-quasi-(n,m)-power normal operator.

Remark 2.3. We give an example which is k-quasi-(n,m)-power normal, but not (n,m)-power normal.

Example 2.4. Let $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. Then T is k-quasi-$(2,1)$-power normal, but not $(2,1)$-power normal.

In fact, we have $T^2T^* - T^*T^2 \neq 0$. Hence T is not $(2,1)$-power normal. However $T^{*k}(T^2T^* - T^*T^2)T^k = 0$ for $k = 1, 2, \cdots$. Therefore T is a k-quasi-$(2,1)$-power normal.

Lemma 2.5. *$T \in \mathcal{B}(\mathcal{H})$ is k-quasi-(n,m)-power normal if and only if it is (n,m)-power normal on $\overline{\mathcal{R}(T^k)}$.*

Proof.

$$\begin{aligned}
T \text{ is a } k\text{–quasi–}(n,m)\text{–power normal} \quad &\Leftrightarrow \quad T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0 \\
&\Leftrightarrow \quad \langle T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^kx \mid x\rangle = 0 \ , \forall\, x \in \mathcal{H} \\
&\Leftrightarrow \quad \langle \Big(T^nT^{*m} - T^{*m}T^n\Big)T^kx \mid T^kx\rangle = 0, \quad \forall\, x \in \mathcal{H} \\
&\Leftrightarrow \quad T^nT^{*m} - T^{*m}T^n = 0, \ \text{ on } \ \overline{T^k(\mathcal{H})}.
\end{aligned}$$

$\square$

Proposition 2.6. *If $T \in \mathcal{B}(\mathcal{H})$, then T is k-quasi-(n,m)-power normal if and only if T is k-quasi-(m,n)-power normal.*

Proof. This assertion is obvious. We omit this proof. $\square$

Proposition 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi-(n,m)-power normal. If $\mathcal{N}(T^{*k}) \subset \mathcal{N}(T)$, then T^* is a k-quasi- (m,n)-power normal.*

Proof. As T is a k-quasi-(n,m)-power normal,

$$T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0.$$

Under the assumption $\mathcal{N}(T^{*k}) \subset \mathcal{N}(T)$, we obtain

$$T\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0,$$

and hence

$$T^{*k}\Big(T^mT^{*n} - T^{*n}T^m\Big)T^* = 0.$$

So,

$$T\Big(T^mT^{*n} - T^{*n}T^m\Big)T^* = 0,$$

which implies

$$T^k\Big(T^mT^{*n} - T^{*n}T^m\Big)T^{*k} = 0.$$

Thus T^* is a k-quasi-(m,n)-power normal. □

In the following theorem, we give a sufficient condition for a k-quasi-(n,m)-power normal operator to be a q-quasi-(n,m)-power normal.

Theorem 2.8. *Let T be a k-quasi-(n,m)-power normal operator. If $\mathcal{N}(T^{*q}) = \mathcal{N}(T^{*(q+1)})$ for some $1 \leq q \leq k-1$, then T is a q-quasi-(n,m)-power normal.*

Proof. Under the assumption that $\mathcal{N}(T^{*q}) = \mathcal{N}(T^{*(q+1)})$ we have $\mathcal{N}(T^{*q}) = \mathcal{N}(T^{*k})\ \ \forall\, k \in \mathbb{N},\ \ k \geq 2$. From the hypothesis, T is a k-quasi-(n,m)-power normal, then

$$T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0.$$

Since $\mathcal{N}(T^{*q}) = \mathcal{N}(T^{*k})$, then a direct computation shows that

$$T^{*q}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^q = 0,$$

which implies that T is a q-quasi-(n,m)-power normal as required. □

Remark 2.9. In the following example, we show that Theorem 2.8 is not true in general if $\mathcal{N}(T^{*q}) \neq \mathcal{N}(T^{*(q+1)})$.

Example 2.10. Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on the two dimensional Hilbert space $\mathbb{C}^2$. Then a direct calculation shows that T is a 2-quasi-$(1,1)$-power normal but it is not a quasi-$(1,1)$-power normal. However $\mathcal{N}(T^*) \neq \mathcal{N}(T^{*2})$.

Proposition 2.11. *Let T be a k-quasi-(n,m)-normal operator and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which reduces T. Then $T|\mathcal{M}$ is a k-quasi-(n,m)-power normal.*

Proof. Under the assumption that $\mathcal{M}$ is a reducing subspace of T, then

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

From the fact that T is k-quasi-(n,m)-power normal, we have

$$T^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^k = 0.$$

Hence

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}^{*k} \left\{ \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}^{n} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}^{*m} - \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}^{*m} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}^{n} \right\} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}^{k} = 0.$$

Therefore

$$\begin{pmatrix} T_1^{*k}(T_1^n T_1^{*m} - T_1^{*m}T_1^n)T_1^k & 0 \\ 0 & V \end{pmatrix} = 0,$$

for some operator V. This means that

$$T_1^{*k}(T_1^n T_1^{*m} - T_1^{*m}T_1^n)T_1^k = 0.$$

Consequently, $T_1 = T|\mathcal{M}$ is k-quasi-(n,m)-power normal. □

Proposition 2.12 ([18]). *Let $T \in \mathcal{B}(\mathcal{H})$ be n-power normal. The following assertions hold.*

(1) *$\overline{T^n(\mathcal{H})}$ reduces T.*

(2) *T has the following matrix representation*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T^n(\mathcal{H})} \oplus \mathcal{N}(T^{*n}),$$

where $T_1 = T|\overline{T^n(\mathcal{H})}$ is also n-power normal, T_2 is nilpotent. Furthermore $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Now we give an equivalent condition for T to be k-quasi-(n,m)-power normal operator. Using this result we obtained several important properties of this class of operators.

