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# The rise and fall of L-spaces

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### Abstract

For a quite long period, the so-called L-structure or L-spaces have been studied by some authors. They have several trivial misconceptions such as their L-spaces extend the well-known generalized convex (G-convex) spaces. In order to clarify this matter and others, we show that our KKM theory on abstract convex spaces improves typical results in L-spaces. Main topics in this paper are related to extensions of the Himmelberg fixed point theorem. Since such studies are beyond of L-spaces, we cordially claim that now is the proper time to give up the useless study on L-spaces and their variants FC-spaces.

*Keywords:* Abstract convex space, KKM theorem, Multimap classes  $\mathfrak{KC}$ ,  $\mathfrak{KD}$ , (Partial) KKM space, Fixed point theorem, L-space, FC-space.

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### 1. Introduction

The KKM theory, first called by the author in 1992, is the study on applications of equivalent formulations or generalizations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Since the theorem and its many equivalent formulations or extensions are powerful tools in showing the existence of solutions of a lot of problems in pure and applied mathematics, many scholars have been studying its further extensions and applications.

The KKM theory was first devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde, to  $c$ -spaces (or H-spaces) by Horvath and

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others, to generalized convex (G-convex) spaces mainly by the present author. Since 2006, we proposed new concepts of abstract convex spaces and (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory. Now the theory becomes the study of (partial) KKM spaces and we obtained a large number of new results in such frame. For the history of the KKM theory, see our recent article [43] in 2017.

In 1998, we derived new concept of G-convex spaces removing the monotonicity restriction. Motivated our original G-convex spaces  $(E, D; \Gamma)$  in 1993, Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [4, 5] in 1998 introduced L-spaces  $(E, \Gamma)$  and stated incorrectly that certain forms of G-convex spaces are particular to their L-spaces. Since then a number of authors followed the misconception of Ben-El-Mechaiekh et al. and published incorrect obsolete articles even after we established the KKM theory on abstract convex spaces in 2006–2010.

In this article, we recall the history of relation between G-convex spaces and L-spaces, and show that other authors' main works on L-spaces are consequences of our KKM theory on abstract convex spaces. Consequently, the study on L-spaces and related FC-spaces are not necessary, and hence, we conclude that now is the proper time to give up such useless study on L-spaces and their variants FC-spaces.

This article is organized as follows: Section 2 is a preliminary on our abstract convex spaces. In Section 3, we recall some history on the relation between G-convex spaces and L-spaces. Here we clarify the cause of why many authors had misconception that L-spaces generalize G-convex spaces. Section 4 deals with the paper of Altwaijry, Ounaies and Chebbi [1] in 2018 on the KKM theory on L-spaces. We clarify that its contents are already well-known in much more generalized form. In Sections 5–8, we deal with the Himmelberg type fixed point theorems in several types of abstract convex uniform spaces. We show that some typical theorems in L-spaces are simple consequences of our previous results. In Section 9, as a byproduct of the above study, we show that Shioji's main theorem [50] on recent unification of a fixed point theorem and a minimax inequality can be extended to a Hausdorff KKM  $L\Gamma$ -space. Finally, in Section 10, we conclude that now is the proper time to give up such useless study on L-spaces and their variants FC-spaces.

## 2. Abstract convex spaces

In order to upgrade the KKM theory, in 2006–2010, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of various known types of particular spaces and adequate to establish the KKM theory.

Multimaps are also called simply *maps*. Let  $\langle D \rangle$  denote the set of all nonempty finite subsets of a set  $D$ .

For the concepts on abstract convex spaces, KKM spaces and the KKM classes  $\mathfrak{RC}$ ,  $\mathfrak{RD}$ , we follow [39, 43, 44] and others with some modifications and the references therein.

**Definition 2.1.** *Let  $E$  be a topological space,  $D$  a nonempty set, and  $\Gamma : \langle D \rangle \multimap E$  a multimap with nonempty values  $\Gamma_A := \Gamma(A)$  for each  $A \in \langle D \rangle$ . The triple  $(E, D; \Gamma)$  is called an abstract convex space whenever the  $\Gamma$ -convex hull of any  $D' \subset D$  is denoted and defined by*

$$\text{co}_\Gamma D' := \bigcup \{ \Gamma_A \mid A \in \langle D' \rangle \} \subset E.$$

A subset  $X$  of  $E$  is called a  $\Gamma$ -convex subset of  $(E, D; \Gamma)$  relative to some  $D' \subset D$  if for any  $N \in \langle D' \rangle$ , we have  $\Gamma_N \subset X$ , that is,  $\text{co}_\Gamma D' \subset X$ .

When  $D \subset E$ , a subset  $X$  of  $E$  is said to be  $\Gamma$ -convex if  $\text{co}_\Gamma(X \cap D) \subset X$ ; in other words,  $X$  is  $\Gamma$ -convex relative to  $D' := X \cap D$ .

In case  $E = D$ , let  $(E; \Gamma) := (E, E; \Gamma)$ .

**Definition 2.2.** *Let  $(E, D; \Gamma)$  be an abstract convex space and  $Z$  a topological space. For a multimap  $F : E \multimap Z$  with nonempty values, if a multimap  $G : D \multimap Z$  satisfies*

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all } A \in \langle D \rangle,$$

then  $G$  is called a KKM map with respect to  $F$ . A KKM map  $G : D \multimap E$  is a KKM map with respect to the identity map  $1_E$ .

A multimap  $F : E \multimap Z$  is called a  $\mathfrak{KC}$ -map [resp. a  $\mathfrak{KO}$ -map] if, for any closed-valued [resp. open-valued] KKM map  $G : D \multimap Z$  with respect to  $F$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. We denote

$$\mathfrak{KC}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{KC}\text{-map}\}.$$

Similarly,  $\mathfrak{KO}(E, Z)$  is defined.

**Definition 2.3.** The partial KKM principle for an abstract convex space  $(E, D; \Gamma)$  is the statement  $1_E \in \mathfrak{KC}(E, E)$ , that is, for any closed-valued KKM map  $G : D \multimap E$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. The KKM principle is the statement  $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$ , that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (partial) KKM space if it satisfies the (partial) KKM principle, resp.

There are plenty of examples of KKM spaces; see [44] and the references therein.

