POLİTEKNİK DERGİSİ

JOURNAL of POLYTECHNIC

ISSN: 1302-0900 (PRINT), ISSN: 2147-9429 (ONLINE)
URL: http://dergipark.gov.tr/politeknik

On metric contact pairs with certain semi-symmetry conditions

Belirli yarı-simetri koşulları altında metrik kontakt çiftler üzerine

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Bu makaleye şu şekilde atıfta bulunabilirsiniz(To cite to this article): Ünal, İ., "On metric contact pairs with certain semi-symmetry conditions", ***Politeknik Dergisi***, 24(1): 333-338, (2021).

Erişim linki (To link to this article): http://dergipark.org.tr/politeknik/archive

DOI: **10.2339/politeknik.769662**

On Metric Contact Pairs with Certain Semi-Symmetry Conditions

Highlights

- *Some significant contributions to the Riemannian geometry of contact pair manifolds were obtained.*
- *Semi-symmetry conditions on Riemannian and Ricci curvature tensors were considered for normal metric contact pair manifolds.*
- *Many classifications were given based on the semi-symmetry conditions of the concircular curvature tensor.*

Aim

The aim of this paper is to study on the normal metric contact pair manifolds under some semi-symmetry conditions.

Design & Methodology

The theoretical methodology of mathematics was used to obtain results.

Originality

All obtained results are original. They could be used for future works about contact pair geometry.

Findings

Normal metric contact pair manifolds are classified as a generalized quasi-Einstein manifold with certain semi-symmetry conditions.

Conclusion

It is seen that normal metric contact pair manifold could be special case of generalized quasi-Einstein manifolds which arises from general relativity.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

On Metric Contact Pairs with Certain Semi-Symmetry Conditions

Araştırma Makalesi / Research Article

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(Geliş/Received : 14.07.2020 ; Kabul/Accepted : 20.08.2020)

ABSTRACT

Blair et al. [7] introduced the notion of bicontact manifold in the context of Hermitian geometry. Bande and Hadjar [1] studied on this notion under the name of contact pairs. These type of structures have important properties and their geometry is some different from classical contact structures. In this paper, we study on some semi-symmetry properties of the normal contact pair manifolds. We prove that a Ricci semi-symmetric (or concircularly Ricci semi-symmetric) normal metric contact pair manifold is a generalized quasi-Einstein manifold. Also, we classify normal metric contact pair manifolds as a generalized quasi-Einstein manifold with certain semi-symmetry conditions $\mathrm{L}(X,Z).R=0,\ R(X,Z).\mathrm{L}=0$ and $\mathrm{L}(X,Z).\mathrm{L}=0$ for the concircular curvature tensor L , the Riemannian curvature tensor R , and an arbitrary vector field X .

Keywords: Contact metric pair, bicontact, curvature properties, symmetry conditions.

Belirli Yarı-Simetri Koşulları Altında Metrik Kontakt Çiftler Üzerine

ÖZ

Blair vd. [7] Hermityan geometri durumu için bikontakt manifold tanımını verdiler. Bande ve Hadjar [1] bu kavramı kontakt çift ismi ile çalıştılar. Bu yapılar bazı önemli özelliklere sahip olmakla birlikte geometrileri, klasik kontakt geometriden farklılıklar göstermektedir. Bu çalışmada normal metrik kontakt çiftler üzerinde bazı yarı-simetri koşulları altında çalışılmıştır. Bir Ricci yarı-simetrik (ve concircular Ricci yarı-simetrik) normal metrik kontakt çift manifoldun genelleştirilmiş yarı-Einstein manifold olduğu ispatlanmıştır. Ayrıca, normal kontakt çift manifoldlar, L concircular eğrilik tensörü kullanılarak R Riemann eğrilik tensörü olmak üzere her X keyfi vektör alanı için $\mathrm{L}(X,Z).R=0,\ R(X,Z).\mathrm{L}=0$ ve $\mathrm{L}(X,Z).\mathrm{L}=0$ yarı-simetri koşulları altında genelleştirilmiş yarı-Einstein manifoldları olarak sınıflandırılmıştır.