Theorem 2.13. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\overline{T^k(\mathcal{H})} \neq \mathcal{H}$. The following properties are equivalent.*

(1) *T is a k-quasi-(n,m)-power normal.*

(2) *$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k})$, where T_1 is an (n,m)-power normal operator and $T_3^k = 0$. Therefore $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. Since $\overline{T^k(\mathcal{H})} \subsetneqq \mathcal{H}$ is an closed invariant subspace of T, T can be written as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, relative to $\mathcal{H} = \overline{T^k(\mathcal{R})} \oplus \mathcal{N}(T^{*k})$.

Assume that T is a k-quasi-(n,m)-power normal operator and let $P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$ be the projection onto $\overline{T^k(\mathcal{H})}$, where $I_1 = I|\overline{T^k(\mathcal{H})}$. It follows that

$$PT^{*k}\Big(T^nT^{*m} - T^{*m}T^n\Big)T^kP = 0$$

and so that

$$T_1^nT_1^{*m} - T_1^{*m}T_1^n = 0.$$

Hence T_1 is an (n,m)-power normal operator.
On the other hand, let $u = u_1 + u_2 \in \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k})$. A simple computation shows that

$$\begin{aligned} \langle T_3^k u_2, u_2 \rangle &= \langle T^k(I-P)u, (I-P)u \rangle \\ &= \langle (I-P)u, T^{*k}(I-P)u \rangle = 0. \end{aligned}$$

So, $T_3^k = 0$.

In view of [16, Corollary 7], it follows that $\sigma(T)\cup\mathcal{V}=\sigma(T_1)\cup\sigma(T_3)$, where $\mathcal{V}$ is the union of certain of the holes in $\sigma(T)$ which is a subset of $\sigma(T_1)\cap\sigma(T_3)$. Further $\sigma(T_3)=\{0\}$ and $\sigma(T_1)\cap\sigma(T_3)$ has no interior points. So we have by [16, Corollary 8]

$$\sigma(T)=\sigma(T_1)\cup\sigma(T_3)=\sigma(T_1)\cup\{0\}.$$

(2) $\Rightarrow$ (1) Suppose that $T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}$ onto $\mathcal{H}=\overline{\mathcal{R}(T^k)}\oplus\mathcal{N}(T^{*k})$, such that

$$\Delta_n^m(T):=T_1^nT_1^{*m}-T_1^{*m}T_1^n=0 \ \text{ and } \ T_3^k=0.$$

Since $T^k=\begin{pmatrix} T_1^k & \sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\\ 0 & T_3^k\end{pmatrix}$ we have

$$T^{*k}\Big(T^nT^{*m}-T^{*m}T^n\Big)T^k$$

$$=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}^{*k}\left\{\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}^{n}\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}^{*m}-\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}^{*m}\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}^{n}\right\}\times\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}^{k}$$

$$=\begin{pmatrix} T_1^{*k} & 0\\ \sum\limits_{j=0}^{k-1}T_3^{*k-1-j}T_2^*T_1^{*j} & T_3^{*k}\end{pmatrix}\times\left\{\begin{pmatrix} T_1^n & \sum\limits_{j=0}^{n-1}T_1^jT_2T_3^{n-1-j}\\ 0 & T_3^n\end{pmatrix}\begin{pmatrix} T_1^{*m} & 0\\ \sum\limits_{j=0}^{m-1}T_3^{*m-1-j}T_2^*T_1^{*j} & T_3^{*m}\end{pmatrix}\right.$$

$$\left.-\begin{pmatrix} T_1^{*m} & 0\\ \sum\limits_{j=0}^{m-1}T_3^{*m-1-j}T_2^*T_1^{*j} & T_3^{*m}\end{pmatrix}\begin{pmatrix} T_1^n & \sum\limits_{j=0}^{n-1}T_1^jT_2T_3^{n-1-j}\\ 0 & T_3^n\end{pmatrix}\right\}\times\begin{pmatrix} T_1^k & \sum\limits_{j=1}^{n-1}T_1^jT_2T_3^{k-1-j}\\ 0 & T_3^k\end{pmatrix}$$

$$=\begin{pmatrix} T_1^{*k} & 0\\ \sum\limits_{j=1}^{k-1}T_3^{*k-1-j}T_2^*T_1^{*j} & 0\end{pmatrix}\times\left\{\begin{pmatrix} \Delta_n^m(T_1) & C\\ D & B\end{pmatrix}\right\}\times\begin{pmatrix} T_1^k & \sum\limits_{j=1}^{k-1}T_1^jT_2T_3^{k-1-j}\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix} T_1^{*k}\Delta_n^m(T_1)T_1^k & T_1^{*k}\Delta_n^m(T_1)\sum\limits_{j=1}^{n-1}T_1^jT_2T_3^{n-1-j}\\ \Big(\sum\limits_{j=1}^{n-1}T_1^jT_2T_3^{n-1-j}\Big)^*\Delta_n^m(T_1)T_1^k & \Big(\sum\limits_{j=1}^{k-1}T_1^jT_2T_3^{k-1-j}\Big)^*\Delta_n^m(T_1)\Big(\sum\limits_{j=1}^{k-1}T_1^jT_2T_3^{k-1-j}\Big)\end{pmatrix}.$$

The condition $\Delta_n^m(T_1)=0$ implies that $T^{*k}\Big(T^nT^{*m}-T^{*m}T^n\Big)T^k=0$. Hence T is a k-quasi-(n,m)-power normal. □

Proposition 2.14. *Let $T\in\mathcal{B}(\mathcal{H})$. The following properties hold.*

(1) *If T is k-quasi-(n,m)-power normal and k-quasi-$(n+1,m)$-power normal, then T is k-quasi-$(n+2,m)$-power normal.*

(2) *If T is k-quasi-(n,m)-power normal and k-quasi-$(n,m+1)$-power normal, then T is k-quasi-$(n,m+2)$-power normal.*

Proof. (1) If $\overline{T^k(\mathcal{H})}=\mathcal{H}$, then by Lemma 2.5, T is (n,m)-power normal and $(n+1,m)$-power normal. From [1, Proposition 1.11] we deduce that T is $(n+2,m)$-power normal. So, T is a k-quasi-$(n+2,m)$-power normal operator.

