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Research Article

An extended life time distribution: theory, properties and applications

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Abstract

This paper is an addition to the series of modification and improvement of the extant distributions which enable them to analyze the new emerging situations efficiently. We develop a new lifetime distribution by generalizing the Erlang truncated exponential distribution using the Topp Leone family of distributions. A comprehensive account of mathematical characteristics such as quantile function, moments, probability weighted moments, moment generating function, and probability generating function of the proposed distribution is presented. Some reliability measures such as hazard rate function, residual life function, and reversed residual life function are also provided. Several entropy measures including Rèyni entropy, Tsallis entropy, cumulative Tsallis entropy, and dynamic cumulative Tsallis entropy are obtained. Besides, the extropy, residual extropy, and cumulative residual extropy are explored. The unknown parameters of the proposed distribution are estimated by using the maximum likelihood method. The stability of the model parameters is examined through the simulation study. The application of our proposed distribution is explained through three real-life examples and its performance is illustrated through its comparison with the competent existing distributions.

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1. Introduction

Probability distributions comprehensively analyze different emerging situations and provide possible solutions leading toward important inferences. Besides the development of new distributions, the generalization of the classical distributions has been a very attractive practice. The resulting generalized distributions are applied in the modeling of real-life data sets effectively in almost all the disciplines such as biology, environmental sciences, survival analysis, medical science, engineering, reliability analysis etc.

The exponential model is widely used in reliability engineering because it has simple expression and analytical tractability. It is one of the well known classical distributions used for generalization due to its interesting "lack of memory property". Many generalizations

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of the exponential distribution are available in the literature. Some prominent generalized distributions are mentioned as; an extended exponentiated exponential distribution by [2], the extended exponential distribution by [3], generalized exponential distributions by [9], the beta exponential distribution by [17], and moment distribution by [6].

El-Alosey [7] introduces a new two parameters exponential distribution named as Erlang truncated exponential (ETEx) distribution. The distribution function (cdf) of the ETEx distribution is given as

$$H(y) = 1 - e^{-\alpha(1-e^{-\gamma})y}, \quad y > 0,\ \alpha, \gamma > 0, \tag{1.1}$$

the corresponding density function (pdf) is obtained by differentiating (1.1) and given as

$$h(y) = \alpha(1 - e^{-\gamma})e^{-\alpha(1-e^{-\gamma})y}, \quad y > 0,\ \alpha, \gamma > 0, \tag{1.2}$$

where α is the shape parameter and γ is the scale parameter.

Many generalizations of ETEx distribution exist in the literature. Okorie et al. [20] define Transmuted Erlang truncated exponential distribution, Nasiru et al. [18] present generalized Erlang truncated exponential distribution, Okorie et al. [22] develop Marshall-Olkin generalized Erlang truncated exponential distribution, Okorie [21] derives the extended Erlang truncated exponential distribution, Nasiru et al. [19] introduce Poisson exponentiated Erlang truncated exponential distribution and Mohsin et al. [14] study the characterization of the Erlang truncated exponential distribution.

Nadarajah and Kotz [16] propose Topp Leone distribution which is taken as a replacement of Beta distribution. The cdf of Topp Leone distribution is in closed form which not only increases its scope but also makes it analytically pliable. Al-Shomrani et al. [4] construct a generalized family of Topp Leone distribution having the following cdf

$$F_{TL-H}(y) = [H(y)]^b[2 - H(y)]^b = [1 - (\bar{H}(y))^2]^b, \quad x\epsilon\Re,\ b > 0, \tag{1.3}$$

with the corresponding pdf as

$$f_{TL-h}(y) = 2bg(t)\bar{H}(y)[1 - (\bar{H}(y))^2]^{b-1},\ b > 0, \tag{1.4}$$

where $g(y) = H'(y)$ and $\bar{H}(y) = 1 - H(y)$.

Since ETEx distribution has a constant failure rate, therefore, it cannot model the intricate situations having a non-constant failure rate as given by [22]. This paper aims to provide a new lifetime distribution by generalizing the ETEx distribution using Equation (1.3). The resulting generalized distribution is named as Topp Leone Erlang truncated exponential (TL-ETEx) distribution. This distribution is capable of modeling the situations having either constant failure rate or non-constant failure rate. Moreover, we present three real-life examples from hydrology, reliability and environmental sciences which support the better fitting of the proposed distribution. We hope this generalization might help to model the complex situations in real life.

The article is unfolded as: In Section 2, the model is developed and its characteristics are derived. In Section 3, measures of entropy and extropy are explored. In Section 4, model parameters are estimated along with the Fisher information matrix. In Section 5, a simulation study is performed. In Section 6, applicability of the proposed model is establisheded in three different fields. In Section 7, some concluding remarks are stated.

2. TL-ETEx distribution and its characteristics

In this section, the expressions for the cdf and the pdf of the Topp Leone Erlang truncated exponential (TL-ETEx) distribution are derived along with several of its characteristics and reliability measures.

The cdf of the TL-ETEx distribution is obtained by inserting Equation (1.3) in Equation (1.1) as

$$F(y) = \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^b, \tag{2.1}$$

and the corresponding pdf is derived by differentiating Equation (2.1) as

$$f(y) = 2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})y}\left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1}, \quad y > 0,\ \alpha, \gamma, b > 0, \tag{2.2}$$

where α and b are the shape parameters and γ is the scale parameter.

