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Research Article

Some betweenness relation topologies induced by simplicial complexes

Abd El Fattah El Atik*, Ashgan Wahba

Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

Abstract

This article aims to create an approximation space from any simplicial complex by representing a finite simplicial complex as a union of its components. These components are arranged into levels beginning with the highest-dimensional simplices. The universal set of the approximation space is comprised of a collection of all vertices, edges, faces, and tetrahedrons, and so on. Moreover, new types of upper and lower approximations in terms of a betweenness relation will be defined. A betweenness relation means that an element lies between two elements: an upper bound and a lower bound. In this work, based on Zhang et al.'s concept, a betweenness relation on any simplicial complex, which produces a set of order relations, is established and some of its topologies are studied.

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Keywords. simplicial complexes, approximation spaces, order relation, topological spaces, betweenness relation

1. Introduction and preliminaries

Simplices are the building blocks of simplicial complexes. Finite simplicial complexes are widely used to represent multidimensional objects, such as faces, which are 2-dimensional complexes, or graphs, which are 1-dimensional complexes. This means that a 0-dimensional simplex is a point, a 1-dimensional simplex is a line segment, a 2-dimensional simplex is a filled triangle, and a 3-dimensional simplex is a tetrahedron. Higher-dimensional simplices live comfortably in the Euclidean space of the appropriate dimension. In other words, drawing them or imagining what they look like is not possible. In general, a k-simplex $S = [u_0, u_1, \cdots, u_k]$ is a convex hull of $k+1$ affinely independent of $u_0, u_1, \cdots, u_k$ points in $\mathbb{R}^d$, where k denotes the dimension of the simplex [20]. An r-face is a convex hull of any subset of $r+1$ vertices of the k-simplex [3, 9], $r \leq k$. The 0-face, 1-face, and 2-face, for example, are points, edges, and triangles of the k-simplex, respectively, where the k-face is the k-simplex. The boundary of an n-simplex is made of $n+1$ simplices of the $n-1$ dimension. For instance, a 1-simplex has two 0-simplicies as boundaries, a 2-simplex has three 1-simplicies as boundaries, and so on. The simplicial complex σ is a finite class of simplices, in which each face belongs to σ and the intersection

*Corresponding Author.
Email addresses: aelatik@science.tanta.edu.eg (A. El Atik), ashganshebl71@gmail.com (A. Wahba)
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of two simplices $S_1, S_2 \in \sigma$ is either empty or a face of both. A simplicial complex σ is n-dimensional if the highest dimension of its simplices is n [3,9,22].

A collection τ of a nonempty set X is said to be a topology [14] on X if $X, \emptyset \in \tau$, a finite intersection of elements of τ belongs to τ, and an arbitrary union of elements of τ belongs to τ. For two topologies τ_1 and τ_2 on X, if $\tau_1 \subseteq \tau_2$, then τ_2 is finer than τ_1. The intersection of a collection of topologies on the same set is a topology, while their union is not always a topology. The collection $\mathcal{B} \subseteq \tau$ is a basis for τ on X if each element of τ is a union of elements in $\mathcal{B}$. Furthermore, $\mathcal{S} \subseteq \tau$ is a subbase for τ on X if each element belongs to τ is a union of intersections of elements of $\mathcal{S}$. Although $\bigcup_{i \in I} \tau_i$ of a collection of topologies $\{\tau_i\}_{i \in I}$ on X is not a topology, in general, the supremum [21] of topologies $\{\tau_i\}_{i \in I}$, which is considered the coarsest topology on X and finer than topologies $\{\tau_i\}_{i \in I}$, denoted by $\bigvee_{i \in I} \tau_i$, is assured to exist. Obviously, this can be represented as $\bigcup_{i \in I} \tau_i \subseteq \bigvee_{i \in I} \tau_i$. The equality holds if and only if $\bigcup_{i \in I} \tau_i$ is a topology on X. Moreover, $\bigcup_{i \in I} \tau_i$ is a subbase of $\bigvee_{i \in I} \tau_i$. In Alexandroff spaces [1], each open set means a smallest open set which is the intersection of all open sets.

A betweenness relation is a multiple of order three, which was introduced by Pasch [15] and Klein [12] and investigated by several researchers [2,4,10,11,19]. In [5], Düvelmeyer and Wenzel have studied the betweenness relation and its relationship with binary relations. Recently, Zhang et al. [21] have defined the betweenness relation as a set of order relations and studied a set of persuasive topologies. Furthermore, Lashin et al. in [13] have generated other topologies using a general binary relation. Some researchers have used a topology to represent structures such as fractals [6,8] and nanotopology with ideals and graphs [7] in terms of binary relations.