If $\overline{T^k(\mathcal{H})}\neq\mathcal{H}$, we can write $T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}$, relative to $\mathcal{H}=\overline{T^k(\mathcal{H})}\oplus\mathcal{N}(T^{*k})$, where $T_1=T|\overline{T^k(\mathcal{H})}$ is both (n,m)- and $(n+1,m)$-power normal. Moreover $T_3^k=0$. In view of [1, Proposition 1.11], T_1 is $(n+2,m)$-power normal. By applying Theorem 2.13, we deduce that T is k-quasi-$(n+2,m)$-power normal.

(2) This proof is similar to the statement (1). So, we omit this proof. $\square$

Corollary 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi-(n,m)-power normal operator. Then T^j is k-quasi- $(1,1)$-power normal, where j is the least common multiple (LCM) of n and m.*

Proof. If $T^k(\mathcal{H})$ is dense then T is an (n,m)-power normal operator and therefore T^j, $j = \mathrm{LCM}(n,m)$, is normal by [4, Lemma 4.2]. Since T^j is normal it is $(1,1)$- power normal and hence k-quasi-$(1,1)$-power normal. Now assume that $T^k(\mathcal{H})$ is not dense, by Theorem 2.13 we have $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k})$, where $T_1 = T|\overline{T^k(\mathcal{H})}$ is an (n,m)-power normal operator and $T_3^k = 0$. We notice that

$$T^j = \begin{pmatrix} T_1^j & \sum\limits_{r=0}^{j-1} T_1^r T_2 T^{j-1-r} \\ 0 & T_3^j \end{pmatrix},$$

where T_1^j is a normal operator ([4, Lemma 4.2]) and $(T_3^j)^k = 0$. Hence T^j is an k-quasi-$(1,1)$-normal operator by Theorem 2.13. $\square$

Recall that two operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ are said to be similar if there exists an operator $Z \in \mathcal{B}(\mathcal{H})$ which is invertible such that $ZT = SZ$, i.e, $T = Z^{-1}SZ$ or $S = ZTZ^{-1}$.

Corollary 2.16. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi-(n,m)-power normal operator such that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$. If T_1 is invertible, then T is similar to a direct sum of an (n,m)-power normal and a nilpotent operator.*

Proof. Since $0 \notin \sigma(T_1)$ and T_3 is nilpotent, then $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. Then from [22] there exists an operator S satisfying $T_1S - ST_3 = T_2$. Hence

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

Consequently, the desired result follows from Theorem 2.13. $\square$

Proposition 2.17. *Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$(2,m)$-power normal and k-quasi-$(3,m)$-power normal for some $m \in \mathbb{N}$, then T is k-quasi-(n,m)-power normal for all $n \geq 4$.*

Proof. Indeed, under the assumptions that T is a $(2,m)$-power normal and a k-quasi-$(3,m)$-power normal operator, we have the following two cases.

If $\overline{T^k(\mathcal{H})} = \mathcal{H}$, then T is a $(2,m)$-power normal and $(3,m)$-power normal and hence T is (n,m)-power normal by [25, Proposition 2.4]. If $\overline{T^k(\mathcal{H})} \neq \mathcal{H}$, then T on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k})$ may be written as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is $(2,m)$-power normal and $(3,m)$-power normal. Hence T is (n,m)-power normal by [25, Proposition 2.4]. Moreover $T_3^k = 0$. Consequently, T is k-quasi-(n,m)-power normal by Theorem 2.13. $\square$

Proposition 2.18. *Let T be k-quasi-$(n,2)$-power normal and k-quasi-$(n,3)$-power normal for some $n \in \mathbb{N}$. Then T is k-quasi-(n,m)-power normal for all integer $m \geq 4$.*

Proof. We omit the proof since the techniques are similar to the proof of Proposition 2.17. $\square$

Theorem 2.19. *Let $T \in \mathcal{B}(\mathcal{H})$ such that T^k does not have dense range. Assume that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is k-quasi-(n, m)-power normal for some positive integers n and m such that $n \geq m$. If T_1^m is a partial isometry, then T is k-quasi-$(n + m, m)$-power normal.*

Proof. Since T is k-quasi-$(n.m)$-power normal, it follows from Theorem 2.13 that $T_1 = T|\overline{T^k(\mathcal{H})}$ is (m, n)-power normal and $T_3^k = 0$. The assumption that T_1^m is a partial isometry implies that T_1 is $(n+m, m)$-power normal operator by [25, Theorem 2.4]. Hence, by Theorem 2.13, T is k-quasi-$(n + m, m)$-power normal. □

Theorem 2.20. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ be commuting k-quasi-(n, m)-power normal operators, then TS is k-quasi-(j, p)-power normal for every $p \in \mathbb{N}$, where j is the least common multiple of n and m.*

Proof. (i) If $\overline{T^k(\mathcal{H})} = \mathcal{H} = \overline{S^k(\mathcal{H})}$, then T and S are (n, m)-power normal operators. It follows from [4, Theorem 4.4] that TS is (j, p)-power normal for every $p \in \mathbb{N}$, where j is the least common multiple of n and m. Hence, TS is k-quasi-(j, p)-power normal.