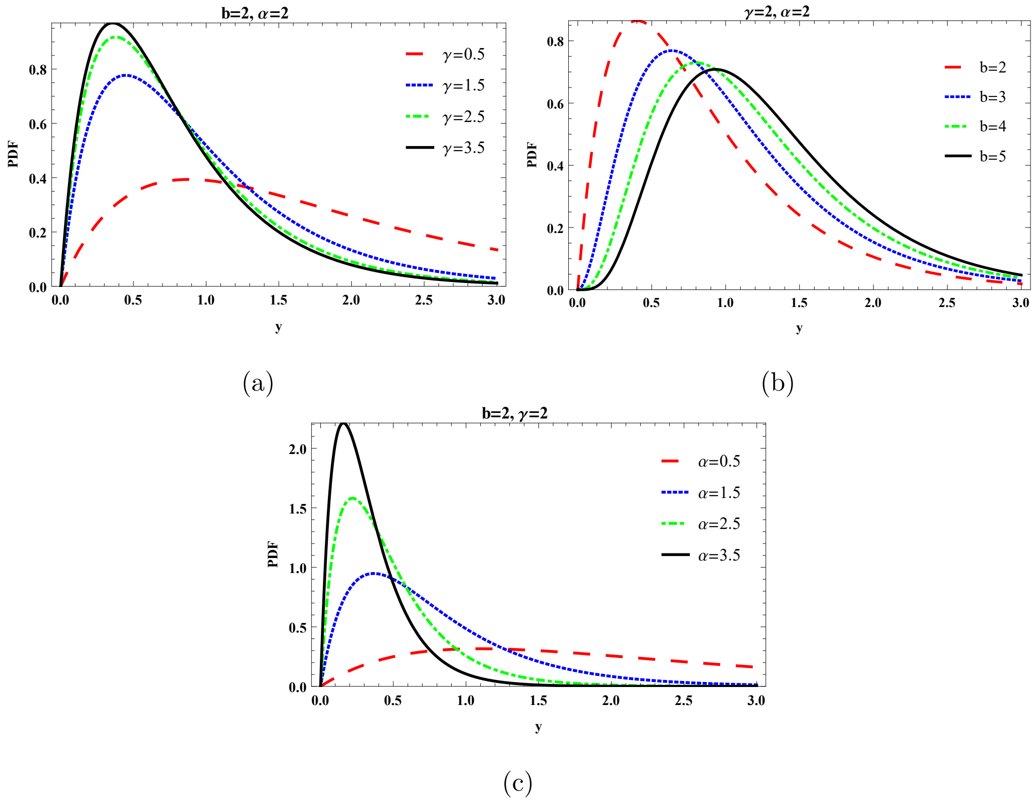


Figure 1. Plots of the pdf of TL-ETEx for different combinations of the parametric values.

Figure 1 shows the effect of different values of the parameters on the shape of the proposed distribution. For the increasing values of the shape parameter α, the curves get more peaked and distribution becomes right-skewed. It is also observed from the Figure 1(a) that α affects the tail weight of TL-ETEx distribution. As values of α increase, the tail weight also increases. For increasing values of b, the peaks of curves start decreasing and the curves move towards the right. It is also obvious from Figure 1(c) that there appears an abrupt shift in the peak of curves and they start increasing as the values of γ increase. The cdf and the pdf of the proposed model can be expressed in terms of the weighted sum of the exponentiated class of distributions. Using the following series representation by [23]

$$(1-z)^a = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(a+1)}{i!\Gamma(a+1-i)} z^i,$$

the distribution function of the TL-ETEx distribution is written as

$$F(y) = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b+1)}{i!\Gamma(b+1-i)} (e^{-2\alpha(1-e^{-\gamma})y})^i.$$

Similarly, the pdf of the proposed distribution is written as

$$f(y)=\sum_{i=0}^{\infty}\frac{(-1)^{i}2\Gamma(b+1)}{i!\Gamma(b-i)}\alpha(1-e^{-\gamma})(e^{-2\alpha(1-e^{-\gamma})y})^{i+1}. \tag{2.3}$$

Several characteristics such as moments, generating function, entropy, and order statistics can easily be computed by using the exponentiated form of the density function.

2.1. Quantile function

The q^{th} quantile function of the TL-ETEx distribution is given as

$$y_q=\frac{-\log(1-q^{\frac{1}{b}})}{2\alpha(1-e^{-\gamma})},\ \ q\in(0,1).$$

The quantile function is used to generate the random data. The median of the proposed model can be obtained by setting $q=\frac{1}{2}$ as

$$Median=\frac{-\log(1-0.5^{\frac{1}{b}})}{2\alpha(1-e^{-\gamma})},\ \ q\in(0,1).$$

2.2. Moments

Generally, the moments are an essential part of the distribution. They are used to find several important characteristics such as mean, variance, skewness, and kurtosis. The common method to find the raw moments or crude moments of any distribution is given as

$$\mu_k'=\int_{-\infty}^{\infty}y^k dF(x).$$

Using density function given in Equation (2.3) and assuming $w=-2\alpha(i+1)(1-e^{-\gamma})y$ we obtain the expression of moment of the proposed model given as

$$\mu_k'=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(b+1)}{i!\Gamma(b-i)}\left\{\frac{1}{i+1}\right\}^{k+1}\left\{\frac{1}{2\alpha(1-e^{-\gamma})}\right\}^{k}\Gamma(k+1). \tag{2.4}$$

For $k>0$, we can find different moments. The mean of TL-ETEx distribution is obtained by inserting $k=1$ expressed as

$$\mu_1'=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(b+1)}{i!\Gamma(b-i)}\left\{\frac{1}{i+1}\right\}^{2}\left\{\frac{1}{2\alpha(1-e^{-\gamma})}\right\}\Gamma(2).$$