Throughout this paper, a set of simplices of a simplicial complex is presented as a universal set U_σ, which is used to establish new types of approximation spaces beginning with its highest-dimensional components. A special kind of a binary relation called $R_\Delta(<)$ on U_σ is constructed. For each pair of two distinct points $x, y \in U_\sigma$, x is the upper bound of y, denoted as $y \leq x$ if the dimension of y is less than the dimension of x. Moreover, some properties of $R_\Delta(<)$ are investigated. Finally, a new approximation space Δ of a simplicial complex via the betweenness relation on U_σ is introduced. Some properties of Δ are obtained and some examples are considered.

Definition 1.1. [3] A k-simplex S is a set of independent abstract vertices $[u_0, u_1, \cdots, u_k]$ that constitutes a convex hull of $k+1$ points; an r-face is an r-simplex $[u_{j_0}, u_{j_1}, \cdots, u_{j_r}]$ whose vertices are a subset of $[u_0, u_1, \cdots, u_k]$ with cardinality $r+1$.

Definition 1.2. [3] The finite simplicial complex σ is a finite set of simplices that satisfies the following conditions:

(i) Any face of a simplex from σ is also in σ.
(ii) The intersection of any two simplices $S_1, S_2 \in \sigma$ is a face of both S_1 and S_2.

Definition 1.3. [16,17] Let U be a finite universal set and R an equivalence relation on U. $U/R = \{[x]_R : x \in U\}$ denotes the family of equivalence classes of R. Then, the pair (U, R) is called an approximation space. For any $X \subseteq U$, the lower and upper approximations of X are defined, respectively, by the following:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\},$$
$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

From Pawlak's definition, X is said to be rough if $\underline{R}(X) \neq \overline{R}(X)$.

Definition 1.4. [18] A ternary relation B_Δ on an approximation space $(U_\sigma, R_\Delta(<))$ ((U_σ, Δ), for short) is a betweenness relation if the following hold:

(i) Symmetric: $(u,v,w) \in B_\Delta \Leftrightarrow (w,v,u) \in B_\Delta$ for any $u,v,w \in U_\sigma$.
(ii) Closure: $(u,v,w) \in B_\Delta \wedge (u,w,v) \in B_\Delta \Leftrightarrow v = w$ for any $u,v,w \in U_\sigma$.
(iii) End-point transitivity: $((o,u,v) \in B_\Delta \wedge (o,v,w) \in B_\Delta) \Rightarrow (o,u,w) \in B_\Delta$ for any $o,u,v,w \in U_\sigma$.

2. Order relations on complexes and their topologies

In this section, we approximate finite simplicial complexes of different dimensions to topological structures.

Definition 2.1. Let σ be a simplicial complex. Each k-simplex approximates to an element in the universal set U_σ. Moreover, each 0-simplex in σ transforms into v_i in U_σ, each 1-simplex in σ transforms into e_j in U_σ, each 2-simplex in σ transforms into f_k in U_σ, each 3-simplex in σ transforms into t_m in U_σ, and so on. Then, U_σ can be written as $U_\sigma = \{v_i : i \in I_1\} \cup \{e_j : j \in I_2\} \cup \{f_k : k \in I_3\} \cup \{t_m : m \in I_4\} \cup \cdots$, where $I_1, I_2, I_3, I_4, \cdots$ are indices. The approximation space (U_σ, Δ) begins with its highest-dimensional simplices.

Definition 2.2. The relation R_Δ on U_σ is called a preorder if the following conditions hold:

(i) $xR_\Delta x$, $\forall\ x \in U_\sigma$.
(ii) If $xR_\Delta y$ and $yR_\Delta z$, then $xR_\Delta z$.

It is called a total order if for any $u, v \in U_\sigma$ either $uR_\Delta v$ or $vR_\Delta u$.

Now, we give an order relation $R_\Delta(<)$ on U_σ of a simplicial complex σ.

Definition 2.3. The order relation $R_\Delta(<)$ is reflexive on U_σ and has the form $R_\Delta(<) = \{(x,y) : x,y \in U_\sigma \text{ and } y < x\}$, where $y < x$ means that x has a dimension greater than y and x is an upper bound of y. Also, $a \in U_\sigma$ is called a minimum element related to $R_\Delta(<)$ if $a\ R_\Delta(<)\ x$, $\forall\ x \in U_\sigma$, where $a\ R_\Delta(<)\ x$ means that a is in a relation with x with respect to $R_\Delta(<)$. In other words, a is an upper bound for all elements of U_σ.