Anahtar Kelimeler: Kontakt metrik çift, bikontakt, eğrilik özellikleri, simetri koşulları.

1. INTRODUCTION

A contact manifold is a $(2n+1)-$dimensional differentiable manifold with a contact 1-form η which satisfies $\eta\wedge(d\eta)^n\neq 0$ [9]. In 1960s this type of manifolds were studied with tensorial approach. For this aim, the almost contact structure was given as similar to almost complex structures. After that, more researcher took their attention to this subject, and the contact geometry has been developed dramatically. An almost contact structure is defined as follow;

Let M be a $(2n+1)$-odd dimensional differentiable manifold and ϕ be a (1,1)-tensor field on M. If we have,

$$\phi^2=-I+\eta\otimes\xi,\ \eta(\xi)=1,\ \phi\xi=0$$

for a vector field ξ on M , then (ϕ,η,ξ) is called an almost contact structure [9]. (M,ϕ,η,ξ) is called as an almost contact manifold.

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An almost complex manifold is a complex manifold if the Nijenhuis tensor of almost complex structure J is zero, that means J is integrable. We have a complex structure J on the product manifold $M\times\mathbb{R}$. Similarly if J is integrable then (M,ϕ,η,ξ) is called normal contact manifold or Sasaki manifold. Also there are many subclasses of almost contact manifolds which are normal or not. We refer to reader [9,17] for more details on the contact geometry.

Blair et al. [7] worked on Calabi-Eckman manifold similar to complex manifolds. They gave a new kind of manifolds with a new structure, which was called by bicontact manifolds. In 2000s, this type of manifolds have been studied by several authors with the name of "contact pairs"([1], [2]). Bande and Hadjar [3] gave the normality of contact pairs and they studied on curvature properties [4] with Blair. In [5] Blair et al. worked on locally symmetric normal metric contact pair (NMCP) manifolds and they obtained some classifications. Presented author worked on normal metric contact pair manifolds under some flatness conditions [13]. Also he has presented some applications of generalized quasi-Einstein manifolds in contact geometry [14].

In this paper, we examine NMCP manifolds under the semi-symmetry conditions $\mathrm{L}(X_1,Z).R=0$, $R(X_1,Z).\mathrm{L}=0$, $\mathrm{L}(X_1,Z).Ric=0$ and $\mathrm{L}(X_1,Z).\mathrm{L}=0$ for the concircular curvature tensor L, Riemannian curvature tensor R and the Ricci curvature tensor Ric. In the first section, we give some fundamental facts on NMCP manifolds. In the second section, we consider a Ricci-semi symmetric and a concircular semi-symmetric NMCP manifold. Finally, in the last two sections, we give some results on NMCP manifolds under certain semi-symmetry conditions with related to concircular curvature tensor.

2. A SHORT REVIEW ON CONTACT PAIRS

Definition 2.1.

Let M be a $(2p+2q+2)$- dimensional differentiable manifold and α_1,α_2 be two 1-forms on M. If the following conditions are satisfied, then (α_1,α_2) is called contact pair of type (p,q) and (M,α_1,α_2) is called contact pair manifold;

$$\alpha_1\wedge(d\alpha_1)^p\wedge\alpha_2\wedge(d\alpha_2)^q\neq 0$$

$$(d\alpha_1)^{p+1}=0,\ (d\alpha_2)^{q+1}=0$$

where p,q are positive integers [1].