Now we have the following diagram for subclasses of abstract convex spaces  $(E, D; \Gamma)$ :

$$\begin{aligned} \text{Simplex} &\implies \text{Convex subset of a t.v.s.} \implies \text{Lassonde's convex space} \\ &\implies \text{c-space of Horvath} \implies \text{L-space} \implies \text{G-convex space} \\ &\implies \phi_A\text{-space} \implies \text{KKM space} \implies \text{Partial KKM space} \\ &\implies \text{Abstract convex space.} \end{aligned}$$

This is a quite old version. Later versions replaced c-spaces and L-spaces by Horvath spaces, which are new class including c-spaces due to Horvath; see [45].

### 3. G-convex spaces and L-spaces

In this section, we recall some history on the relation between G-convex spaces and L-spaces. Here we clarify the cause of why many authors had misconception that L-spaces generalize G-convex spaces.

(1) In 1993, Park and Kim [46] defined: “A *generalized convex space* or a *G-convex space*  $(X, D; \Gamma)$  consists of a topological space  $X$ , a nonempty subset  $D$  of  $X$ , and a map  $\Gamma : \langle D \rangle \multimap X$  with nonempty values such that

(1) for each  $A, B \in \langle D \rangle$ ,  $A \subset B$  implies  $\Gamma(A) \subset \Gamma(B)$ ; and

(2) for each  $A \in \langle D \rangle$  with  $|A| = n + 1$ , there exists a continuous function  $\phi_A : \Delta_n \rightarrow \Gamma(A)$  such that  $J \in \langle A \rangle$  implies  $\phi_A(\Delta_J) \subset \Gamma(J)$ .”

Here, if  $n$  is any natural number and if  $J \subset \{0, 1, \dots, n\}$ ,  $\Delta_n$  will denote the unit-simplex of  $\mathbb{R}^{n+1}$  and  $\Delta_J$  the face of  $\Delta_n$  corresponding to  $J$ , i.e.  $\Delta_J = \text{co}\{e_j : j \in J\}$  where  $\{e_0, e_1, \dots, e_n\}$  is the canonical basis of  $\mathbb{R}^{n+1}$ .

In [46], we listed various examples of G-convex spaces and gave their fundamental properties.

The condition (1) will be called the *monotonicity* and is the cause of big trouble later.

(2) In 1998, Park [28] : “At first, a G-convex space is defined under the extra restriction that

(0) for each  $A, B \in \langle D \rangle$ ,  $A \subset B$  implies  $\Gamma(A) \subset \Gamma(B)$ ;

which was shown later to be superfluous.”

(3) In 1998, apparently motivated by [46], Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [5] gave:

“Definition (3.1). An L-structure on  $E$  is given by a nonempty set-valued map  $\Gamma : \langle E \rangle \rightarrow E$  verifying:

(\*) For every  $A \in \langle E \rangle$ , say  $A = \{x_0, x_1, \dots, x_n\}$ , there exists a continuous function  $f^A : \Delta_n \rightarrow \Gamma(A)$  such that for all  $J \subset \{0, 1, \dots, n\}$ ,  $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$ .

The pair  $(E, \Gamma)$  is then called an L-space and  $X \subset E$  is said to be L-convex if  $\forall A \in \langle X \rangle$ ,  $f(A) \subset X$ .

In particular, if  $\Gamma$ , as in Definition 3.1, verifies the additional condition

(\*\*) For each  $A, B \in \langle E \rangle$ ,  $A \subset B$  implies  $\Gamma(A) \subset \Gamma(B)$ ,

then the pair  $(E, \Gamma)$  is what is called by Park and Kim [46], a *G-convex space*."

This statement is incorrect. Our G-convex space is a triple  $(X, D; \Gamma)$  and L-space is a pair  $(E, \Gamma)$ . This statement leads many naive peoples to think L-spaces generalize G-convex spaces without checking [46, 28] and scores of later works on G-convex spaces.

Note that the L-spaces are motivated by [46]: In fact, [5] states "As noted by Park and Kim [46], it follows from Theorem 1, Section 1 of Horvath [21] that if  $\Gamma$  defines an H-structure, then  $(X, \Gamma)$  is an L-space."

(4) In 1999, Ding [9] first adopted the concept of the original generalized convex spaces in our [47, 48] and applied them to variational inequalities and equilibrium problems by applying a coincidence theorem involving the admissible class of multimaps due to Park.

(5) Abstract of Ding [11] submitted in 1999: "Some new generalized G-KKM and generalized S-KKM theorems are proved under the noncompact setting of generalized convex spaces. As applications, some new minimax inequalities, saddle point theorems, a coincidence theorem, and a fixed point theorem are given in generalized convex spaces. These theorems improve and generalize many important known results in recent literature."

(6) In 2001, Ding [10] wrote: "If an L-convex space  $(X, \Gamma)$  satisfies the monotonicity condition, then the pair  $(X, \Gamma)$  is called by Park and Kim [47, 48] a generalized convex (or G-convex) space. It is clear that the notion of L-convex space reduces the corresponding G-convex spaces as a special case."

And then Ding quote our definition of the class  $\mathfrak{A}_c^k(X, Y)$  in [47, 48]. This is the evidence that Ding read [47, 48] where G-convex spaces are triples  $(X, D; \Gamma)$  not pairs  $(X, \Gamma)$ . Then his careless claim on G-convex spaces are definitely false and contrary to his [9, 11]. Moreover, we found that many of his works are unreliable.

In 2002, Ding [12] repeated his incorrect claim,

(7) In 2002, Ding and Park, J.Y. [16] called L-convex spaces instead of L-spaces and stated: "If an L-convex space  $(X, \Gamma)$  satisfies the following additional condition

(2) for each  $A, B \in \mathfrak{F}(X)$ ,  $A \subset B$  implies  $\Gamma(A) \subset \Gamma(B)$ ,

then the pair  $(X, \Gamma)$  is called by Park and Kim [48] a generalized convex (or G-convex) space. Recently, Park [29] has removed the isotony condition (2) and considered the G-convex space  $(X, D; \Gamma)$ , where  $D$  need not be  $X$ . If  $D = X$ , then a G-convex space in [29] is an L-convex space."

This seems to be the only almost correct statement given by Ding on L-spaces and G-convex spaces. Moreover, in 2001–2003, Ding (with some coauthors) defined G-convex spaces of the form  $(X; \Gamma)$  and claimed some particular, not general enough, results on them.