In the following, the order relation $R_\Delta(<)$ is used to construct a topology $\tau_{R_\Delta(<)}$ from a simplicial complex σ whose approximation space is (U_σ, Δ).

Definition 2.4. Let $V \subseteq U_\sigma$. V is called an upper set if for all $x, y \in U_\sigma$, $x \in V$ such that $x\ R_\Delta(<)\ y$; then, $y \in V$.

Definition 2.5. The right neighborhood of any element $x \in U_\sigma$ is defined by $xR_\Delta(<) = \{y \in U_\sigma : x\ R_\Delta(<)\ y\}$. Moreover, the collection $\{xR_\Delta(<) : x \in U_\sigma\}$ forms a basis $\mathcal{B}_{R_\Delta(<)}$ for a topology called $\tau_{R_\Delta(<)}$. $xR_\Delta(<)$ is said to be the smallest neighborhood (or smallest upper set) of x with respect to $\tau_{R_\Delta(<)}$.

Proposition 2.6. *$\mathcal{B}_{R_\Delta(<)}$ is a basis for the topology $\tau_{R_\Delta(<)}$ on U_σ.*

Proof. Let $U_\sigma = \bigcup_{x \in U_\sigma} xR_\Delta(<)$, using Definition 2.5. Then, for each $x \in U_\sigma$, we put $B = xR_\Delta(<)$ and so $U_\sigma = \bigcup\{B : B \in \mathcal{B}_{R_\Delta(<)}\}$. To prove that $\mathcal{B}_{R_\Delta(<)}$ is a basis for $\tau_{R_\Delta(<)}$, it is sufficient to prove that $\tau_{R_\Delta(<)}$ is a topology on U_σ. It is clear that $U_\sigma, \emptyset \in \tau_{R_\Delta(<)}$ since $\emptyset = \bigcup\{B : B \in \emptyset \subseteq \mathcal{B}_{R_\Delta(<)}\}$. Now, let $\{G_i : i \in I\}$ be a collection of members of $\tau_{R_\Delta(<)}$. Then, each $G_i = \bigcup_{x \in U_\sigma} xR_\Delta(<)$, $x \in G_i$, for each i. So, each G_i is a union of members of $\mathcal{B}_{R_\Delta(<)}$. Therefore, $\bigcup_{i \in I} G_i$ is a union of members of $\mathcal{B}_{R_\Delta(<)}$. In the same way, $G_1 \cap G_2$ is a union of members of $\mathcal{B}_{R_\Delta(<)}$, for each $G_1, G_2 \in \tau_{R_\Delta(<)}$. □

Now, the topologies in terms of upper sets in Definition 2.4 are established from Examples 2.7, 2.8, and 2.9.

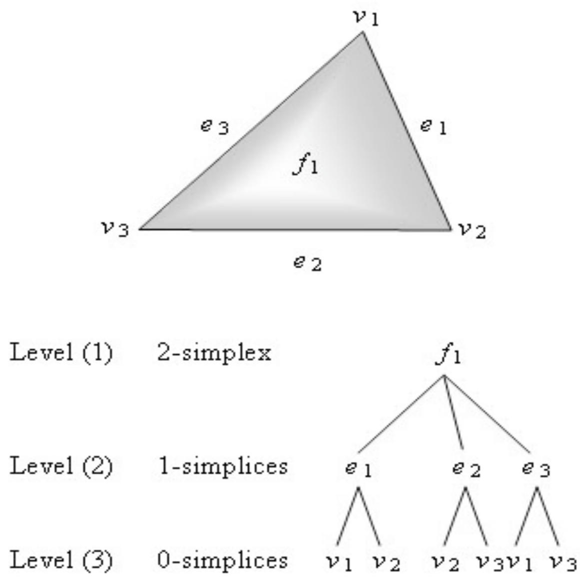

Figure 1. The 2-simplicial complex σ_1 and its approximation space Δ_1.

Example 2.7. In Figure 1, σ_1 has only one 2-simplex, three 1-simplices, and three 0-simplices. The universal set is $U_{\sigma_1} = \{v_1, v_2, v_3, e_1, e_2, e_3, f_1\}$. Let (U_{σ_1}, Δ_1) be an approximation space of a simplicial complex σ_1 in Figure 1. The order relation $R_{\Delta_1}(<)$ on U_{σ_1} is as follows:

$$R_{\Delta_1}(<) =\{(f_1, f_1), (e_1, e_1), (e_2, e_2), (e_3, e_3), (v_1, v_1), (v_2, v_2), (v_3, v_3), (f_1, e_1), (f_1, e_2), (f_1, e_3), (f_1, v_1), (f_1, v_2), (f_1, v_3), (e_1, v_1), (e_1, v_2), (e_2, v_2), (e_2, v_3), (e_3, v_1), (e_3, v_3)\}.$$ It is clear that f_1 is the minimum element in $R_{\Delta_1}(<)$.