The kernels of these two forms defined two integrable distributions D_1 and D_2. Also we have two characteristic foliations of M by $F_1=D_1\cap\ker d\alpha_1$ and $F_2=D_2\cap\ker d\alpha_2$

F_1 and F_2 are contact manifolds with 1-forms which are induced from α_2,α_1. We have the Reeb vector fields Z_1 and Z_2 are given by

$$\alpha_1(Z_1)=\alpha_2(Z_2)=1,\ \alpha_1(Z_2)=\alpha_2(Z_1)=0.$$

For the subbundles

$$TG_i=\ker d\alpha_i\cap\ker\alpha_1\cap\ker\alpha_2$$

we have

$$TF_i=TG_i\oplus\mathbb{R}Z_j,\ 1\le i,j\le 2, i\neq j.$$

Thus, we get the decomposition of TM by

$$TM=TG_1\oplus TG_2\oplus\mathbb{R}Z_1\oplus\mathbb{R}Z_2.$$

We call $H=TG_1\oplus TG_2$ as horizontal subbundle and $V=\mathbb{R}Z_1\oplus\mathbb{R}Z_2$ as vertical subbundle.

Definition 2.2.

Let M be $(2p+2q+2)$-dimensional differentiable manifold and g be a Riemannian metric on M. Then M is called metric almost contact pair manifold with a triple α_1,α_2,ϕ such that

$$\phi^2=-I+\alpha_1\otimes Z_1+\alpha_2\otimes Z_2, \qquad (2.1)$$
$$\phi Z_1=\phi Z_2=0$$

where X_1,X_2 are arbitrary vector fields on M [2].

Also the compatible metric of almost contact pair structure is given by

$$g(\phi X_1,\phi X_2)=g(X_1,X_2)-\alpha_1(X_1)\alpha_1(X_2)-\alpha_2(X_1)\alpha_2(X_2).$$

$(M,\alpha_1,\alpha_2,\phi,g,Z_1,Z_2)$ is said to be a metric almost contact pair manifold. ϕ is called decomposable if $\phi TF_i\subset TF_i$ for $1\le i\le 2$. When ϕ is decomposable then (α_1,Z_1,ϕ_1) (resp. (α_2,Z_2,ϕ_2)) is an almost contact structure on TF_2 $(resp.TF_1)$.

Remark: In the whole of our paper, we suppose that ϕ is decomposable.

From (2.1) we obtain

$$g(X_1,Z_i)=\alpha_i(X_1),\quad g(Z_i,Z_j)=\delta_{ij},\quad g(Z,Z)=2$$

For $Z=Z_1+Z_2$. Also, on M we have

$$\nabla_{Z_i}^{Z_j}=0,\nabla_{Z_i}^{\phi}=0$$

and for every $X_1\in TF_i$ we have

$$\nabla_{X_1}^{Z_1}=-\phi_1X_1,\ \nabla_{X_1}^{Z_2}=-\phi_2X_1$$

where $\phi=\phi_1+\phi_2$ and $Z=Z_1+Z_2$.

As known the normality of almost contact structure is an important notion for the Riemannian geometry of contact manifolds. The normality of metric almost contact pair structure was given by Bande and Hadjar [3]. If the following two tensors are integrable then metric almost contact metric pair manifold $(M,\alpha_1,\alpha_2,\phi,Z_1,Z_2,g)$ is called normal;

$$J=\phi-\alpha_2\otimes Z_1+\alpha_1\otimes Z_2,$$

$$T=\phi+\alpha_2\otimes Z_1-\alpha_1\otimes Z_2.$$

Thus we recall the manifold by NMCP manifold.

The curvature properties of a normal metric contact pair manifold were given by Bande at al. [4]. For X_1,X_2 horizontal vector fields we have

$$R(X_1,Z)X_2=-g(X_1,X_2)Z \qquad (2.2)$$

$$R(X_1,Z)Z=X_1 \qquad (2.3)$$

where R is the Riemann curvature tensor of M which is given by

$$R(X_1,X_2)X_3=\nabla_{X_1}\nabla_{X_2}^{X_3}-\nabla_{X_2}\nabla_{X_1}^{X_3}-\nabla_{[X_1,X_2]}^{X_3}$$

for all vector fields X_1, X_2, X_3 on M. The Ricci curvature of M has following properties [5];