(8) In 2002, Llinares [23] noted: "it is obvious that the notion of G-convex spaces, used in [46], is a particular case of L-spaces since, to define the G-convex spaces, it is required that all of the conditions of Definition 9 [on L-structure and L-space] be satisfied, together with a monotonicity condition on the set-valued map."

Here we found another people who can not distinguish pairs and triples.

(9) In 2004, Ding and Xia [18] and Ding, Yao, and Lin [19]: "The notion of a generalized convex (or G-convex) space was introduced under an extra isotonic condition by Park and Kim [48]. Recently Park [35] gave the following definition of a G-convex space by removing the extra condition.

A G-convex space  $(X, D; \Gamma)$  consists of a topological space  $X$ , a nonempty set  $D$  and a set-valued mapping  $\Gamma : \langle D \rangle \rightarrow 2^X \setminus \{\emptyset\}$  such that for each  $A = \{a_0, a_1, \dots, a_n\} \in \langle D \rangle$  with  $|A| = n + 1$ , there exists a continuous mapping  $\varphi_A : \Delta_n \rightarrow (A)$  such that  $J \subseteq \{0, 1, \dots, n\}$  implies  $\varphi_A(\Delta_J) \subseteq \Gamma(\{a_j : j \in J\})$ , where  $\Delta_J = \text{co}\{e_j : j \in J\}$ , the convex hull of the set  $\{e_j : j \in J\}$ ."

It should be noted that the so-called L-spaces  $(X, \Gamma)$  are G-convex spaces  $(X, D; \Gamma)$  for  $X = D$ . However, Ding and others repeatedly claim the converse in scores of his later papers.

**(10)** In 2004, Ding and Park, J. Y. [17]: The notion of generalized convex (G-convex) space was introduced as in the preceding papers. It is strange that only three papers of Ding in 2004 correctly stated the concept of G-convex spaces. After that Ding invented FC-spaces and claimed incorrectly that they are generalizations of G-convex spaces. Note that FC-spaces are pairs.

In this paper and many others, Ding introduced the concept of compact closure (ccl), compact interior (cint), transfer compactly open, etc. Note that if we consider compactly generated topology (as in  $k$ -spaces) instead of the original one, then these concepts become the usual ones. They are not practical and useless.

**(11)** In 2006, Ding [13, Theorem 3.2] obtained the following fixed point theorem:

“Theorem 3.2. *Let  $(X, \mathcal{U}, \{\varphi_N\})$  be a locally FC-space, and  $T \in KKM(X, X)$  be a u.s.c. compact mapping with closed values. Then  $T$  has a fixed point in  $X$ .*”

Comments on this will be given in Section 8.

**(12)** In 2008, Cain and González [6] considered relationship among some subclasses of the class of G-convex spaces and introduced a subclass of L-spaces. In [6, Theorems 3.2 and 3.4], it was shown that L-spaces are G-convex spaces.

**(13)** In 2009, Abstract of Park [38]: “We show that FC-spaces due to Ding are particular types of L-spaces due to Ben-El-Mechaiekh et al., and hence particular types of G-convex spaces. Some counterexamples are given and related matters are also discussed.”

In the text: “For the definition of a G-convex space  $(X, D; \Gamma)$ , at first we assumed  $X \supset D$  and an additional monotonicity condition. This monotonicity was removed since 1998 and the restriction  $X \supset D$  since 1999. However, note that most of useful examples of G-convex spaces satisfy monotonicity, but, examples not satisfying monotonicity seem to be artificial.”

**(14)** In 2010, Abstract of Park [40]: “In the KKM theory, various types of  $\phi_A$ -spaces  $(X, D; \{\phi_A\}_{A \in (D)})$  due to other authors are simply G-convex spaces. Various types of generalized KKM maps on  $\phi_A$ -spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of  $\phi_A$ -spaces. In 2006-09, G-convex spaces are extended to KKM spaces. In the present paper, we review the recent transition from G-convex spaces to KKM spaces, and introduce a basic KKM theorem on abstract convex spaces satisfying the partial KKM principle.”

**(15)** In 2011, Abstract of Chebbi, Gourdél, and Hammami [8]: “We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend Fan’s matching theorem.”

They also stated: “It should be noticed that the L-convexity is different from the G-convexity defined by Park and Kim in [46] in 1993 which assumes in addition the following condition:

$$\text{For all } A, B \in X, A \subset B \text{ implies } \Gamma(A) \subset \Gamma(B).”$$

**(16)** In 2012, Ding [15] wrote: “it is easy to see that each H-space must be a G-convex space, each G-convex space must be a L-convex space and each L-convex space must be a FC-space.”

“some critiques on L-spaces and FC-spaces given by Park [38] are not fair. I believe that the readers will give the fairest judgment.”

Still Ding confused L-spaces and L-convex spaces, and did not recognize the difference between a pair and a triple. Ding repeated this kind of statements in several of his papers.

**(17)** In 2013, Abstract of Park [41]: “In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano obtained a generalization of Ky Fan’s 1984 KKM theorem on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. In 2011, Chebbi, Gourdél, and Hammami introduced a generalized

coercivity type condition for multimaps defined on topological spaces endowed with a generalized convex structure and extended Fan's KKM theorem. In this paper, we show that better forms of above-mentioned theorems can be deduced from a KKM theorem on abstract convex spaces in Park's sense."

(18) In 2014, Fakhari and Zafarani [20] gave the definition of L-convex spaces and then wrote: "If  $\Gamma$  as the above definition, verifies the additional condition: For each  $A, B \in \mathfrak{F}(X)$ ,  $A \subseteq B$  implies  $\Gamma(A) \subseteq \Gamma(B)$ , then the pair  $(X, \Gamma)$  is what called by Park and Kim [46], a G-convex spaces. Recently, Park [49] has removed the above condition and considered the G-convex space  $(X, D; \Gamma)$ , where  $D$  need not be a subset of  $X$ . If  $D = X$ , then a G-convex space in [49] is an L-convex space. ... It is clear that, the notion of L-convex spaces includes the G-convex spaces of Park and Kim [46]."

This paper [20] has some incorrect statements: As we have noted several times, our G-convex spaces were  $(X, D; \Gamma)$  from the beginning, and hence L-spaces  $(X, \Gamma)$  can not include them. Moreover, the class  $\mathfrak{RC}(X, Y)$  was introduced by ourselves, and we showed that  $\mathcal{A}(X, Y) \subset \mathfrak{A}_c^c(X, Y) \subset \mathfrak{RC}(X, Y)$ , where  $\mathcal{A}$  is approachable maps.