The coefficient of variation (CV), the coefficient of skewness (CS), and the coefficient of kurtosis (CK) of the TL-ETEx distribution are obtained as

$$CV=\sqrt{\frac{\mu_2}{\mu_1}-1},$$

$$CS=\frac{\mu_3-3\mu_2\mu_1+2\mu_1^3}{(\mu_2-\mu_1)^{\frac{3}{2}}},$$

and

$$CK=\frac{\mu_4-4\mu_3\mu_1+6\mu_2\mu_1^2}{(\mu_2-\mu_1^2)^2}.$$

Now, the first incomplete moment is used to derive the mean deviation, Bonferroni, and Lorenz curves. These curves have great influences in economics, reliability, demography, insurance, and medicine. The incomplete moment of a distribution is defined by

$$\varphi_p(y)=\int_0^t y^p dF(y).$$

Using density function given in Equation (2.3), the incomplete moment of the TL-ETEx distribution is given as

$$\varphi_p = \sum_{i=0} \frac{(-1)^i \Gamma(b+1)}{i! \Gamma(b-i)} \left\{ \frac{1}{i+1} \right\}^{p+1} \left\{ \frac{1}{2\alpha(1-e^{-\gamma})} \right\}^{p} \gamma(p+1, 2\alpha(i+1)(1-e^{-\gamma})).$$

The mean deviation of Y about mean $[m_1 = E(|Y-\mu_1'|)]$ and mean deviation of Y about median $[m_2 = E(|Y-M|)]$ are given as $m_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $m_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(Y)$, $M = Median(Y) = Q(0.5)$, and $F(\mu_1')$ is calculated from Equation (2.4) and $\varphi_1(t)$ is the first incomplete moment with $p = 1$. These equations for $\varphi_1(t)$ can be used to obtain Bonferroni and Lorenz curves for the given probability π as $B(\pi) = \frac{\varphi_1(q)}{\pi \mu_1'}$ and $L(\pi) = \frac{\varphi_1(q)}{\mu_1'}$, respectively, where $\mu_1' = E(Y)$ and $q = Q(\pi)$ is quantile function of Y at π. The (q, s) probability weighted moment (PWM) of Y is defined by

$$\rho_{q,s} = \int_0^\infty y^q F(y)^s f(y) dy.$$

Following the above expression, the PWM of the TL-ETEx distribution is derived as

$$\rho_{q,s} = A_{j,k} b^* \Gamma(q+1),$$

where

$$A_{j,k} = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k} \Gamma(s+1) \Gamma(b+1)}{j!, k! \Gamma(s+1-j) \Gamma(b-k)},$$

and

$$b^* = \left\{ \frac{1}{j+k+1} \right\}^{q+1} \left\{ \frac{1}{2\alpha(1-e^{-\gamma})} \right\}^{q}.$$

2.3. An important distributional property

We present an important distributional property of TL-ETEx distribution as discussed by [13].

Theorem 2.1. *Let Y be a non-negative and absolutely continuous random variable with pdf*

$$f(y) = 2b\alpha(1-e^{-\gamma}) e^{-2\alpha(1-e^{-\gamma})y} \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1}, \quad y > 0, \ \alpha, \gamma, b > 0,$$

and n-th conditional moment as

$$E(Y^n | Y \leq z) = \frac{1}{F(z)} \int_0^z y^n 2b\alpha(1-e^{-\gamma}) e^{-2\alpha(1-e^{-\gamma})y} \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1} dy,$$

if and only if

$$F(z) = \left[1 - e^{-2\alpha(1-e^{-\gamma})z}\right]^b.$$

Proof of Theorem 2.1. For the necessary condition it can be easily proved that

$$E(Y^n | Y \leq z) = \int_0^z y^n \frac{h(y)}{F(z)} dy.$$

For the sufficient condition, we will proceed as

$$\int_0^z y^n \tfrac{h(y)}{F(z)} dy = \tfrac{1}{\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^b} \int_0^z y^n 2b\alpha(1-e^{-\gamma}) e^{-2\alpha(1-e^{-\gamma})y} \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1} dy. \quad (2.5)$$

From Equation (2.5), we have

$$\int_0^z y^n h(y) dy = F(z) \tfrac{1}{\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^b} \int_0^z y^n 2b\alpha(1-e^{-\gamma}) e^{-2\alpha(1-e^{-\gamma})y} \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1} dy. \quad (2.6)$$

Differentiating both sides of Equation (2.6) with respect to z, we get

$$z^n f(z) = f(z)I_1 + F(z)I_2 I^* + F(z)\frac{1}{F(y)}I_3.$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^b}\int_0^z y^n 2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})y}\left[1-e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1},\\
I_2 &= -\frac{1}{2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})z}\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^{b-1}},\\
I_3 &= z^n 2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})z}\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^{b-1},\\
I^* &= \int_0^z y^n 2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})y}\left[1-e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1}.
\end{aligned}$$

After some simplification, we have

$$f(z)\{z^n - W^*\} = F(z)B^*\{z^n - W^*\}$$

$$\frac{f(z)}{F(z)} = B^*, \tag{2.7}$$

where

$$B^* = \frac{2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})z}\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^{b-1}}{\left[1-e^{-2\alpha(1-e^{-\gamma})z}\right]^b}.$$

Integrating both sides of Equation (2.7) with respect to z from $z = y$ to ∞, we get

$$-\ln F(y) = -\ln\left[1-e^{-2\alpha(1-e^{-\gamma})y}\right]^b.$$

Finally, we have

$$F(y) = \left[1-e^{-2\alpha(1-e^{-\gamma})y}\right]^b.$$

□

2.4. Moment generating function and probability generating function

The moment generating function (mgf) of Y is defined as

$$M_t(Y) = \int_{-\infty}^{\infty} e^{yt}dF(y),$$

considering $e^{yt} = \sum_{r=0}^{\infty}\frac{t!y^r!}{r!}$ and using pdf given in Equation (2.3), the mgf of the proposed distribution is given as

$$M_t(Y) = \sum_{i,r=0}^{\infty}\frac{t!(-1)^i\Gamma(b+1)}{r!i!\Gamma(b-i)}\left\{\frac{1}{i+1}\right\}^{r+1}\left\{\frac{1}{2\alpha(1-e^{-\gamma})}\right\}^r\Gamma(r+1).$$