$$Ric(X_1, Z) = 0, \text{ for } X_1 \in \Gamma(\mathrm{H}), \tag{2.4}$$

$$Ric(Z, Z) = 2p + 2q,\ Ric(Z_1, Z_1) = 2p, \tag{2.5}$$

$$Ric(Z_2, Z_2) = 2q, Ric(Z_1, Z_2) = 0.$$

Also from the decomposition of TM, we have

$$X_1 = X_1^{\,1} + X_1^{\,2}$$

such that

$$X_1^{\,1} = X_1^{1^{\mathrm{H}}} + \alpha_2(X_1)Z_2,\quad X_1^{\,2} = X_1^{2^{\mathrm{H}}} + \alpha_1(X_1)Z_1,$$

$$X_1^{\mathrm{H}} = X_1^{1^{\mathrm{H}}} + X_1^{2^{\mathrm{H}}},\quad X_1^{1^{\mathrm{H}}} \in T\mathcal{G}_1 \text{ and } X_1^{2^{\mathrm{H}}} \in T\mathcal{G}_2,$$

(for details see[13]).

Generalized quasi-Einstein manifolds are an important class of Riemannian manifolds. This type of manifolds were defined by De and Ghosh [10]. The definition of a generalized quasi-Einstein NMCP manifold was given by Ünal as follow:

Definition 2.3.

An NMCP manifold $(M, \alpha_1, \alpha_2, \phi, Z_1, Z_2, g)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor is not identically zero and satisfies;

$$\begin{aligned} \mathrm{Ric}(X_1, X_2) &= \lambda \mathrm{g}(X_1, X_2) \\ &+ (2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) \\ &+ (2q - \lambda)\alpha_2(X_1)\alpha_2(X_2) \end{aligned} \tag{2.6}$$

for all vector fields X_1, X_2 on M and λ is non-zero scalar [14].

The following lemma comes from the decomposition of TM :

Lemma 2.4.

A normal metric contact pair manifold is generalized quasi-Einstein manifold if and only if the horizontal bundle is Einstein i.e for a non-zero scalar λ on M, we have $Ric(X_1^{\mathrm{H}}, X_2^{\mathrm{H}}) = \lambda g(X_1^{\mathrm{H}}, X_2^{\mathrm{H}})$ [14].

The concircular curvature tensor on Riemannian manifold was defined by Yano in[15]. On a $(2\mathrm{p} + 2\mathrm{q} + 2)$ -dimensional normal metric contact pair manifold is defined by

$$\begin{aligned} \mathrm{L}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 \\ &- K[g(X_2, X_3)X_1 \\ &- g(X_1, X_3)X_2] \end{aligned} \tag{2.7}$$

for all X_1, X_2, X_3 vector fields on M, where $K = \dfrac{scal}{(2p + 2q + 2)(2p + 2q + 1)}$.

On the other hand for two $(1,3)-$ type tensors $\mathrm{T}_1, \mathrm{T}_2$ we have

$$\begin{aligned} (\mathrm{T}_1(X_1, X_2).\mathrm{T}_2)(X_3, X_4)X_5 &= \mathrm{T}_1(X_1, X_2)\mathrm{T}_2(X_3, X_4)X_5 \\ &- \mathrm{T}_2(\mathrm{T}_1(X_1, X_2)X_3, X_4)X_5 \\ &- \mathrm{T}_2(X_3, \mathrm{T}_1(X_1, X_2)X_4)X_5 \\ &- \mathrm{T}_2(X_3, X_4)\mathrm{T}_1(X_1, X_2)X_5 \end{aligned} \tag{2.8}$$

and for $(0,2)-$type tensor ω we have

$$\begin{aligned} (\mathrm{T}_1(X_1, X_2).\omega)(X_3, X_4) &= \omega(\mathrm{T}_1(X_1, X_2)X_3, X_4) \\ &+ \omega(X_3, \mathrm{T}_1(X_1, X_2)X_4). \end{aligned} \tag{2.9}$$

For all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$.