(19) In 2015, Abstract of Park [42]: "Recently, Kulpa and Szymanski published an article entitled 'Some remarks on Park's abstract convex spaces' [Top. Meth. Nonlinear Anal. 44(2) (2014) 369–379]. The present short note is to trace out the history of that article and to respond to some remarks given there."

Actually, some of their remarks are inadequate.

(20) In 2018, Altwaijry, Ounaies and Chebbi [1]: "We note also that to define the structure of L-convexity, we do not require that  $\Gamma$  satisfies a monotony type condition, i.e. if  $A \subset B$ , then  $\Gamma(A) \subset \Gamma(B)$  as it is used by Park and Kim in [46] to define G-convex spaces and by many other papers in the literature dealing with G-convex spaces."

This statement is definitely incorrect. Since 1998, scores of papers on G-convex spaces adopted new definition without the monotonicity. Certain authors adopting the monotonicity can erase it in order to improve their results since it does not needed. Some of them also could not recognize the difference between triples and pairs.

Note that our 1993 paper, a quarter century old, is still quoted in 2018 incorrectly.

#### 4. Basic results of the KKM theory

Since we introduced the concept of abstract convex spaces in 2006 [31], its theory had been developed rapidly by ourselves. For example, in our work [39] in 2010, we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we added more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

Later in 2018, Altwaijry, Ounaies and Chebbi [1] gave L-space versions of the Fan-KKM Principle, a Browder-Fan type fixed point theorem and some applications. In this section, their L-space results are already well-known in much more generalized form for abstract convex spaces as in [39].

The following type of the KKM theorem for abstract convex spaces is known by ourselves [31] in 2006 as follows:

**Theorem 4.1.** *Let  $(E, D; \Gamma)$  be a partial KKM space (resp. a KKM space), and  $G : D \multimap E$  be a multimap satisfying*

(1)  *$G$  has closed (resp. open) values; and*

(2)  *$\Gamma_N \subset G(N)$  for any  $N \in \langle D \rangle$  (that is,  $G$  is a KKM map).*

*Then  $\{G(y)\}_{y \in D}$  has the finite intersection property. Further, if*

(3)  $\bigcup_{y \in M} G(y)$  is compact for some  $M \in \langle D \rangle$ ,  
then we have  $\bigcap_{y \in D} G(y) \neq \emptyset$ .

**Theorem 4.2.** Let  $(E, D; \Gamma)$  be an abstract convex space, and  $G : E \multimap D$ ,  $H : E \multimap E$  maps satisfying

(1) for each  $x \in E$ ,  $M \in \langle G(x) \rangle$  implies  $\Gamma_M \subset H(x)$ ; and

(2)  $E = G^-(N)$  for some  $N \in \langle D \rangle$ .

(3)  $G^-$  has open [resp. closed] values.

If  $(E, D; \Gamma)$  is a partial KKM space [resp. a KKM space], then  $H$  has a fixed point  $x_0 \in E$ , that is,  $x_0 \in F(x_0)$ .

Recall that Theorem 4.1 is the abstract form of Ky Fan's 1961 KKM lemma and Theorem 4.2 originates from the Fan-Browder fixed point theorem.

After giving L-space versions of Theorems 4.1 and 4.2, the authors of [1] stated: "The concept of KF-majorization due to Borglin and Keiding in 1976 is then easily extended to L-spaces and a result on the existence of a maximal element for such correspondence is deduced. As an application, we prove an equilibrium existence result for qualitative games defined in an L-space and an equilibrium result for an abstract economy."

However, there are too many generalizations of the Borglin-Keiding results. The results in [1] can be easily extended to several types of abstract convex spaces and hence their application seems to be useless.

Actually, we have far reaching generalizations of the Borglin-Keiding results for abstract convex spaces in [35]. Its abstract is as follows: In [35], KKM theorems or coincidence theorems on abstract convex spaces are applied to obtain the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtain generalizations or improvements of a number of known equilibria results, especially, in a recent work of Ding and Wang on the so-called FC-spaces.

Finally, in this section, consider the following given in Chebbi et al. [8]:

**Lemma 4.1.** (Chebbi et al. [8]) Let  $(X, \Gamma)$  be an L-space,  $Z$  a nonempty subset of  $X$  and  $F : Z \multimap X$  a KKM map with quasi-compactly closed values. Suppose that for some  $z \in Z$ ,  $F(z)$  is quasi-compact. Then  $\bigcap_{x \in Z} F(x) \neq \emptyset$ .

This is a modification of the 1961 KKM lemma of Ky Fan. If we choose a quasi-compactly generated topology (as in  $k$ -spaces) on  $X$ , then Lemma 4.1 is easily accessible. More generally, Lemma 4.1 can be extended scores of examples for partial KKM spaces. But they are useless under the quasi-compactness.

## 5. Himmelberg type theorems for t.v.s.

In our previous survey [36] in 2008, we reviewed various generalizations of the Himmelberg fixed point theorem within the category of topological vector spaces. We considered the Lassonde type, the Idzik type, and the KKM type generalizations for Kakutani maps, and other types of generalizations for acyclic maps. Finally, generalizations for various 'better' admissible maps on admissible almost convex domains to Klee approximable ranges were discussed.

In this section, we recall some of them in order to compare with later results.

In our early work [24] in 1992, we obtained the following generalization of the Himmelberg fixed point theorem:

**Theorem 5.1.** Let  $X$  and  $C$  be nonempty convex subsets of a Hausdorff locally convex t.v.s.  $E$ . Let  $F : X \rightarrow \text{ca}(X + C)$  be a compact u.s.c. multifunction. Suppose that one of the following conditions holds:

(i)  $X$  is closed and  $C$  is compact.

(ii)  $X$  is compact and  $C$  is closed.

(iii)  $C = \{0\}$ .

Then there is an  $x_0 \in X$  such that  $Fx_0 \cap (x_0 + C) \neq \emptyset$ .

**Example.** 1. Here  $\text{ca}(X + C)$  denotes the set of all closed acyclic subsets of  $X + C$ .

2. Moreover, in case when  $\text{ca}(X + C)$  is replaced by the set  $\text{cc}(X + C)$  of all closed convex subsets of  $X + C$ , Theorem 3.1 was obtained by Lassonde in 1983. Note that Case (iii) of Lassonde's is the Himmelberg fixed point theorem in 1972.