(ii) If $\overline{T^k(\mathcal{H})} \neq \mathcal{H} \neq \overline{S^k(\mathcal{H})}$, in view of Theorem 2.13, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k})$$

with $T_1 = T|\overline{T^k(\mathcal{H})}$ is (n, m)-power normal and $T_3^k = 0$.
Similarly,

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{S^k(\mathcal{H})} \oplus \mathcal{N}(S^{*k})$$

where $S_1 = S|\overline{S^k(\mathcal{H})}$ is (n, m)-power normal and $S_3^k = 0$.
By observing that $T_1|\overline{(ST)^k(\mathcal{H})}$ and $S_1|\overline{(ST)^k(\mathcal{H})}$ are (n, m)-power normal operators, it follows from [4, Theorem 4.4] that T_1S_1 is (j, p)-power normal. Moreover, $(T_3S_3)^k = 0$. Hence we have for the decomposition

$$TS = \begin{pmatrix} T_1S_1 & * \\ 0 & T_3S_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{(TS)^k(\mathcal{H})} \oplus \mathcal{N}((TS)^{*k}),$$

$T_1S_1|\overline{(ST)^k(\mathcal{H})}$ is (j, p)-power normal and $(T_3S_3)^k = 0$. Therefore TS is a k-quasi-(j, p)-power normal operator by Theorem 2.13.

(iii) If $\overline{T^k(\mathcal{H})} = \mathcal{H} \neq \overline{S^k(\mathcal{H})}$, we can write

$$T = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{S^k(\mathcal{H})} \oplus \mathcal{N}(S^{*k}),$$

where T is (n, m)-power normal and S_1 is (n, m)-power normal on $\overline{S^k(\mathcal{H})}$. Clearly, $TS = \begin{pmatrix} TS_1 & TS_2 \\ 0 & 0 \end{pmatrix}$ and moreover TS_1 is (n, m)-power normal. By a similar argument as in (ii) we can see that TS is k-quasi-(n, m)-power normal.

(iv) We omit the case when $\overline{T^k(\mathcal{H})} \neq \mathcal{H} = \overline{S^k(\mathcal{H})}$ because the proof is similar to the one given in (iii) since $ST = TS$. □

Theorem 2.21. *If $T \in \mathcal{B}(\mathcal{H})$ is k-quasi-(n, m)-power normal, then T has Bishop's property (β).*

Proof. We consider two cases:

(1) If $T^k(\mathcal{H})$ is dense, then T is an (n,m)-power normal operator and hence T is decomposable (by [4, Lemma 4.2]. So, T has Bishop's property (β).

(2) If $T^k(\mathcal{H})$ is not dense, by Theorem 2.13, we write T on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k})$ as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

with T_1 is an (n,m)-power normal operator on $\overline{T^k(\mathcal{H})}$ and $T_3^k = 0$.
Let $g_k(\mu)$ be analytic on $\mathbb{D} \subseteq \mathbb{C}$ with $(T-\mu)g_k(\mu) \to 0$ uniformly on each compact K of $\mathbb{D}$. Then

$$\begin{pmatrix} T_1 - \mu & T_2 \\ 0 & T_3 - \mu \end{pmatrix} \begin{pmatrix} g_{k_1}(\mu) \\ g_{k_2}(\mu) \end{pmatrix} = \begin{pmatrix} (T_1-\mu)g_{k_1}(\mu) + T_2 g_{k_2}(\mu) \\ (T_3-\mu)g_{k_2}(\mu) \end{pmatrix} \to 0.$$

Since T_3 is nilpotent, T_3 satisfies Bishop's property (β). Thus, $g_{k_2}(\mu) \to 0$ uniformly on each compact K of $\mathbb{D}$. Therefore, $(T_1-\mu)g_{k_1}(\mu) \to 0$ as $k_1 \to \infty$. Since T_1 satisfies Bishop's property, it follows that $g_{k_1}(\mu) \to 0$ and so T has Bishop's property (β) as required. □

Corollary 2.22. *Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-(n,m)-power normal, then T has SVEP.*

In [18], it was proved that if T is quasinilpotent n-normal operator, then T is nilpotent and in [4] it was proved that a quasinilpotent (n,m)-normal operator is nilpotent. In the following theorem, we extend this result to k-quasi-(n,m)-power normal operators.

Theorem 2.23. *If $T \in \mathcal{B}(\mathcal{H})$ is k-quasi-(n,m)-power normal and quasinilpotent, then T is nilpotent, and hence subscalar.*

Proof. Assume that $\overline{T^k(\mathcal{H})} = \mathcal{H}$, then T is (n,m)-power normal. By [4, Theorem 4.3], T is nilpotent. Therefore T is algebraic and hence T is subscalar by [15]. So we may assume that $\overline{T^k(\mathcal{H})} \neq \mathcal{H}$. Hence by Theorem 2.13, we write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k}),$$

with T_1 is an (n,m)-power normal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.
Since $\sigma(T_1) \neq \emptyset$ and $\sigma(T) = \{0\}$, we see that $\sigma(T_1) = \{0\}$. Therefore T_1 is quasinilpotent (n,m)-power normal. Hence T_1 is nilpotent. Then $T_1^q = 0$, for some positive integer q. An easy computation yields

$$T^{k+q} = T^q T^k \begin{pmatrix} 0 & U \\ 0 & T_3^q \end{pmatrix} \begin{pmatrix} T_1^k & V \\ 0 & 0 \end{pmatrix} = 0.$$

Consequently, T is nilpotent and hence algebraic. So, T is subscalar ([15]). □

Recall that an operator $X \in \mathcal{B}(\mathcal{H})$ satisfying $X^{-1}(0) = \{0\}$ and $\overline{X(\mathcal{H})} = \mathcal{H}$ is called quasiaffinity. Let $T, S \in \mathcal{B}(\mathcal{H})$. S is said to be a quasiaffine transform of T if there is a quasiaffinity X such that $XS = TX$. Furthermore, S and T are quasisimilar if there are quasi-affinities X and Y such that $XS = TX$ and $SY = YT$.