3. SOME SEMI-SYMMETRY CONDITIONS ON NMCP MANIFOLDS

A Riemann manifold is called a locally symmetric or a semi-symmetric manifold if $R.R = 0$ [14]. The contact manifolds are characterized by using semi-symmetry properties. Blair et al.[8] classified $N(k)$-contact metric manifolds under the semi-symmetry condition $\mathrm{L}(X,\xi).\mathrm{L} = 0$. Ünal and Altın [15] studied on $N(k)$-contact metric manifolds under the semi-symmetry conditions $\mathrm{L}(X,\xi).R = 0$ and $R(X,\xi).\mathrm{L} = 0$ with a special connection. In [12] Turgut Vanlı and Ünal worked on complex contact manifolds with semi-symmetry conditions $\mathrm{L}(X,\xi).R = 0$ and $R(X,\xi).\mathrm{L} = 0$. Blair and Bande [6] studied on semi-symmetric NCMP manifolds. With motivated all this studies we consider the conditions $\mathrm{L}(X, Z).R = 0$, $R(X, Z).\mathrm{L} = 0$ and $\mathrm{L}(X, Z).\mathrm{L} = 0$ for NMCP manifolds.

3.1. NMCP Manifolds Satisfying $R(X, Z).Ric = 0$ and $\mathrm{L}(X, Z).Ric = 0$

An NMCP manifold is called by a Ricci-semi symmetric, if $R(X, Z).Ric = 0$. For a Ricci-semi symmetric NMCP manifold we have following result.

Theorem 3.1

A Ricci semi-symmetric NMCP manifold is a generalized quasi-Einstein manifold.

Proof.

Let $(M, \alpha_1, \alpha_2, \phi, Z_1, Z_2, g)$ *be a Ricci semi-symmetric NMCP manifold. Then from* (2.9) *we have*

$$\mathrm{Ric}(\mathrm{R}(X_1, X_2)X_3, X_4) + \mathrm{Ric}(X_3, \mathrm{R}(X_1, X_2)X_4) = 0.$$

By setting $X_2 = X_3 = Z$ *and* $X_1 = X_1^{\mathrm{H}}, X_4 = X_4^{\mathrm{H}}$ *and from* (2.2), (2.3) *we get*

$$\mathrm{Ric}\left(X_1^{\mathrm{H}}, X_4^{\mathrm{H}}\right) + \mathrm{Ric}\left(Z, -g(X_1^{\mathrm{H}}, X_4^{\mathrm{H}})Z\right) = 0 .$$

Thus, we have

$$Ric(X_1^{\mathrm{H}}, X_4^{\mathrm{H}}) + -g(X_1^{\mathrm{H}}, X_4^{\mathrm{H}})Ric(Z,Z) = 0 .$$

Thus from (2.5), *we obtain*

$$Ric(X_1^{\mathrm{H}}, X_4^{\mathrm{H}}) = (2p+2q)\, g(X_1^{\mathrm{H}}, X_4^{\mathrm{H}}) .$$

From Lemma 2.4 we get

$$\begin{aligned} Ric(X_1, X_4) &= (2p+2q)\, g(X_1, X_4) \\ &+ 2p\alpha_1(X_1)\alpha_1(X_4) \\ &+ 2q\alpha_2(X_1)\alpha_2(X_4) \end{aligned}$$

for arbitrary vector fields X_1, X_4 on M. Thus M is a generalized quasi-Einstein manifold, the proof is completed.

Lemma 3.2

Let M be an NMCP manifold. For all X_1, X_2, X_3 vector fields on M, we have

$$\begin{aligned} \mathrm{L}(X_1, Z,)X_3 &= (Kg(X_1, X_3) - g(\phi X_1, \phi X_3))Z \\ &- K(g(X_3, Z)X_1, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \mathrm{L}(X_1, X_2)Z &= R(X_1, X_2)Z \\ &- K\left(g(X_2, Z)X_1 - g(X_1, Z)X_2\right), \end{aligned} \quad (3.2)$$

$$\mathrm{L}(X_1, Z)Z = \phi^2 X_1 - K(2X_1 - g(X_1, Z)Z) \quad (3.3)$$

where $K = \dfrac{scal}{(2p+2q+2)(2p+2q+1)}$.