3. Note that Case (iii) implies the extension of the Himmelberg fixed point theorem for multimaps having closed acyclic values as follows:

**Corollary 5.2.** *Let  $X$  be a nonempty convex subset of a Hausdorff locally convex t.v.s.  $E$ . Let  $F : X \rightarrow \text{ca}(X)$  be a compact u.s.c. multifunction. Then there is an  $x_0 \in X$  such that  $x_0 \in Fx_0$ .*

**Remark 5.3.** 1. *This is the extension of the Himmelberg fixed point theorem for multimaps having closed acyclic values.*

2. *Recall that Corollary 5.2 implies a large number of historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Ky Fan, Glicksberg, Mazur, Bohnenblust and Karlin, Hukuhara, Browder, Singbal, and others; see [24].*

Recall that we have introduced the following class of multimaps in several occasions:

**Definition 5.4.** *Let  $X$  and  $Y$  be topological spaces. An admissible class  $\mathfrak{A}_c^k(X, Y)$  of maps  $T : X \multimap Y$  is the one such that, for each compact subset  $K$  of  $X$ , there exists a map  $S \in \mathfrak{A}_c(K, Y)$  satisfying  $S(x) \subset T(x)$  for all  $x \in K$ ; where  $\mathfrak{A}_c$  is consisting of finite compositions of maps in  $\mathfrak{A}$ , and  $\mathfrak{A}$  is a class of maps satisfying the following properties:*

- (i)  $\mathfrak{A}$  contains the class  $\mathbb{C}$  of (single-valued) continuous functions;
- (ii) each  $F \in \mathfrak{A}$  is u.s.c. and compact-valued; and
- (iii) for any polytope  $P$ , each  $F \in \mathfrak{A}(P, P)$  has a fixed point.

It is well-known that the most of multimaps belong to admissible class.

We have the following generalization of Theorem 5.1 in [25]:

**Theorem 5.5.** *Let  $X$  and  $C$  be nonempty convex subsets of a locally convex Hausdorff t.v.s.  $E$ , and  $F \in \mathfrak{A}_c(X, X + C)$  a compact multifunction. Suppose that one of the following conditions holds:*

- (i)  $X$  is closed and  $C$  is compact.
- (ii)  $X$  is compact and  $C$  is closed.
- (iii)  $C = \{0\}$ .

*Then there is an  $x_0 \in X$  such that  $Fx_0 \cap (x_0 + C) \neq \emptyset$ .*

Since approachable maps  $\mathcal{A}$  belong to  $\mathfrak{A}$ , we immediately have the following case (iii):

**Corollary 5.6.** *(Ben-El-Mechaiekh [3]) If  $X$  is a nonempty convex subset of a locally convex topological vector space and if  $\Phi \in \mathcal{A}(X)$  is upper semicontinuous with nonempty closed values, then  $\Phi$  has a fixed point provided  $\Phi(X)$  is contained in a compact subset  $K$  of  $X$ .*

Note that this is not appeared in [36]. Here  $\mathcal{A}(X)$  denotes the class of approachable selfmaps on  $X$ . It is well-known that  $\mathcal{A} \subset \mathfrak{A}_c^k \subset \mathfrak{B}$ . Moreover, every nonempty convex subset of a locally convex t.v.s. is admissible; see below.



## 6. Himmelberg type theorems for KKM spaces

In our previous work [37] in 2009, we established fixed point theorems for multimaps in abstract convex uniform spaces. Our new results generalize corresponding ones in topological vector spaces (t.v.s.), convex spaces due to Lassonde,  $c$ -spaces due to Horvath, and  $G$ -convex spaces due to Park. We showed that fixed point theorems on multimaps of the Fan-Browder type, multimaps having ranges of the Zima-Hadžić type, and multimaps whose ranges are  $\Phi$ -sets or Klee approximable sets can be established in abstract convex spaces or KKM spaces.

In this section, we introduce a few results in [37].

**Definition 6.1.** *An abstract convex uniform space  $(E, D; \Gamma; \mathcal{U})$  is the one with a basis  $\mathcal{U}$  of a uniform structure of  $E$ .*

*A KKM uniform space  $(E, D; \Gamma; \mathcal{U})$  is a KKM space with a basis  $\mathcal{U}$  of a uniform structure of  $E$ .*

*A KKM uniform space  $(E \supset D; \Gamma; \mathcal{U})$  is called an  $L\Gamma$ -space or locally  $\Gamma$ -convex space if  $D$  is dense in  $E$  and, for each  $U \in \mathcal{U}$ , the  $U$ -neighborhood  $U[A] := \{x \in E : A \cap U[x] \neq \emptyset\}$  around a given  $\Gamma$ -convex subset  $A \subset E$  is  $\Gamma$ -convex.*

For abstract convex spaces, we have the following extension of the Himmelberg theorem due to ourselves [37, Corollary 4.5]:

**Theorem 6.2.** *Let  $(E \supset D; \Gamma; \mathcal{U})$  be a Hausdorff KKM  $L\Gamma$ -space and  $T : E \multimap E$  a compact u.s.c. map with nonempty closed  $\Gamma$ -convex values. Then,  $T$  has a fixed point.*

**Corollary 6.3.** *Let  $(X, D; \Gamma)$  be an Hausdorff  $LG$ -space and  $T : X \multimap X$  a compact u.s.c. multimap with closed  $\Gamma$ -convex values. Then  $T$  has a fixed point.*

This is given in 2002.

Six years later in 2008, Ding [14] derived the class of locally FC-uniform spaces, and claimed that these includes the classes of locally convex topological vector spaces, LC-spaces of Horvath, locally H-convex uniform spaces of Tarafdar and locally  $G$ -convex spaces of Park as *true subclasses*.

The last statement is false as the following shows that our  $LG$  space is a quadruple  $(X, D; \Gamma; \mathcal{U})$  and his one is a triple  $(X, \mathcal{U}, \{\psi_N\})$ .

**Corollary 6.4.** *(Ding [14]) Let  $(X, \mathcal{U}, \{\psi_N\})$  be a locally FC-uniform space, and  $F : X \rightarrow 2^X$  be a compact upper semicontinuous set-valued mapping with closed values such that for each  $x \in X$ ,  $F(x)$  is an FC-subspace of  $X$ . Then  $F$  has a fixed point  $x_0 \in X$ , i.e.  $x_0 \in F(x_0)$ .*

**Corollary 6.5.** *Let  $X$  be a nonempty convex subset of a Hausdorff locally convex t.v.s.  $E$ . Let  $F : X \rightarrow cc(X)$  be a compact u.s.c. multifunction. Then there is an  $x_0 \in X$  such that  $x_0 \in Fx_0$ .*

This is the original Himmelberg theorem.