The general expression to find the probability generating function (pgf) is given as

$$\pi_y = \int_0^{\infty} t^y f(y)dy.$$

Using $t^y = \sum_{m=0}^{\infty}\frac{(\ln t)^m y^m}{m!}$, the pgf of the TL-ETEx distribution is given as

$$\pi_y = \sum_{i,r=0}^{\infty}\frac{(\ln t)^m(-1)^i\Gamma(b+1)}{m!i!\Gamma(b-i)}\left\{\frac{1}{i+1}\right\}^{m+1}\left\{\frac{1}{2\alpha(1-e^{-\gamma})}\right\}^m\Gamma(m+1).$$

2.5. Survival function

The survival function which is used to explain the probability of survival of an item beyond a given time t for any distribution is given by

$$R(y) = 1 - F(y),$$

so the $R(y)$ of the TL-ETEx distribution is derived as

$$R(y) = 1 - \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^b. \tag{2.8}$$

2.6. Hazard rate function

The hazard rate function of any probability distribution is given as

$$h(y) = \frac{f(y)}{1 - F(y)}.$$

The hazard rate function of TL-ETEx distribution is obtained as

$$h(y) = \frac{2b\alpha(1 - e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})y}\left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^{b-1}}{1 - \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^b}.$$

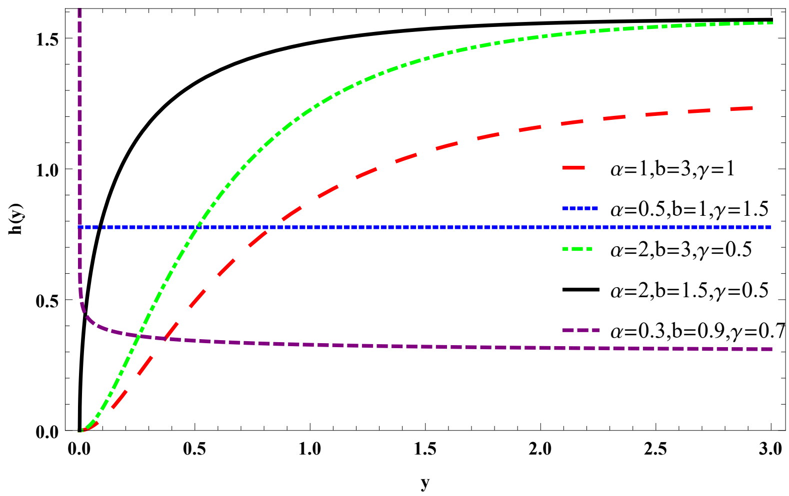


Figure 2. Hazard rate plots of TL-ETEx for different combinations of the parametric values.

The cumulative hazard rate of the proposed distribution is also given as

$$H(y) = -\log\left\{1 - \left[1 - e^{-2\alpha(1-e^{-\gamma})y}\right]^b\right\}.$$

2.7. Residual life and reversed residual life

The mean residual life (MRL) is the expected additional life length for a unit which survived till time y. The nth moment of the residual life is defined as

$$\pi_n(t) = E\left[(Y - t)^n | Y > t\right] = \frac{1}{R(t)} \int_t^\infty (Y - t)^n f(y) dy, \quad n = 1, 2, 3, \ldots$$

After some simplifications, the above expression reduces to

$$\pi_n(t) = \frac{1}{R(t)} \sum_{k,r=0}^{\infty} (-t)^{n-r} t_k \binom{n}{r} \int_t^\infty y^r f(y) dy.$$

We use Equation (2.3) to solve $\int_t^{\infty} y^r f(y)dy$. After some simplifications, we obtain

$$\pi_n(t) = \frac{1}{R(t)} \sum_{k,r,i,j=0}^{\infty} (-t)^{n-r} t_k \binom{n}{r} \delta_i \Gamma(p+1, 2\alpha(i+1)(1-e^{-\gamma})),$$

where

$$\delta_i = \sum_{i=0} \frac{(-1)^i \Gamma(b+1)}{i!\Gamma(b-i)} \left\{\frac{1}{i+1}\right\}^{p+1} \left\{\frac{1}{2\alpha(1-e^{-\gamma})}\right\}^{p}.$$

We can find the mean residual life (MRL) from the above equation by replacing $n = 1$. The nth moment of reserved residual life of any distribution is given as

$$\kappa_n(t) = E\big[(t-Y)^n \big| Y \le t\big] = \frac{1}{F(t)} \int_0^x (Y-t)^n f(y)dt, \quad n = 1, 2, 3, \ldots$$

The above expression can be written as

$$\kappa_n(t) = \frac{1}{R(t)} \sum_{k,r=0}^{\infty} (-1)^r (t)^{n-r} t_k \binom{n}{r} \int_0^t y^r f(y)dy.$$

The nth moment of the reversed residual life of the TL-ETEx distribution is given by

$$\kappa_n(t) = \frac{1}{R(t)} \sum_{k,r,j,i=0}^{\infty} (-1)^r (t)^{n-r} t_k \binom{n}{r} \delta_i \gamma(p+1, 2\alpha(i+1)(1-e^{-\gamma})).$$

3. Entropy and extropy

In this section, we compute some measures of entropy and extropy. Entropy measures the average amount of information provided by the findings of a random experiment whereas extropy is the complementary measure of entropy.