Proof.

Using (2.2) and (2.3) in the definition of L *one can obtain (3.1), (3.2) and (3.3).*

We recall an NMCP manifold by concircular Ricci-semi symmetric if $\mathrm{L}(X_1, Z).Ric = 0$.

Theorem 3.3.

A concircular Ricci-semi symmetric NMCP manifold is a generalized quasi-Einstein manifold.

Proof.

Suppose that M satisfies $\mathrm{L}(X_1, Z).Ric = 0$ for all X_1. Then, from (2.9) we get

$$0 = (Ric(\mathrm{L}(X_1, Z)X_3, X_4) + Ric(X_3, \mathrm{L}(X_1, Z)X_4).$$

Let take $X_1, X_4 \in \Gamma(\mathrm{H})$ and $X_3 = Z$. Thus, from (3.1) we obtain

$$0 = (Ric((1-2K)X_1, X_4) + (K-1)g(X_1, X_4)Ric(Z,Z) .$$

By using (2.5), we get

$$Ric(X_1, X_4) = \frac{(1-K)(2p+2q)}{1-2K} g(X_1, X_4) .$$

Therefore, by consider the Lemma 2.4 the manifold becomes a generalized quasi-Einstein manifold.

3.2. NMCP Manifolds Satisfying $\mathrm{L}(X_1, Z).R = 0$, $R(X_1, Z).\mathrm{L} = 0$ and $\mathrm{L}(X_1, Z).\mathrm{L} = 0$

Theorem 3.4.

Let M be an NMCP manifold satisfying $\mathrm{L}(X_1, Z).R = 0$. Then, the scalar curvature of M is $scal = (2p+2q+2)(2p+2q+1)$ or the sectional curvature of M is 1.

Proof.

Let M be a normal metric contact pair manifold satisfying $\mathrm{L}(X_1, Z).R = 0$. From (2.8), we have

$$\begin{aligned} 0 &= \mathrm{L}(X_1, Z)R(X_3, X_4)X_5 - R(\mathrm{L}(X_1, Z)X_3, X_4)X_5 \\ &- R(X_3, \mathrm{L}(X_1, Z)X_4)X_5 - R(X_3, X_4)\mathrm{L}(X_1, Z)X_5 \end{aligned}$$

for all $X_1, X_3, X_4, X_5 \in \Gamma(TM)$. By using (3.1) and (3.2) we obtain

$$\begin{aligned} 0 = (K-1)\{ & g(X_1, R(X_3, X_4)X_5)Z \\ & - g(X_1, X_3)g(X_3, X_4)Z \\ & - g(X_1, X_4)g(X_3, X_5)Z \\ & - g(X_1, X_5)R(X_3, X_4)Z\}. \end{aligned}$$

Thus we have two case. If $K = 1$ then the scalar curvature is $scal = (2p+2q+2)(2p+2q+1)$.

Suppose that $K \neq 1$, then we have

$$\begin{aligned} 0 &= g(X_1, R(X_3, X_4)X_5)Z - g(X_1, X_3)g(X_3, X_4)Z \\ &- g(X_1, X_4)g(X_3, X_5)Z - g(X_1, X_5)R(X_3, X_4)Z. \end{aligned}$$

By taking inner product with Z, we obtain

$$\begin{aligned} R(X_3, X_4, X_5, X_1) &= g(X_1, X_3)g(X_3, X_4) \\ &- g(X_1, X_4)g(X_3, X_5) \end{aligned}$$

which provides that the sectional curvature is 1.

Theorem 3.5.

A normal metric contact pair manifold satisfying $R(X_1, Z).\mathrm{L} = 0$ is a space of constant curvature.

Proof.