Theorem 2.24. *Let $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. If A is a surjective (n,m)-power normal operator and $C^k = 0$ for some integer $k \in \mathbb{N}$, then T is similar to a k-quasi-(n,m)-power normal operator.*

Proof. Under the conditions $A(\mathcal{H}) = \mathcal{H}$ and C is nilpotent, we have $\sigma_s(A) \cap \sigma_a(C) = \emptyset$. In view of the statement (c) in [17, Theorem 3.5.1], it follows that there exists some operator $R \in \mathcal{B}(\mathcal{H})$ for which $AR - RC = B$. Since

$$\begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & R \\ 0 & I \end{pmatrix},$$

it easy to see that T is similar to $S = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$.

From the assumptions that A is (n,m)-power normal and $C^k = 0$, we get

$$\begin{aligned} & S^{*k}\Big(S^n S^{*m} - S^{*m} S^n\Big) S^k \\ &= \begin{pmatrix} A^{*k} & 0 \\ 0 & C^{*k} \end{pmatrix} \left\{ \begin{pmatrix} A^n & 0 \\ 0 & C^n \end{pmatrix} \begin{pmatrix} A^{*m} & 0 \\ 0 & C^{*m} \end{pmatrix} - \begin{pmatrix} A^{*m} & 0 \\ 0 & C^{*m} \end{pmatrix} \begin{pmatrix} A^n & 0 \\ 0 & C^n \end{pmatrix} \right\} \\ &\quad \times \begin{pmatrix} A^k & 0 \\ 0 & C^k \end{pmatrix} \\ &= \begin{pmatrix} A^{*k}(A^n A^{*m} - A^{*m} A^n) A^k & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Thus T is similar to a k-quasi-(n,m)-power normal operator. □

Question 2.25. If A, B and $C \in \mathcal{B}(\mathcal{H})$ are such that A is an (n,m)-power normal operator and $C^k = 0$ for some $k \in \mathbb{N}$, then is the operator matrix $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ a k-quasi-(n,m)-power normal operator?

The following example gives a negative answer to the Question 2.25.

Example 2.26. Let $T = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Obviously, $A = I$ is (n,m)-power normal and $C^k = 0^k = 0$. On the other hand, we observe that $T^q = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$ for all $q \geq 1$ and easy calculation shows that

$$T^{*k}\Big(T^n T^{*m} - T^{*m} T^n\Big) T^k = \begin{pmatrix} I & I \\ I & I \end{pmatrix} \neq 0.$$

Therefore, T is not a k-quasi-(n,m)-power normal operator for all positive integers n, m and k.

It was observed in [6, Lemma 4.1] that quasisimilar normal operators are unitarily equivalent. Therefore quasisimilar normal operators have equal spectra and essential spectra. The following theorem extended these properties to k-quasi-(n,m)-power normal operators.

Theorem 2.27. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ be quasisimilar k-quasi-(n,m)-power normal operators, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$.*

Proof. In view of Theorem 2.21, we have that T and S satisfy the Bishop's property (β). The proof follows from [21]. □

Definition 2.28 ([18])**.** Let $T \in \mathcal{B}(\mathcal{H})$.
(1) The ascent of T is the smallest nonnegative integer $p = p(T)$ such that $\mathcal{N}(T^p) = \mathcal{N}(T^{p+1})$. If such integer does not exist, then we put $p(T) = \infty$.
(2) The descent of T is defined as the smallest nonnegative integer $q = q(T)$ such that $T^q(\mathcal{H}) = T^{q+1}(\mathcal{H})$. If such integer does not exist, then we put $q(T) = \infty$.

If $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ by [9, Proposition 38.6].

Recall that for $\mu \in \operatorname{iso}\sigma(T)$, the Riesz idempotent (spectral projection) P_μ of T relative to μ is given by $P_\mu = \dfrac{1}{2i\pi}\displaystyle\int_{\partial\mathbb{D}} (z - T)^{-1} dz$, where $\mathbb{D}$ is a closed disk with center at μ and radius small such that $\mathbb{D} \cap \sigma(T) = \{\mu\}$.

Theorem 2.29. *Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-(n,m)-power normal for some integers n, m and $k \in \mathbb{N}$. Let $0 \neq \lambda_0 \in \operatorname{iso}\sigma(T)$ and P_{λ_0} the Riesz idempotent for λ_0. Then $\lambda_0 \in \pi(T)$ and*

$$P_{\lambda_0}\mathcal{H} = \mathcal{N}(T - \lambda_0 I).$$

Therefore $\lambda_0 \in \sigma_p(T)$.