3.1. Renyi entropy

A generalized Renyi entropy of order θ introduced by [25] is given as

$$H_\theta(Y) = \frac{1}{1-\theta} \log \int_0^{\infty} (f(y))^\theta dy.$$

Using $f(y)$ given by Equation (2.2), the above expression for the TL-ETEx distribution is obtained as

$$H_\theta(Y) = \frac{1}{1-\theta} \log \int_0^{\infty} b^\theta (2\alpha(1-e^{-\gamma}))^\theta (e^{-2\alpha(1-e^{-\gamma})y})^\theta \big[1-e^{-2\alpha(1-e^{-\gamma})y}\big]^{\theta(b-1)} dy.$$

Using the series representation given in [23], the above expression is reduced as

$$H_\theta(Y) = \frac{1}{1-\theta} \log \int_0^{\infty} M_j b^\theta (2\alpha(1-e^{-\gamma}))^\theta (e^{-2\alpha(1-e^{-\gamma})y})^{\theta+j} dy,$$

where

$$M_j = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\theta(b-1)+1)}{j!\Gamma(\theta(b-1)+1-j)}.$$

Transforming $z = 2(\theta+j)\alpha(1-e^{-\gamma})y$, and performing some calculation the Renyi entropy of TL-ETEx distribution is given as

$$H_\theta(Y) = \frac{1}{1-\theta} \log \Big(M_j b^\theta (2\alpha(1-e^{-\gamma}))^{\theta-1}\Big).$$

3.2. Tsallis entropy

A generalized Tsallis entropy of order θ introduced by[27] is given as

$$S_\theta(Y) = \frac{1}{1-\theta}\left(1 - \int_0^\infty (f(y))^\theta dy\right).$$

Using Equation (2.2) and simplifying the above expression, the Tsallis entropy of the proposed distribution is obtained as

$$S_\theta(Y) = \frac{1}{1-\theta}\left(1 - \left(M_j b^\theta \left(2\alpha(1-e^{-\gamma})\right)^{\theta-1}\right)\right).$$

Both entropies have strong relationship with Shannon entropy. For $\theta \to 0$, both Renyi and Tsallis entropies approache to Shannon entropy.

Tsallis entropy plays a key role in several areas such as physics, chemistry, biology, medicine, and economics. Different applications of Tsallis entropy are discussed by [5]. The major difference between Renyi and Tsallis entropies is that the Tsallis entropy is nonextensive and nonlogarithmic whereas Renyi is extensive. The relationship between the both entropies can be expressed as

$$H_\theta(Y) = \frac{1}{1-\theta}\log[1 - (\theta-1)S_{\theta(Y)}].$$

3.3. Cumulative residual Tsallis entropy

The Cumulative Residual Tsallis Entropy (CRTE) of order θ defined by [26] is given as

$$\eta_\theta(Y) = \frac{1}{1-\theta}\left(1 - \int_0^\infty (R(y))^\theta dy\right),$$

where $R(y) = \bar{F}(y) = 1 - F(y)$. Considering Equation (2.8), the above expression is written as

$$\eta_\theta(Y) = \frac{1}{1-\theta}\left(1 - \int_0^\infty \left(1 - [1 - e^{-2\alpha(1-e^{-\gamma})y}]^b\right)^\theta dy\right).$$

Using the series representation given by [23], the above expression becomes

$$\eta_\theta(Y) = \frac{1}{1-\theta}\left(1 - \int_0^\infty L^*_{j,k} e^{-2\alpha(1-e^{-\gamma})yk} dy\right), \tag{3.1}$$

where

$$L^*_{j,k} = \sum_{j,k=0}^{\infty} (-1)^{j+k} \frac{\Gamma(\theta+1)(bj+1)}{j!k!\Gamma(\theta+1-j)(bj+1-k)}.$$

After solving Equation (3.1), the CRTE of TL-ETEx distribution is given as

$$\eta_\theta(Y) = \frac{1}{1-\theta}\left(1 - L^*_{j,k}\frac{1}{2k\alpha(1-e^{-\gamma})}\right).$$

3.4. Dynamic cumulative residual Tsallis entropy

For a random variable Y_t with survival function $R(y)$, the dynamic cumulative residual Tsallis entropy (DCRTE) of order θ is defined as

$$\eta_\theta(Y;t) = \frac{1}{1-\theta}\left(1 - \frac{1}{R(t)}\int_t^\infty (R(y))^\theta dy\right).$$

Considering Equation (2.8), the DCRTE of TL-ETEx distribution is given as

$$\eta_\theta(Y;t) = \frac{1}{1-\theta}\left(1 - \frac{1}{R(t)} L^*_{j,k} \frac{e^{\gamma+2\alpha\left(e^{-\gamma}-1\right)kt}}{2\alpha\left(e^\gamma - 1\right)k}\right).$$

3.5. Residual extropy

The residual extropy proposed by [24] as the measure of residual uncertainty of a random variable is

$$J(Y;t) = -\frac{1}{2R(t)^2}\int_t^{+\infty} f(y)^2 dy.$$

The residual extropy of TL-ETEx distribution is obtained by using Equation (2.2) and is given as

$$J(Y) = -T_j b^2 \alpha(1-e^{-\gamma})e^{-t}.$$

3.6. Cumulative residual extropy

A new measure of extropy called cumulative residual extropy (CRE) introduced by [11] is

$$\xi J(X) = -\frac{1}{2}\int_0^{\infty} R(y)^2 dy.$$

Using Equation (2.8), the CRE of the proposed distribution is obtained as

$$\xi J(X) = -t^*_{j,k}\frac{1}{4\alpha(1-e^{-\gamma})k},$$

where

$$t^*_{j,k} = \sum_{k=0}^{\infty}\sum_{j=0}^{3}(-1)^{j+k}\frac{\Gamma(bj+1)(3)}{j!k!\Gamma(bj+1-k)(3-j)}.$$