Let M be an NMCP manifold satisfying $R(X_1, Z).\mathrm{L} = 0.$ From (2.8) *we have*

$$\begin{aligned} 0 &= R(X_1, Z)\mathrm{L}(X_3, X_4)X_5 - \mathrm{L}(R(X_1, Z)X_3, X_4)X_5 \\ &- \mathrm{L}(X_3, R(X_1, Z)X_4)X_5 - \mathrm{L}(X_3, X_4)R(X_1, Z)X_5 \end{aligned}$$

for all $X_1, X_3, X_4, X_5 \in \Gamma(TM)$. By using (3.1), (3.2) *and taking inner product with Z, we get*

$$g(X_1, \mathrm{L}(X_3, X_4)X_5) = -(K-1)g(X_1, X_3)g(X_4, X_5) + (K-1)g(X_1, X_4)g(X_3, X_5).$$

Also from the definition of concircular curvature tensor, we obtain

$$R(X_3, X_4, X_5, X_1) = (-2K+1)\big(\big(g(X_1, X_4)g(X_3, X_5) - g(X_1, X_3)g(X_4, X_5)\big) \quad (3.4)$$

which provides that M is a space of constant curvature.

By using curvature properties of an NMCP manifold we get the following Lemma.

Lemma 3.6

Let M be an NMCP manifold. Then, we have

$$d\alpha_j(\phi X, Z_i) = 0 \quad 1 \le i, j \le 2 \quad (3.5)$$

$$R(X, Z_1)Z_1 = \phi_1 X_1 \quad (3.6)$$

$$R(X, Z_1)Z_2 = -\phi_2\phi_1 X_1 \quad (3.7)$$

$$R(X, Z_2)Z_1 = -\phi_1\phi_2 X_1 \quad (3.8)$$

$$R(Z_i, Z_j)Z_k = 0, \quad 1 \le i, j, k \le 2 \quad (3.9)$$

$$R(X_1, Z_1, Z_2, X_2) + R(X_1, Z_1, Z_2, X_2) = g(\phi_1 X_1, \phi_1 X_2) + g(\phi_2 X_1, \phi_2 X_2) \quad (3.10)$$

for all $X_1, X_2 \in \Gamma(TM)$.

Let $S = \{e_1, e_2, ..., e_n, \phi e_1, \phi e_2, ..., \phi e_n, Z_1, Z_2\}$ be a local orthonormal basis of TM . Then, we have

$$\sum_{i=1}^{2p+2q} g(X_1, E_i)g(X_2, E_i) = g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2) \quad (3.11)$$

and

$$Ric(X_1, X_2) = \sum_{i=1}^{2p+2q} R(X_1, E_i, E_i, X_2) + R(X_1, Z_1, Z_1, X_2) + R(X_1, Z_2, Z_2, X_2)$$

for $E_i \in S$, $X_1, X_2 \in \Gamma(TM)$. By using (3.10) we obtain

$$\sum_{i=1}^{2p+2q} R(X_1, E_i, E_i, X_2) = Ric(X_1, X_2) - g(\phi_1 X_1, \phi_1 X_2) - g(\phi_2 X_1, \phi_2 X_2). \quad (3.12)$$

Corollary 3.7

Let M be an NMCP manifold satisfying $R(X_1, Z).\mathrm{L} = 0$ for any $X_1 \in \Gamma(TM)$. Then, M is a generalized quasi-Einstein manifold.

Proof.