Proof. If the range $T^k(\mathcal{H})$ is dense, then T is an (n,m)-power normal operator and the theorem holds by Theorem 3.6 in [5]. Assume that $\overline{T^k(\mathcal{H})} \neq \mathcal{H}$. Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \mathcal{N}(T^{*k}).$$

Theorem 2.13 implies that T_1 is (n,m)-power normal on $\overline{T^k(\mathcal{H})}$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. If $0 \neq \lambda_0 \in \operatorname{iso}\sigma(T)$, then $\lambda_0 \in \operatorname{iso}\sigma(T_1)$. Therefore λ_0 is a simple pole of the resolvent of T_1 and T_1 has the representation

$$T_1 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & S \end{pmatrix} \text{ on } \overline{T^k(\mathcal{H})} = \mathcal{N}(T_1 - \lambda_0 I) \oplus \overline{\mathcal{R}(T_1 - \lambda_0)},$$

where $\lambda_0 \notin \sigma(S)$. Therefore,

$$T - \lambda_0 = \begin{pmatrix} 0 & 0 & T_{21} \\ 0 & S - \lambda_0 & T_{22} \\ 0 & 0 & T_3 - \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{N}(T_1 - \lambda_0 I) \oplus \overline{\mathcal{R}(T_1 - \lambda_0)} \oplus \mathcal{N}(T^{*k}),$$

where $B = \begin{pmatrix} S - \lambda_0 & T_{22} \\ 0 & T_3 - \lambda_0 \end{pmatrix}$.

Since B is an invertible operator on $\overline{\mathcal{R}(T_1 - \lambda_0)} \oplus \mathcal{N}(T^{*k})$, a direct calculation shows that $p(T - \lambda_0) = q(T - \lambda_0) = 1$. Thus λ_0 is a simple pole of the resolvent of T. By observing that P_{λ_0} is the Riesz idempotent of T relative to λ_0 we have

$$P_{\lambda_0}\mathcal{H} = \mathcal{N}(T - \lambda_0).$$

□

Following [5, Corollary 4.3], it was observed that if $T \in \mathcal{B}(\mathcal{H})$ is (n,m)-power normal, then T is isoloid and polaroid.

Corollary 2.30. *Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-(n,m)-power normal. If $0 \notin \operatorname{iso}\sigma(T)$, then T is isoloid.*

Proof. Assume that $0 \neq \lambda \in \operatorname{iso}\sigma(T)$. In view of Theorem 2.29, we have

$$P_\lambda\mathcal{H} = \mathcal{N}(T - \lambda) \neq \{0\}, \quad \text{for } P_\lambda \neq 0.$$

Therefore $\lambda \in \sigma_p(T)$. Hence, every nonzero isolated point of T is an eigenvalue of T. Therefore T is isoloid. □

3. Algebraically k-quasi-(n,m)-power normal operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be algebraically (n,m)-power normal if there exists a nonconstant polynomial $Q \in \mathbb{C}[z]$ such that $Q(T)$ is an (n,m)-power normal operator. In general, the following implications hold:

normal $\Rightarrow$ n-normal $\Rightarrow$ (n,m)-power normal $\Rightarrow$ algebraically (n,m)-power normal.

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be (n,m)-power normal and $\mu \in \mathbb{C}$. If $\sigma(T) = \{\mu\}$, then there exists a positive integer j such that $T^j = \mu^j I$.*

Proof. We consider two cases:

(i) $\mu = 0$. Under the assumption that T is (n,m)-power normal, it follows that T^j is normal where j is the least common multiple of n and m. Hence T^j is normaloid. Hence $T^j = 0$.

(ii) $\mu \neq 0$. Obviously, T is invertible, and (n,m)-power normal. So T^{-1} is also (n,m)-power normal. Therefore T^{-j} is normaloid. Moreover, $\sigma(T^{-j}) = \left\{\dfrac{1}{\mu^j}\right\}$. Hence

$$\|T^j\|\|T^{-j}\| = |\mu^j||\frac{1}{\mu^j}| = 1.$$

In view of [19, Lemma 3], we deduce that T^j is convexoid, so $W(T^j) = \{\mu^j\}$, where $W(T^j)$ is the numerical range of T^j. Therefore $T^j = \mu^j I$. □

Lemma 3.2. *If $T \in \mathcal{B}(\mathcal{H})$ is quasinilpotent algebraically (n,m)-power normal, then T is nilpotent.*

Proof. Suppose that $Q(T)$ is (n,m)-power normal for some nonconstant polynomial Q. From the fact that $\sigma(Q(T)) = Q(\sigma(T)) = \{Q(0)\}$, by Lemma 3.1 there exists a positive integer j such that $Q(T)^j - Q(0)^j = 0$. Set $P(z) = Q(z)^j - Q(0)^j$.
We observe that $P(0) = 0$ and $P(T) = 0$. Hence

$$P(T) = aT^k(T-\mu_1)(T-\mu_2)\cdots(T-\mu_r) = Q(T)^j - Q(0)^j = 0 \text{ (where } k \geq 1).$$

By observing that $T - \mu_s$ is invertible for each $\mu_s \neq 0$, we must have $T^k = 0$. □

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically (n,m)-power normal operator. If $T - \mu$ is an algebraically (n,m)-power normal operator for $\mu \in \operatorname{iso}\sigma(T)$, then T is isoloid.*

Proof. Assume that $\mu \in \operatorname{iso}\sigma(T)$ and consider $P_\mu := \dfrac{1}{2i\pi}\displaystyle\int_{\partial\mathbb{D}(\mu,r)^-} (\lambda - T)^{-1} d\lambda$ the Riesz idempotent of T associated to μ, where $\mathcal{D}(\mu,r)^-$ is a closed disk centered at μ which contains no other point of $\sigma(T)$. Then, T can be written as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ with } \sigma(T_1) = \{\mu\} \text{ and } \sigma(T_2) = \sigma(T) - \{\mu\}.$$