Right neighborhoods (smallest upper sets) of U_{σ_1} are as follows:

$f_1R_{\Delta_1}(<) = \{f_1, e_1, e_2, e_3, v_1, v_2, v_3\}$, $v_1R_{\Delta_1}(<) = \{v_1\}$,
$e_1R_{\Delta_1}(<) = \{e_1, v_1, v_2\}$, $v_2R_{\Delta_1}(<) = \{v_2\}$,
$e_2R_{\Delta_1}(<) = \{e_2, v_2, v_3\}$, $v_3R_{\Delta_1}(<) = \{v_3\}$.
$e_3R_{\Delta_1}(<) = \{e_3, v_1, v_3\}$,

The basis is $\mathcal{B}_{R_{\Delta_1}(<)} = \{U_{\sigma_1}, \emptyset, \{e_1, v_1, v_2\}, \{e_2, v_2, v_3\}, \{e_3, v_1, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$, and the topology is $\tau_{R_{\Delta_1}(<)} = \{U_{\sigma_1}, \emptyset, \{e_1, v_1, v_2\}, \{e_2, v_2, v_3\}, \{e_3, v_1, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}, \{e_1, e_2, v_1, v_2, v_3\}, \{e_1, e_3, v_1, v_2, v_3\}, \{e_1, v_1, v_2, v_3\}, \{e_2, e_3, v_1, v_2, v_3\}, \{e_2, v_1, v_2, v_3\}, \{e_3, v_1, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{e_1, e_2, e_3, v_1, v_2, v_3\}\}$.

Example 2.8. In Figure 2, σ_2 has two 2-simplices, six 1-simplices, and five 0-simplices. The universal set is $U_{\sigma_2} = \{v_1, \cdots v_5, e_1 \cdots e_6, f_1, f_2\}$. Let (U_{σ_2}, Δ_2) be an approximation space of a simplicial complex σ_2 in Figure 2. The order relation $R_{\Delta_2}(<)$ on U_{σ_2} is as follows:

$$R_{\Delta_2}(<) =\{(f_1, f_1), (f_2, f_2), (e_1, e_1), (e_2, e_2), (e_3, e_3), (e_4, e_4), (e_5, e_5), (e_6, e_6), (v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4), (v_5, v_5), (f_1, e_1)(f_1, v_1), (f_1, v_2), (f_1, e_2), (f_1, v_3), (f_1, e_3), (e_1, v_1), (e_1, v_2), (e_2, v_2), (e_2, v_3), (e_3, v_1), (e_3, v_3), (f_2, e_4), (f_2, v_3), (f_2, v_5), (f_2, e_5), (f_2, v_4), (f_2, e_6), (e_4, v_3), (e_4, v_5), (e_5, v_3), (e_5, v_4), (e_6, v_4), (e_6, v_5)\}.$$ It is clear that there is no minimum element in $R_{\Delta_2}(<)$.

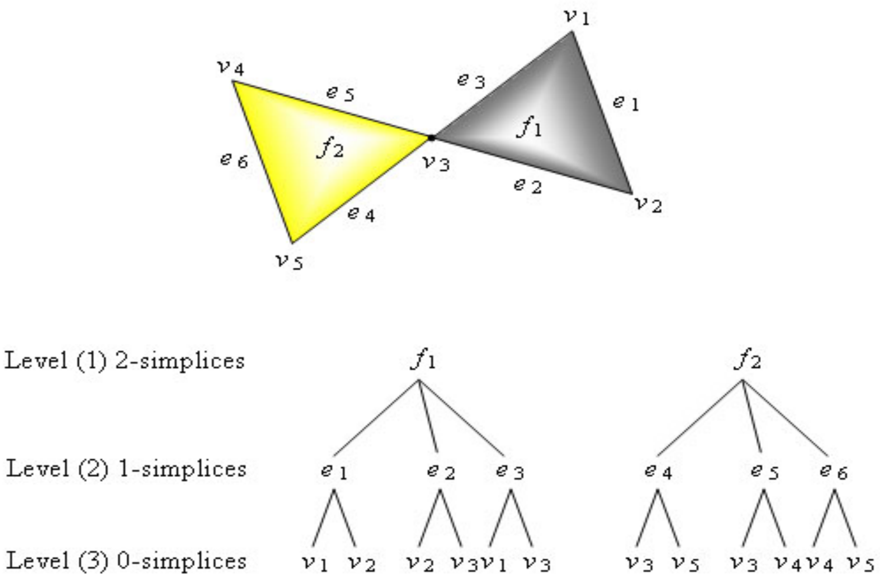

Figure 2. The 2-simplicial complex σ_2 and its approximation space Δ_2.