## 7. Himmelberg type theorems for admissible spaces

In our work [37] in 2009, we introduced particular types of subsets of abstract convex uniform spaces adequate to establish our fixed point theory:

**Definition 7.1.** *For an abstract convex uniform space  $(E, D; \Gamma; \mathcal{U})$ , a subset  $X$  of  $E$  is said to be admissible (in the sense of Klee) if, for each nonempty compact subset  $K$  of  $X$  and for each entourage  $U \in \mathcal{U}$ , there exists a continuous function  $h : K \rightarrow X$  satisfying*

- (1)  $(x, h(x)) \in U$  for all  $x \in K$ ;
- (2)  $h(K) \subset \Gamma_N$  for some  $N \in \langle D \rangle$ ; and
- (3) there exist continuous functions  $p : K \rightarrow \Delta_n$  and  $\phi_N : \Delta_n \rightarrow \Gamma_N$  with  $|N| = n + 1$  such that  $h = \phi_N \circ p$ .

**Definition 7.2.** Let  $(E, D; \Gamma)$  be an abstract convex space,  $X$  a nonempty subset of  $E$ , and  $Y$  a topological space. We define the better admissible class  $\mathfrak{B}$  of maps from  $X$  into  $Y$  as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$  is a map such that, for any  $\Gamma_N \subset X$ , where  $N \in \langle D \rangle$  with the cardinality  $|N| = n + 1$ , and for any continuous function  $p : F(\Gamma_N) \rightarrow \Delta_n$ , there exists a continuous function  $\phi_N : \Delta_n \rightarrow \Gamma_N$  such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that  $\Gamma_N$  can be replaced by the compact set  $\phi_N(\Delta_n) \subset X$ .

The following is [37, Theorem 8.5]:

**Theorem 7.3.** Let  $(X, D; \Gamma; \mathcal{U})$  be an admissible abstract convex uniform space. Then any compact closed map  $F \in \mathfrak{B}(X, X)$  has a fixed point.

There are lots of examples of admissible abstract convex uniform spaces and better admissible multimaps, Therefore Theorem 7.3 has plenty of examples. Here we give only a few of them appeared in literature.

Recall that Ben-El-Mechaiekh et al. [4, 5] extended Himmelberg's fixed point theorem replacing the usual convexity in topological vector spaces by their L-convexity. They claimed the existence, under weak hypotheses, of a fixed point for a compact approachable map and provided sufficient conditions under which their result applies to maps whose values are convex in the abstract sense mentioned above.

They provided a generalization of the Himmelberg theorem to a class of uniform L-spaces as a main theorem of [4, 5], but this follows from our Theorem 7.3:

**Corollary 7.4.** (Ben-El-Mechaiekh et al. [4, 5]) Assume that  $(X, \mathcal{U}, \Gamma)$  is a uniform L-space such that for every  $U \in \mathcal{U}$ , there exist two correspondences  $S : X \multimap X$  and  $T : X \multimap X$  (depending on  $U$ ) satisfying:

- (i)  $\forall x \in X, S(x) \subset T(x)$
- (ii)  $\forall x \in X, \forall A \in \langle S(x) \rangle, \Gamma(A) \subset T(x)$
- (iii)  $X = \bigcup \{\text{int} S^{-1}(y) : y \in X\}$
- (iv)  $\forall x \in X, T(x) \subset U[x]$ .

Assume also that  $\Phi \in \mathcal{A}(X, X)$  is u.s.c. with nonempty closed values. Then  $\Phi$  has a fixed point, provided  $\Phi(X)$  is contained in a compact subset  $K$  of  $X$ .

*Proof.* Let  $U \in \mathcal{U}$  be arbitrary but fixed and consider a cover of  $K$  by a finite collection  $\{\text{int} S^{-1}(y_i)\}_{i=0}^n$  and a continuous partition of unity  $p = (p_i)_{i=0}^n$  subordinated to this cover. Applying condition (\*) in the definition of L-convexity [See Section 3, (3)] there exists a continuous function  $\phi_N : \Delta_N \rightarrow X$  such that  $\forall J \in \langle N \rangle, \phi_N(\Delta_J) \subset \Gamma(\{y_i : i \in J\})$ . Note that for every  $x \in K$ ,

$$\phi_N \circ p(x) \in \Gamma(\{y_i : x \in \text{int} S^{-1}(y_i)\}) \subset \Gamma(\{y_i : y_i \in S(x)\}) \subset T(x) \subset U[x].$$

Then  $h := \phi_N \circ p$  is continuous and (1)  $h(x) \subset U[x]$ ; (2)  $h(K) \subset \Gamma_N$ ; and (3)  $h$  is continuous. Therefore,  $(X, \mathcal{U}, \Gamma)$  is admissible. Moreover,  $\Phi \in \mathfrak{B}(X, X)$  is compact closed map. Hence it has a fixed point by Theorem 7.3.  $\square$

In the proof, note that Corollary 7.4 holds for  $\mathfrak{B}(X, X)$  instead of  $\mathcal{A}(X, X)$ . Moreover, we have a lot of particular forms of Theorem 7.3. We list only some of them:

**Corollary 7.5.** (Park [27]) Let  $X$  be an admissible convex subset of a Hausdorff topological vector space  $E$ , and let  $T \in \mathfrak{B}(X, X)$ . If  $T$  is compact and closed, then  $T$  has a fixed point in  $X$ .

**Corollary 7.6.** (Park [26]) Let  $X$  be a nonempty convex subset of a Hausdorff locally convex topological vector space  $E$ . Then, any closed compact map  $F \in \mathfrak{B}(X, X)$  has a fixed point.