4. Inference

In this section, we discuss the estimation of the unknown parameters of the proposed distribution by employing the method of maximum likelihood estimation. Let a random sample, $y_1, y_2, \ldots, y_n$ of size n is drawn from the TL-ETEx distribution then the maximum likelihood estimates (MLEs) of its parameters are obtained by using the following likelihood (L) function

$$L = \prod_{i=1}^{n} 2b\alpha(1-e^{-\gamma})e^{-2\alpha(1-e^{-\gamma})y_i}\left[1-e^{-2\alpha(1-e^{-\gamma})y_i}\right]^{b-1},$$

and log-likelihood (l) function

$$\begin{aligned} l =& n\log 2 + n\log b + n\log\alpha + n\log(1-e^{-\gamma}) - 2\alpha(1-e^{-\gamma})\sum_{i=1}^{n} y_i \\ &+ (b-1)\sum_{i=1}^{n}\log\left[1-e^{-2\alpha(1-e^{-\gamma})y_i}\right]. \end{aligned} \tag{4.1}$$

Taking the partial derivatives of Equation (4.1) with respect to α,γ, and b and setting resulting equations equal to zero, we obtain a system of three equations in three unknowns parameters are given as

$$\frac{\partial L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left(1-e^{-2\alpha\left(1-e^{-\gamma}\right)y_i}\right) = 0. \tag{4.2}$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - 2\left(1-e^{-\gamma}\right)\sum_{i=1}^{n} y_i + (b-1)2\left(1-e^{-\gamma}\right)\sum_{i=1}^{n}\frac{y_i e^{-2\alpha\left(1-e^{-\gamma}\right)y_i}}{1-e^{-2\alpha(1-e^{-\gamma})y_i}} = 0. \tag{4.3}$$

$$\frac{\partial L}{\partial \gamma} = \frac{ne^{-\gamma}}{1-e^{-\gamma}} - 2\alpha e^{-\gamma}\sum_{i=1}^{n} y_i + (b-1)2\alpha e^{-\gamma}\sum_{i=1}^{n}\frac{y_i e^{-2\alpha\left(1-e^{-\gamma}\right)y_i}}{1-e^{-2\alpha(1-e^{-\gamma})y_i}} = 0. \tag{4.4}$$

It is important to point out here that the resulting analytical solution of the system of non linear Equations (4.2), (4.3), and (4.4) are unknown. So, the estimates of $\hat{b}$, $\hat{\alpha}$, and $\hat{\gamma}$ are obtained by solving the system of these equations using some numerical methods.

The Fisher Information matrix $I(b, \alpha, \gamma)$, assessed by taking the minus expectations of the second partial derivatives of Equations (4.2), (4.3), and (4.4), is expressed as

$$\begin{pmatrix} \hat{b} \\ \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} \sim N\left(\begin{pmatrix} b \\ \alpha \\ \gamma \end{pmatrix}\begin{pmatrix} \hat{J}_{bb} & \hat{J}_{b\alpha} & \hat{J}_{b\gamma} \\ & \hat{J}_{\alpha\alpha} & \hat{J}_{\alpha\gamma} \\ & & \hat{J}_{\gamma\gamma} \end{pmatrix}\right),$$

$$\frac{1}{J} = -E\begin{pmatrix} J_{bb} & J_{b\alpha} & J_{b\gamma} \\ & J_{\alpha\alpha} & J_{\alpha\gamma} \\ & & J_{\gamma\gamma} \end{pmatrix}.$$

The entries of the Fisher Information matrix are given as

$$\hat{J}_{bb} = -\frac{n}{b^2},$$

$$\hat{J}_{b\alpha} = \sum_{i=1}^{n}\frac{2\,(w)\,y_i e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}},$$

$$\hat{J}_{b\gamma} = \sum_{i=1}^{n}\frac{2\alpha e^{-\gamma} y_i e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}},$$

$$\hat{J}_{\alpha\alpha} = -\frac{n}{\alpha^2} - (b-1)\sum_{i=1}^{n}\frac{4\,(w)^2\,y_i^2 e^{-4\alpha(w)y_i}}{\left(1-e^{-2\alpha(w)y_i}\right)^2} - \sum_{i=1}^{n}\frac{4\,(w)^2\,y_i^2 e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}},$$

$$\begin{aligned}\hat{J}_{\alpha\gamma} =& -2e^{-\gamma}\sum_{i=1}^{n} y_i - (1-b)\sum_{i=1}^{n}\left(\frac{4\alpha e^{-\gamma}\,(w)\,y_i^2 e^{-4\alpha(w)y_i}}{\left(1-e^{-2\alpha(w)y_i}\right)^2}\right)\\ &-(b-1)\sum_{i=1}^{n}\left(\frac{4\alpha e^{-\gamma}\,(w)\,y_i^2 e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}} + \frac{2e^{-\gamma} y_i e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}}\right),\end{aligned}$$

$$\begin{aligned}\hat{J}_{\gamma\gamma} =& \frac{ne^{-\gamma}}{(w)^2} + 2\alpha e^{-\gamma}\left(\sum_{i=1}^{n} y_i\right) - (b-1)\sum_{i=1}^{n}\left(-\frac{4\alpha^2 e^{-2\gamma} y_i^2 e^{-4\alpha(w)y_i}}{\left(1-e^{-2\alpha(w)y_i}\right)^2}\right)\\ &-(b-1)\sum_{i=1}^{n}\left(\frac{4\alpha^2 e^{-2\gamma} y_i^2 e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}} + \frac{2\alpha e^{-\gamma} y_i e^{-2\alpha(w)y_i}}{1-e^{-2\alpha(w)y_i}}\right),\end{aligned}$$

where $1 - e^{-\gamma} = w$.