By the assumption that T is algebraically (n,m)-power normal operator, it follows that there exists a nonconstant polynomial Q for which $Q(T)$ is (n,m)-power normal. From the equality $\sigma(T_1) = \{\mu\}$, we have

$$\sigma(Q(T_1)) = Q(\sigma(T_1)) = \{Q(\mu)\}.$$

Hence $Q(T_1) - Q(\mu)$ is quasinilpotent. Since $Q(T_1)$ is (n,m)-power normal, it follows from Lemma 3.1 that there exists a positive integer j for which

$$Q(T_1)^j - Q(\mu)^j = 0.$$

Put $q(z) := Q(z)^j - Q(\mu)^j$. Then $q(T_1) = 0$, and so T_1 is algebraically (n,m)-power normal. By observing that $T_1 - \mu$ is quasinilpotent and algebraically (n,m)-power normal, we have from Lemma 3.2 that $T_1 - \mu$ is nilpotent. Consequently, $\mu \in \pi(T_1)$, and hence $\mu \in \pi(T)$. This means that T is isoloid. □

An operator $T \in \mathcal{B}(\mathcal{H})$ is called algebraically k-quasi-(n,m)-power normal, if there exists a nonconstant polynomial $Q \in \mathbb{C}[z]$ such that $Q(T)$ is k-quasi-(n,m)-power normal.

Proposition 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasinilpotent algebraically k-quasi-(n,m)-power normal. Then T is nilpotent.*

Proof. Since T is an algebraically k-quasi-(n,m)-power normal operator, there exists a nonconstant polynomial Q for which $Q(T)$ is k-quasi-(n,m)-power normal. If $\big(Q(T)\big)^k$ has dense range, then $Q(T)$ is an (n,m)-power normal operator. Hence T is algebraically (n,m)-power normal and it follows from Lemma 3.2 that T is nilpotent. So we assume that $\mathcal{R}\big(Q(T)\big)^k$ is not dense. From Theorem 2.13 we can write $Q(T)$ on the upper triangular matrix from

$$Q(T) = \begin{pmatrix} S & R \\ 0 & V \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(Q(T)^k)} \oplus \mathcal{N}(Q(T)^{*k}),$$

where $S := Q(T)|\mathcal{R}(Q(T))^k)$ is an (n,m)-power normal operator and $\sigma(Q(T) = \sigma(S) \cup \{0\}$. Since T is quasinilpotent we have from spectral mapping theorem that

$$\sigma(Q(T)) = Q(\sigma(T)) = \{Q(0)\}.$$

Therefore $Q(T) - Q(0)$ is quasinilpotent. Moreover $\sigma(S) \cup \{0\} = \{Q(0)\}$ implies $Q(0) = 0$. Hence $Q(T)$ is quasinilpotent. However $Q(T)$ is a k-quasi-(n,m)-power normal operator, by Theorem 2.23, $Q(T)$ is nilpotent.
On the other hand, by $Q(0) = 0$, we have

$$Q(z) = a.z^{j_0}(z - \lambda_1)^{j_1} \cdots (z - \lambda_q)^{j_q},$$

where $\lambda_r \neq \lambda_s$ for $r \neq s$. Consequently,

$$0 = \big(Q(T)\big)^p = a^p T^{j_0 p}(T - \lambda_1)^{j_1 p} \cdots (T - \lambda_q)^{j_q p}.$$

Since $\sigma(T) = \{0\}$, $(T - \lambda_1), (T - \lambda_2), \cdots, (T - \lambda_q)$ are invertible, we have $T^{j_0 p} = 0$. Hence T is nilpotent. $\square$

Proposition 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically k-quasi-(n,m)-power normal operator. If $T - \mu$ is an algebraically k-quasi-(n,m)-power normal operator for $\mu \in \operatorname{iso}\sigma(T)$, then T is polaroid and isoloid.*

Proof. Since T is algebraically k-quasi-(n,m)-power normal operator, there exists a nonconstant polynomial Q such that $Q(T)$ is a k-quasi-(n,m)-power normal operator. Let $\mu \in \operatorname{iso}\sigma(T)$ and consider the spectral projection,

$$P_\mu = \frac{1}{2i\pi} \int_{\partial\mathbb{D}(\mu,r)^-} (\lambda - T)^{-1} d\lambda,$$

where $\mathbb{D}(\mu,r)^-$ is a closed disk of center μ such that $\mathbb{D}(\mu,r)^- \cap \sigma(T) = \{\mu\}$, we can write T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, with $\sigma(T_1) = \{\lambda_1\}$ and $\sigma(T_2) = \sigma(T) - \{\mu\}$. We have $Q(T) = \begin{pmatrix} Q(T_1) & 0 \\ 0 & Q(T_2) \end{pmatrix}$ and by the fact that $Q(T)$ is a k-quasi-(n,m)-power normal operator, it follows that $Q(T_1)$ is a k-quasi-(n,m)-power normal operator, i.e., T_1 is an algebraically k-quasi-(n,m)-power normal operator, so is $T_1 - \mu$. Since $\sigma(T_1 - \mu) = \{0\}$, it follows that $T_1 - \mu$ is quasinilpotent and hence nilpotent (from Proposition 3.4). This means that $T_1 - \mu$ has finite ascent and descent.