In 1999, Wu and Li [51] introduced the following:

**Corollary 7.7.** (Wu [51]) *Let  $(X; \{\Gamma_A\})$  be a Hausdorff locally convex  $H$ -space and  $D$  a  $H$ -compact subset of  $X$ . If  $T : X \multimap D$  is an u.s.c. multimap with closed acyclic values, then there exists a point  $x_0 \in D$  such that  $x_0 \in T(x_0)$ .*

## 8. Himmelberg type theorems for $\mathfrak{KC}$ -maps

Motivated by our previous works on the KKM theory, Chang and Yen [7] in 1996 stated as follows:

Assume that  $X$  is a convex subset of a linear space and  $Y$  is a topological space. If  $S, T : X \rightarrow 2^Y$  are two set-valued mappings such that  $T(\text{co}A) \subset SA$  for each finite subset  $A$  of  $X$ , then we call  $S$  a generalized KKM mapping w.r.t.  $T$ , where  $\text{co}A$  denotes the convex hull of  $A$ . Let  $T : X \rightarrow 2^Y$  be a set-valued mapping such that if  $S : X \rightarrow 2^Y$  is a generalized KKM mapping w.r.t.  $T$  then the family  $\{\overline{Sx} : x \in X\}$  has the finite intersection property (where  $\overline{Sx}$  denotes the closure of  $Sx$ ), then we say that  $T$  has the KKM property. Denote  $\text{KKM}(X, Y) = \{T : X \rightarrow 2^Y \mid T \text{ has the KKM property}\}$ .

Remark 1. Generalized KKM mappings were first introduced by Park in 1989, and followed by some others.

Many authors adopt the obsolete KKM class nowadays. For example, Amini-Harandi, Farajzadeh, O'Regan and Agarwal [2] in 2009 followed our theory of abstract convex spaces implicitly, but still studied the KKM class. However, we already extended this class to  $\mathfrak{KC}$  class on abstract convex spaces, and note that we have also  $\mathfrak{KD}$  class as shown in Section 2.

In 2006, we derived the following in [31]:

**Definition 8.1.** *For a given abstract convex space  $(E, D; \Gamma)$  and a topological space  $X$ , a map  $H : X \multimap E$  is called a  $\Phi$ -map (or a Fan-Browder map) if there exists a map  $G : X \multimap D$  such that*

- (i) *for each  $x \in X$ ,  $\text{co}_\Gamma G(x) \subset H(x)$ ; and*
- (ii)  *$X = \bigcup \{\text{Int} G^-(y) : y \in D\}$ .*

**Definition 8.2.** *For an abstract convex uniform space  $(E, D; \Gamma; \mathcal{U})$ , a subset  $Z$  of  $E$  is called a  $\Phi$ -set if, for any entourage  $U \in \mathcal{U}$ , there exists a  $\Phi$ -map  $H : Z \multimap E$  such that  $\text{Gr}(H) \subset U$ . If  $E$  itself is a  $\Phi$ -set, then it is called a  $\Phi$ -space. A point  $x \in E$  is called a  $U$ -fixed point of a map  $F : E \multimap E$  if  $F(x) \cap U[x] \neq \emptyset$ . The map  $F$  is said to have the almost fixed point property whenever it has a  $U$ -fixed point for each  $U \in \mathcal{U}$ .*

The following almost fixed point theorem and its corollaries are given by ourselves in [34]:

**Theorem 8.3.** *Let  $(E, D; \Gamma; \mathcal{U})$  be an abstract convex uniform space and  $F \in \mathfrak{KC}(E, E)$  a compact map. If  $F(E)$  is a  $\Phi$ -set, then  $F$  has the almost fixed point property.*

**Corollary 8.4.** *Under the hypothesis of Theorem 8.3, further if  $(E, \mathcal{U})$  is separated and if  $F$  is closed, then it has a fixed point.*

This is quoted by Amini-Harandi, Farajzadeh, O'Regan and Agarwal [2] in 2009 as follows:

**Corollary 8.5.** *Let  $(E, D; \Gamma; \mathcal{U})$  be an abstract convex uniform space, and  $F \in \text{KKM}(E, E)$  be a closed compact map. If  $F(E)$  is a  $\Phi$ -set, then  $F$  has a fixed point.*

Actually, the authors restated some part of our papers without mentioning it.

In 2006, Ding [13, Theorem 3.2] obtained the following fixed point theorem:

**Corollary 8.6.** (Ding [13]) *Let  $(X, \mathcal{U}, \{\varphi_N\})$  be a locally  $FC$ -space, and  $T \in \text{KKM}(X, X)$  be a u.s.c. compact mapping with closed values. Then  $T$  has a fixed point in  $X$ .*

More early, in 2001, Ding [10] obtained the following results similar to the Himmelberg fixed point theorem:

**Corollary 8.7.** (Ding [10]) Let  $(X, \Gamma, \mathcal{U})$  be a locally  $L$ -convex space,  $D$  be an  $L$ -compact subset of  $X$ . If  $T \in KKM(X, D)$ , then for each open entourage  $U \in \mathcal{U}$  there exists  $x_U \in E$  such that  $Tx_U \cap U(x_U) \neq \emptyset$ , where  $E$  is the compact  $L$ -convex subset of  $X$  containing  $D$ .

**Corollary 8.8.** (Ding [10]) Let  $(X, \Gamma, \mathcal{U})$  be a locally  $L$ -convex space,  $D$  be an  $L$ -compact subset of  $X$ , and  $T \in KKM(X, D)$  be upper semicontinuous with closed values. Then  $T$  has a fixed point in  $X$ .

Since any subset of a locally  $G$ -convex space is a  $\Phi$ -set [32], these are particular forms of Theorem 8.3 and Corollary 8.4, resp. Ding's theorems are actually for maps in  $KKM(D, D)$  and hence can not generalize the Himmelberg theorem. Moreover, throughout his paper, Ding claimed typical false statement that his results generalize corresponding ones for  $G$ -convex spaces,

**Corollary 8.9.** (Chang and Yen [7]) Let  $X$  be a convex subset of a locally convex Hausdorff t.v.s.  $E$ , and let  $T \in KKM(X, X)$ . If  $T$  is compact and closed, then  $T$  has a fixed point in  $X$ .

The following should be a corollary of Theorem 7.3. But we put here since it stated for KKM:

**Corollary 8.10.** (Lin and Yu [22]) Let  $E$  be a topological vector space, let  $X$  be an admissible convex subset of  $E$ , and let  $F \in KKM(X, X)$  be compact and closed; then  $F$  has a fixed point.

Recall that, for convex spaces,  $KKM = \mathfrak{B}$  for closed compact multimaps; see [26].

## 9. Unification of fixed points and minimax inequalities

In 2019, Shioji [50] showed a unified form of two fixed point theorems and a unified form of a fixed point theorem and a minimax inequality. He also applied his results to show the existence of solutions of generalized quasi-variational inequalities.