Let M be an NMCP manifold satisfying for any $X_1 \in \Gamma(TM)$. *Then, by setting* $X_4 = X_5 = E_i$ *in* (3.4) *and taking sum over i , we obtain*

$$\sum_{i=1}^{2p+2q} R(X_3, E_i, E_i, X_1) = (1-2K)\sum_{i=1}^{2p+2q}\big(g(X_3, E_i)g(X_1, E_i) - g(X_3, X_1)g(E_i, E_i)\big).$$

Thus from (3.11) *and* (3.12) *, we get*

$$Ric(X_3, X_1) = (1-2K)(1-2p-2q)g(X_3, X_1) + g(\phi_1 X_3, \phi_1 X_1) + g(\phi_2 X_3, \phi_2 X_1).$$

Let take $X_1, X_3 \in T\mathrm{F}_1 \cap T\mathrm{F}_2 \cap \mathrm{H}$, *since* ϕ_i *are the contact structure induced from* ϕ *on the subbundle* $T\mathrm{F}_i$, *we obtain*

$$g(\phi_1 X_3, \phi_1 X_1) + g(\phi_2 X_3, \phi_2 X_1) = 2g(X_3, X_1)$$

and thus we get

$$Ric(X_3, X_1) = \big[(1-2K)(1-2p-2q) - 2\big]g(X_3, X_1).$$

From Lemma 2.4, we obtain that M is a generalized quasi-Einstein manifold.

Theorem 3.8

An NMCP manifold M satisfying $\mathrm{L}(X_1, Z).\mathrm{L} = 0$ is a space of constant curvature or the scalar curvature is $scal = (2p+2q+2)(2p+2q+1)$ **.**

Proof.

Let M be an NMCP manifold satisfying $\mathrm{L}(X_1, Z).\mathrm{L} = 0$. *From* (2.8) *we have*

$$0 = \mathrm{L}(X_1, Z)\mathrm{L}(X_3, X_4)X_5 - \mathrm{L}(\mathrm{L}(X_1, Z)X_3, X_4)X_5 - \mathrm{L}(X_3, \mathrm{L}(X_1, Z)X_4)X_5 - \mathrm{L}(X_3, X_4)\mathrm{L}(X_1, Z)X_5$$

for all $X_1, X_3, X_4, X_5 \in \Gamma(TM)$. *By using* (3.1), (3.2) *and taking inner product with* Z *, we get*

$$0 = (K-1)(g(X_1, \mathrm{L}(X_3, X_4)X_5) - (K-1)(g(X_1, X_3)g(X_4, X_5) + g(X_1, X_4)g(X_3, X_5)))$$

Thus we have $K = 1$ *or*

$$g(X_1, L(X_3, X_4)X_5) = (K-1)(g(X_1, X_3)g(X_4, X_5) + g(X_1, X_4)g(X_3, X_5)).$$

If $K=1$, *then the scalar curvature of* M *is* $scal = (2p+2q+2)(2p+2q+1)$. *On the other hand if* $K \neq 1$, *then from the definition of concircular curvature tensor we obtain*

$$R(X_1, X_3, X_4, X_5) = (1-2K)[(g(X_1, X_3)g(X_4, X_5) + g(X_1, X_4)g(X_3, X_5))].$$

Following same steps in the proof of Corollary 3.7 we have;

Corollary 3.9

Let M be NMCP manifold satisfying $L(X_1, Z).L = 0$. Then, M is a generalized quasi-Einstein manifold.

4. DISCUSSIONS AND CONCLUSIONS

The contact geometry plays a significant role in science. There are many applications of the contact geometry to medical science, technology, physics, geometric optics, geometric quantization, control theory, thermodynamics, and classical mechanics. Researchers have increased studies on this field from different areas in recent years. The improvement of the contact geometry depends on the differential geometry of the manifolds with structures. So, the theoretical backgrounds of the contact geometry are powered by mathematicians and theoretical physicists. In this manner, scientific studies on the Riemannian geometry of contact manifolds will give essential contributions to applications of such manifolds. Another subclass of contact geometry is the contact pair manifolds. The Riemannian geometry of contact pair manifolds is less explored, and there is a shortlist of papers in the mathematical literature on this topic. The works on this subject will be useful tools for the applications of the contact manifolds. In this way, we gave some new results on the NMCP manifolds. Our results will be used for future works on the subject. Some new problems with using different kinds of curvature tensors could be researched for the NMCP manifolds.

DECLARATION OF ETHICAL STANDARDS

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

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