5. Simulation

In this section, we perform a simulation study to evaluate the performance of MLEs of the unknown parameters of the TL-ETEx distribution. We generate $N = 10000$ simulations of the TL-ETEx distribution for different sample sizes $n = 50, 250, 500, 1000$ using the following equation

$$y_q = \frac{-\log(1-q^{\frac{1}{b}})}{2\alpha(1-e^{-\gamma})}, \quad q \in (0, 1),$$

where U is a random variable that follows the uniform distribution on the interval $[0, 1]$. Table 1 lists the average estimated (AE)values of the parameters and their mean square errors (MSEs).

Table 1. Average estimated values of the parameters and their mean square errors for several combinations of the parameters of TL-ETEx distribution.

n	True values			AE			MSEs		
	α	b	γ	$\hat{\alpha}$	$\hat{b}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{b}$	$\hat{\gamma}$
50	0.5	0.5	1	0.5177	0.5313	1.0354	0.0368	0.0470	0.0791
	1.5	1.5	0.5	1.5193	1.5298	0.5414	0.0451	0.0741	0.0950
	2	1.5	2.5	2.0216	1.5222	2.5311	0.0535	0.0307	0.0454
	2.5	2	2.5	2.5181	2.0343	2.5311	0.0247	0.0849	0.0547
	2.5	0.5	2	2.5223	0.5271	2.0433	0.0563	0.0483	0.1024
250	0.5	0.5	1	0.5045	0.5069	1.0096	0.0160	0.01901	0.0338
	1.5	1.5	0.5	1.5051	1.5081	0.5103	0.0191	0.0258	0.0374
	2	1.5	2.5	2.0036	1.5065	2.5098	0.0069	0.0165	0.0443
	2.5	2	2.5	2.5054	2.0077	2.5099	0.0248	0.0297	0.0375
	2.5	0.5	2	2.5042	0.5078	2.0120	0.0091	0.0296	0.0548
500	0.5	0.5	1	0.5030	0.5037	1.0059	0.0159	0.0165	0.0287
	1.5	1.5	0.5	1.5026	1.5039	0.5059	0.0093	0.0157	0.0313
	2	1.5	2.5	2.0022	1.5036	2.5053	0.0060	0.0147	0.0199
	2.5	2	2.5	2.5032	2.0042	2.5055	0.0178	0.0232	0.0202
	2.5	0.5	2	2.5025	0.5044	2.0063	0.0086	0.0196	0.0341
1000	0.5	0.5	1	0.5013	0.5016	1.0034	0.0048	0.0053	0.0235
	1.5	1.5	0.5	1.5014	1.5023	0.5028	0.0054	0.0153	0.0149
	2	1.5	2.5	2.0015	1.5022	2.5033	0.0086	0.0103	0.0263
	2.5	2	2.5	2.5017	2.0023	2.5024	0.012	0.0159	0.0096
	2.5	0.5	2	2.5014	0.5024	2.0026	0.0066	0.0199	0.0094

Table 1 indicates that the estimates are quite stable and are close to the true values for different sample sizes. It is observed that in general the average estimated values of the parameters become close to the true values while the MSEs decrease as n increases.

6. Application

In this section, we study the applicability of the TL-ETEx distribution by means of three real data sets from the fields of hydrology, reliability, and environmental sciences. We compare the results of the proposed distribution with some existing distributions. We consider-2ℓ (maximized log-likelihood), AIC (Akaike information criterion), and BIC (Bayesian Information criterion) as the goodness of fit measures. The compared models are listed below in Table 2.

Table 2. Fitted distributions and their abbreviations.

Model	Abbreviations	Referrence
Topp Leone Erlang Truncated Exponential	TL-ETEx	Proposed
Erlang Truncated Exponential	ETEx	[7]
Exponential	Ex	[4]
Marshall-Oklin Erlang Truncated Exponential	MO-ETEx	[21]
Inverse Weibull	IEx	[12]
Topp Leone Inverse Weibull	TL-IW	[1]

The first data set represents the level of mercury in 34 albacores caught in the Eastern Mediterranean used by [15]. The second data set consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul by [10]. The third data set represents the waiting times (in minutes) before the service of 100 Bank customers which was examined and analyzed by [8].

The estimated parameters of the fitted distributions, -2ℓ, AIC and BIC are listed in Tables [3-5] for the selected three data sets, respectively. The values in these tables reveal that the TL-ETEx distribution provides better fits than other fitted models. The fitted density, empirical cdf, and total time on test (TTT) plots of the TL-ETEx distribution are displayed in Figures 3-5 for the three data sets, respectively. These figures show that the TL-ETEx distribution fits all data sets adequately. The TTT plot shows that all data sets have a monotonically increasing hazard rate function.

Table 3. Estimated parameters, -2ℓ, AIC, and BIC for the fitted distributions for data set I.

Model	Parameters			-2ℓ	AIC	BIC
	α	β	γ			
TL-ETEx	1.38714	17.2665	2.4083	20.2687	46.53747	51.1165
ETEx	1.1229	1.0662		44.40812	92.8162	95.8689
Ex	0.7363			44.40812	90.8162	92.3426
MO-ETEx	155.9432	4.0686	2.6041	20.7923	47.5847	52.1637
IW	1.2301			62.2802	126.5605	128.0869
TL-IW	2.3288	3.9152	0.4059	22.1965	50.3930	54.9721

Table 4. Estimated parameters, -2ℓ, AIC, and BIC for fitted distributions for data set II.