His results are given for a locally convex Hausdorff topological vector space. As a byproduct of the main results of the present article, we show that Shioji's main theorem can be extended to a Hausdorff KKM  $L\Gamma$ -space as follows:

**Theorem 9.1.** Let  $(X; \Gamma; \mathcal{U})$  be a Hausdorff KKM  $L\Gamma$ -space. Let  $S : X \multimap X$  be a continuous multimap such that, for each  $x \in X$ ,  $Sx$  is a nonempty, closed,  $\Gamma$ -convex subset of  $X$ . Let  $A, B : X \multimap X$  be multimaps such that

- (1) for each  $x \in X$ ,  $x \in Bx$ ,  $Bx \subset Ax$  and  $X \setminus Bx$  is  $\Gamma$ -convex,
- (2) for each  $y \in X$ ,  $A^{-1}y$  is closed and  $\Gamma$ -convex,
- (3)  $A$  is upper semicontinuous.

Then there exists  $x \in X$  such that  $x \in Sx$  and  $Sx \subset Ax$ .

*Proof.* We define a multimap  $T : X \multimap X$  by

$$Tx = \{z \in Sx : Sx \subset Az\} \text{ for each } x \in X.$$

Let  $x \in X$ . Note that  $Tx = Sx \cap \bigcap_{y \in Sx} A^{-1}y$  is closed and  $\Gamma$ -convex. We will show that  $Tx$  is nonempty. We claim that the self multimap on  $Sx$  given by  $z \mapsto Sx \cap B^{-1}z$  is a KKM map. Assume that the claim does not hold. Then there exist  $N = \{z_1, \dots, z_n\} \subset Sx$  and  $z \in \text{co}_\Gamma N$  such that  $z \notin \bigcup_{i=1}^n (Sx \cap B^{-1}z_i)$ . Since  $Sx$  is  $\Gamma$ -convex, we have  $z \in Sx$ . Hence  $z_i \in X \setminus Bz$  for each  $i = 1, \dots, n$ . Since  $X \setminus Bz$  is  $\Gamma$ -convex, we have  $z \notin Bz$ , which is a contradiction. Thus we have shown the claim. Since  $Sx \cap B^{-1}z \subset Sx \cap A^{-1}z$  for each  $z \in Sx$ , the self multimap of  $Sx$  given by  $z \mapsto Sx \cap A^{-1}z$  is also a KKM-map. Hence, it is a closed-valued KKM map on the compact partial KKM space  $Sx$ . Therefore, by Theorem 4.1, we have  $Tx = Sx \cap \bigcap_{y \in Sx} A^{-1}y \neq \emptyset$ . Thus we have shown that  $Tx$  is nonempty.

Next, we show that  $T$  is upper semicontinuous. Let  $\{(x_\alpha, z_\alpha)\} \subset X \times X$  be a net such that  $z_\alpha \in Tx_\alpha$  for each  $\alpha$  and  $\{(x_\alpha, z_\alpha)\}$  converges to  $(x, z) \in X \times X$ . Since  $S$  is upper semicontinuous, we have  $z \in Sx$ .

Let  $y$  be an arbitrary element of  $Sx$ . Since  $S$  is lower semicontinuous, there exist a subnet  $\{x_{\alpha_\beta}\}$  of  $\{x_\alpha\}$  and a net  $\{y_\beta\}$  such that  $y_\beta \in Sx_{\alpha_\beta}$  for each  $\beta$  and  $\{y_\beta\}$  converges to  $y$ . By the upper semicontinuity of  $S$  and  $A$ , we have  $y \in Az$ . Since  $y$  is an arbitrary element of  $Sx$ , we have  $Sx \subset Az$ , which yields  $z \in Tx$ . Thus we have shown that  $T$  is upper semicontinuous. Note that  $X$  is also a Hausdorff KKM  $L\Gamma$  space and  $T : X \multimap X$  is a compact u.s.c. map with nonempty closed  $\Gamma$ -convex values. Hence by Theorem 6.2, we find that  $T$  has a fixed point  $x \in X$ , i.e.,  $x \in Sx$  and  $Sx \subset Ax$ .  $\square$

**Corollary 9.2.** (Shioji [50]) *Let  $X$  be a nonempty, compact, convex subset of a locally convex Hausdorff topological vector space. Let  $S : X \multimap X$  be a continuous multimap such that, for each  $x \in X$ ,  $Sx$  is a nonempty, closed, convex subset of  $X$ . Let  $A, B : X \multimap X$  be multimaps such that*

- (1) *for each  $x \in X$ ,  $x \in Bx$ ,  $Bx \subset Ax$  and  $X \setminus Bx$  is convex,*
- (2) *for each  $y \in X$ ,  $A^{-1}y$  is closed and convex,*
- (3)  *$A$  is upper semicontinuous.*

*Then there exists  $x \in X$  such that  $x \in Sx$  and  $Sx \subset Ax$ .*

**Corollary 9.3.** *Let  $(X; \Gamma; \mathcal{U})$  be a Hausdorff KKM  $L\Gamma$ -space such that  $\Gamma\{x\} = \{x\}$  for all  $x \in X$ . Let  $A : X \multimap X$  be a multimap such that*

- (1)  *$A$  is a surjection,*
- (2) *for each  $y \in X$ ,  $A^{-1}y$  is closed and  $\Gamma$ -convex,*
- (3)  *$A$  is upper semicontinuous.*

*Then there exists  $x \in X$  such that  $x \in Ax$ .*

*Proof.* In Theorem 9.1, let  $S = 1_X$ , the identity map of  $X$ , and  $Bx = X$  for all  $x \in X$ . Then the conclusion of Theorem 9.1 follows.  $\square$

## 10. Conclusion

In this article, we showed that a careless statement in [5] leads some people to write useless articles for twenty years. In fact, the so-called L-structure or L-spaces have been studied by some authors. They have several trivial misconceptions such as their L-spaces extend the well-known generalized convex (G-convex) spaces. In order to clarify this matter and others, we show in this paper that our KKM theory on abstract convex spaces implies typical results in L-spaces by several groups of authors. Main topics in this paper are related to extensions of the Himmelberg fixed point theorem. Since our results in this paper show the uselessness of L-spaces and , we cordially express that now is the proper time to give up the study on L-spaces including FC-spaces.

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