Since $T_2 - \mu$ is invertible, clearly it has finite ascent and descent. Hence $T - \mu$ has finite ascent and descent, and hence μ is a pole of the resolvent of T. Thus $\mu \in \operatorname{iso}\sigma(T) \Longrightarrow \mu \in \pi(T)$, and so $\operatorname{iso}\sigma(T) \subset \sigma(T)$. Hence T is polaroid. $\square$

Proposition 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be an algebraically k-quasi-(n,m)-power normal operator. Then T has Bishop's property (β).*

Proof. Since T is algebraically k-quasi-(n,m)-power normal, $Q(T)$ is k-quasi-(n,m)-power normal for some nonconstant polynomial Q, and so it follows from Theorem 2.21 that $Q(T)$ has Bishop's property (β). Therefore T has Bishop's property (β) from [17, Theorem 3.3.9]. $\square$

Corollary 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be algebraically k-quasi-(n,m)-power normal. Then T has SVEP.*

4. Tensor product for k-quasi-(n,m)-power normal operators

Given $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ with $T \neq 0$ and $S \neq 0$, let $T \otimes S$ be the tensor product on T and S. It is known that $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. In [26] it was shown that the tensor product of T and S is normal if and only if T and S are normal and in [23] it was observed that the tensor product of two normaloid operators is normaloid. However there exist paranormal operators such that their tensor product is not paranormal (see [8]). The study of tensor products of members of the class $\mathcal{A}$, class $\mathcal{A}(k)$, and $*$-class $\mathcal{A}$ operators was considered in [10, 11, 14]. Panayappan et al. [20] proved that $T, S \in A_k$ if and only if $T \otimes S \in A_k$ operators.

In this section, we prove an analogues property for k-quasi-(n,m)-power normal operators.

Tensor product of two non-zero operators T and S satisfies the following identities:

(1) $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$.

(2) $(T \otimes S)^k = T^k \otimes S^k,\ k \in \mathbb{N}$.

Proposition 4.1 ([26, Proposition 2.1])**.** *Let $A_j \in \mathcal{B}(\mathcal{H})$ and $B_j \in \mathcal{B}(\mathcal{H})$ for $j = 1, 2$ are nonzero operators, then $A_1 \otimes B_1 = A_2 \otimes B_2$ if and only if there exists $c \in \mathbb{C}\backslash\{0\}$ such that $A_2 = cA_1$ and $B_2 = c^{-1}B_1$.*

Theorem 4.2. *Let $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ such that $S, T \neq 0$. Then $T \otimes S$ is k-quasi-(n,m)-power normal if and only if one of the following conditions holds:*

(i) *T and S are k-quasi-(n,m)-power normal operators.*

(ii) *There exists a constant $c \in \mathbb{C}\backslash\{0\}$ such that*

$$\begin{cases} T^{*k}T^nT^{*m}T^k = cT^{*k+m}T^{k+m}, \\[2mm] S^{*k}S^nS^{*m}S^k = \frac{1}{c}S^{*k+m}S^{k+m}. \end{cases}$$

Proof. A direct calculation shows that

$$\begin{aligned} &(T\otimes S)^{*k}\Big[(T\otimes S)^n(T\otimes S)^{*m} - (T\otimes S)^{*m}(T\otimes S)^n\Big](T\otimes S)^k \\ = \ & T^{*k}T^nT^{*m}T^k \otimes S^{*k}S^nS^{*m}S^k - T^{*k}T^{*m}T^nT^k \otimes S^{*k}S^{*m}S^nS^k. \end{aligned}$$

Hence, if either (i) or (ii) hold, clearly $T \otimes S$ is k-quasi-(n,m)-power normal.

Conversely, assume that $T \otimes S$ is a k-quasi-(n,m)-power normal operator. From the above equality

$$\begin{aligned} T^{*k}T^nT^{*m}T^k \otimes S^{*k}S^nS^{*m}S^k &= T^{*k}T^{*m}T^nT^k \otimes S^{*k}S^{*m}S^nS^k \\ &= T^{*k+m}T^{k+m}S^{*k+m}S^{k+m}. \end{aligned}$$

In view of Proposition 4.1 there is a constant $c \neq 0$ for which

$$\begin{cases} T^{*k}T^nT^{*m}T^k = cT^{*k+m}T^{k+m}, \\[2mm] S^{*k}S^nS^{*m}S^k = \frac{1}{c}S^{*k+m}S^{k+m}. \end{cases}$$

If $c = 1$, then T and S are k-quasi-(n,m)-normal operators and if $c \neq 1$, then T and S satisfy the condition (ii). $\square$

Lemma 4.3. *If $T \in \mathcal{B}(\mathcal{H})$, then T is a k-quasi-(n,m)-power normal operator if and only if $T \otimes I$ (or $I \otimes T$) is k-quasi-(n,m)-power normal.*

Theorem 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ be k-quasi-(n,m)-power normal operators. Then $T \otimes S$ is k-quasi-(j,p)-power normal for every $p \in \mathbb{N}$, where $j = \mathrm{LCM}(n,m)$.*

Proof. It is well known that

$$T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I).$$

Since T and S are k-quasi-(n,m)-power normal, we deduce from Lemma 4.3 that $T \otimes I$ and $I \otimes S$ are k-quasi-(n,m)-power normal operators. Applying Theorem 2.20 it follows that $T \otimes S$ is a k-quasi-(j,p)-power normal operator. □

Corollary 4.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ be k-quasi-(n,m)-power normal operators. Then $T^j \otimes S^j$ is k-quasi-$(1,1)$-power normal, where j is the least common multiple of n and m.*

Proof. From Corollary 2.15, it is known that T^j and S^j are k-quasi-$(1,1)$-power normal, where $j = \mathrm{LCM}(n,m)$. Hence $T^j \otimes S^j$ is a k-quasi-$(1,1)$-power normal operators by Theorem 4.4. □

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