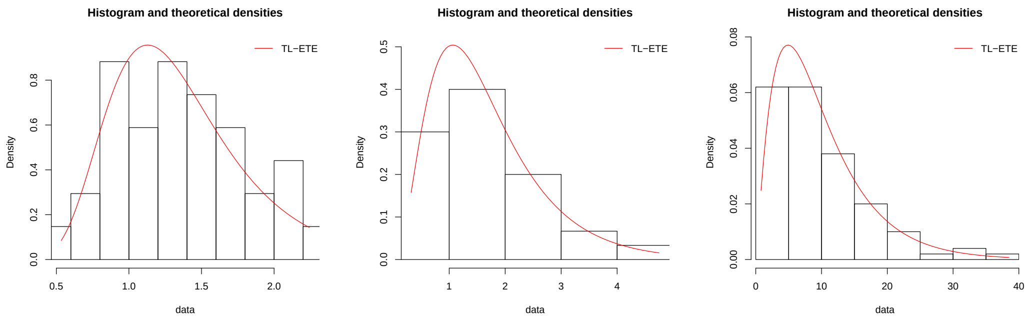
Model	Parameters			-2ℓ	AIC	BIC
	α	β	γ			
TL-ETEx	0.7157728	3.46107	1.6886	38.0942	82.1885	86.3921
ETEx	0.9596	0.9732		45.4744	94.9488	97.7512
Ex	0.7362973			44.40812	90.81625	92.34261
MO-ETEx	9.97431	4.8276	0.3885	39.377	84.754	88.9576
IW	0.8014			69.0502	140.1003	141.5015
TL-IW	1.2951	3.5629	0.3767	40.1633	86.3266	90.5302

Table 5. Estimated parameters, -2ℓ, AIC, and BIC for fitted distributions for data set III.

Model	Parameters			-2ℓ	AIC	BIC
	α	β	γ			
TL-ETEx	0.2508	2.1826	0.3814	317.0953	640.1906	648.0061
ETEx	0.2043	0.6842		329.0209	662.0418	667.2521
Ex	0.1012			329.0209	660.0418	662.6469
MO-ETEx	4.1164	1.3268	0.15662	320.712	647.4241	655.2396
IW	0.1922			554.9761	1111.952	1114.557
TL-IW	0.9567	12.5285	0.5246	327.1056	660.2112	668.0267

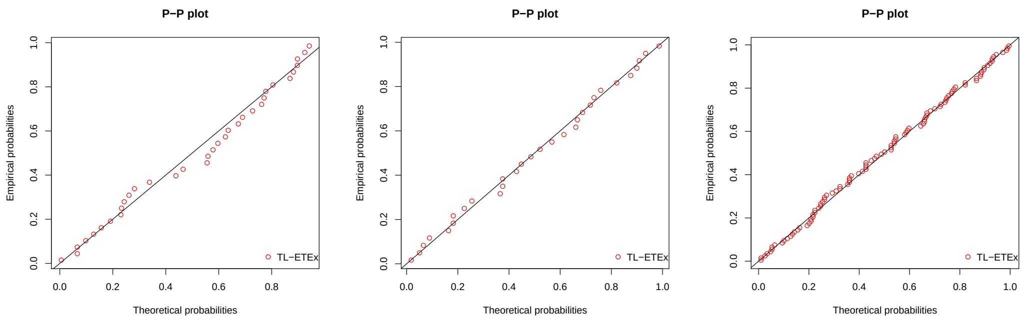
7. Conclusion

We propose a new lifetime distribution called Topp Leone Erlang truncated exponential (TL-ETEx) distribution by generalizing the Erlang truncated exponential distribution using Topp Leone family of distributions. Several important characteristics of the proposed distribution are derived and discussed. The moments of TL-ETEx distribution help to study its behavior while the graphical representation describes the impact of different parametric combinations on the peak and the skewness of the proposed distribution. It is observed from the reliability analysis that the TL-ETEx distribution has increasing, decreasing and constant failure rates which indicates its ability to handle the versatile situations. Various types of entropies and extropies are derived which help to find the average amount of uncertainty in the information provided by the random trials. The



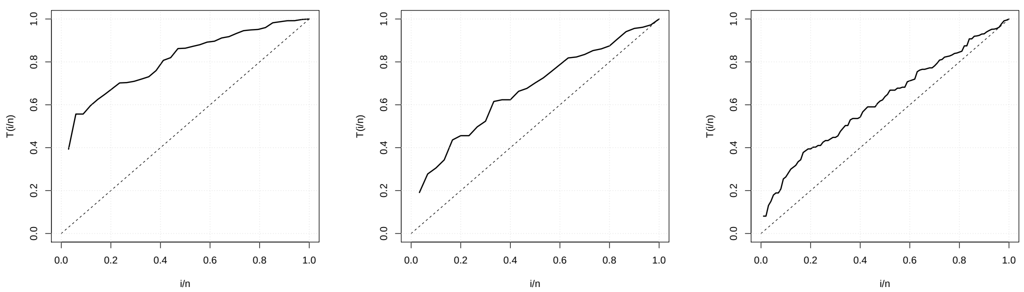
Data set I Data set II Data set III

Figure 3. Fitted density of the TL-ETEx distribution for three data sets.



Data set I Data set II Data set III

Figure 4. Probability plots of the TL-ETEx distribution for three data sets.



Data set I Data set II Data set III

Figure 5. TTT plots of the TL-ETEx distribution for three data sets.

findings of the simulation study reveal that the estimated values of the parameters are close to their true values which establish their stability. It is also noticed that the estimated values become closer to the true values where as the biases and standard errors decrease as the samples sizes increase. Three real data sets from hydrology, reliability, and environmental sciences are modeled by using the TL-ETEx distribution that endorse its vast application.

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