Right neighborhoods (smallest upper sets) of U_{σ_2} are as follows:

$f_1R_{\Delta_2}(<) = \{f_1, e_1, e_2, e_3, v_1, v_2, v_3\}$,
$f_2R_{\Delta_2}(<) = \{f_2, e_4, e_5, e_6, v_3, v_4, v_5\}$,
$e_1R_{\Delta_2}(<) = \{e_1, v_1, v_2\}$,
$e_2R_{\Delta_2}(<) = \{e_2, v_2, v_3\}$,
$e_3R_{\Delta_2}(<) = \{e_3, v_1, v_3\}$,
$e_4R_{\Delta_2}(<) = \{e_4, v_3, v_5\}$,
$e_5R_{\Delta_2}(<) = \{e_5, v_3, v_4\}$,
$e_6R_{\Delta_2}(<) = \{e_6, v_4, v_5\}$,
$v_1R_{\Delta_2}(<) = \{v_1\}$,
$v_2R_{\Delta_2}(<) = \{v_2\}$,
$v_3R_{\Delta_2}(<) = \{v_3\}$,
$v_4R_{\Delta_2}(<) = \{v_4\}$,
$v_5R_{\Delta_2}(<) = \{v_5\}$.

The basis is $\mathcal{B}_{\Delta_2(<)} = \{U_{\sigma_2}, \emptyset, \{f_1, e_1, e_2, e_3, v_1, v_2, v_3\}, \{f_2, e_4, e_5, e_6, v_3, v_4, v_5\}, \{e_1, v_1, v_2\}, \{e_2, v_2, v_3\}, \{e_3, v_1, v_3\}, \{e_4, v_3, v_5\}, \{e_5, v_3, v_4\}, \{e_6, v_4, v_5\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$. A union of members of $\mathcal{B}_{R_{\Delta_2}(<)}$ gives a topology $\tau_{R_{\Delta_2}(<)}$.

Example 2.9. In Figure 3, σ_3 has only one 3-simplex, four 2-simplices, six 1-simplices, and four 0-simplices. The universal set is $U_{\sigma_3} = \{v_1, \cdots v_4, e_1, \cdots e_6, f_1, \cdots f_4, t_1\}$. Let (U_{σ_3}, Δ_3) be an approximation space of a simplicial complex σ_3 in Figure 3. The order relation $R_{\Delta_3}(<)$ on U_{σ_3} is as follows:

$$R_{\Delta_3}(<) = \{(t_1,t_1),(f_1,f_1),(f_2,f_2),(f_3,f_3),(f_4,f_4),(e_1,e_1)(e_2,e_2),(e_3,e_3),(e_4,e_4),(e_5,e_5),(e_6,e_6),(v_1,v_1),(v_2,v_2),(v_3,v_3),(v_4,v_4),(t_1,f_1),(t_1,f_2),(t_1,f_3),(t_1,f_4),(t_1,e_1),(t_1,e_2),(t_1,e_3),(t_1,e_4),(t_1,e_5),(t_1,e_6),(t_1,v_1),(t_1,v_2),(t_1,v_3),(t_1,v_4),(f_1,e_1),(f_1,e_2),(f_1,e_5),(f_1,v_1),(f_1,v_2),(f_1,v_3),(f_2,e_3),(f_2,e_4),(f_2,e_5),(f_2,v_1),(f_2,v_3),(f_2,v_4),(f_3,e_2),(f_3,e_3),(f_3,e_6),(f_3,v_2),(f_3,v_3),(f_3,v_4),(f_4,e_1),(f_4,e_4),(f_4,e_6),(f_4,v_1),(f_4,v_2),(f_4,v_4),(e_1,v_1),(e_1,v_2),(e_2,v_2),(e_2,v_3),(e_5,v_1),(e_5,v_3),(e_3,v_3),(e_3,v_4),(e_4,v_1),(e_4,v_4),(e_6,v_2),(e_6,v_4)\}.$$

It is clear that t_1 is the minimum element in $R_{\Delta_3}(<)$.

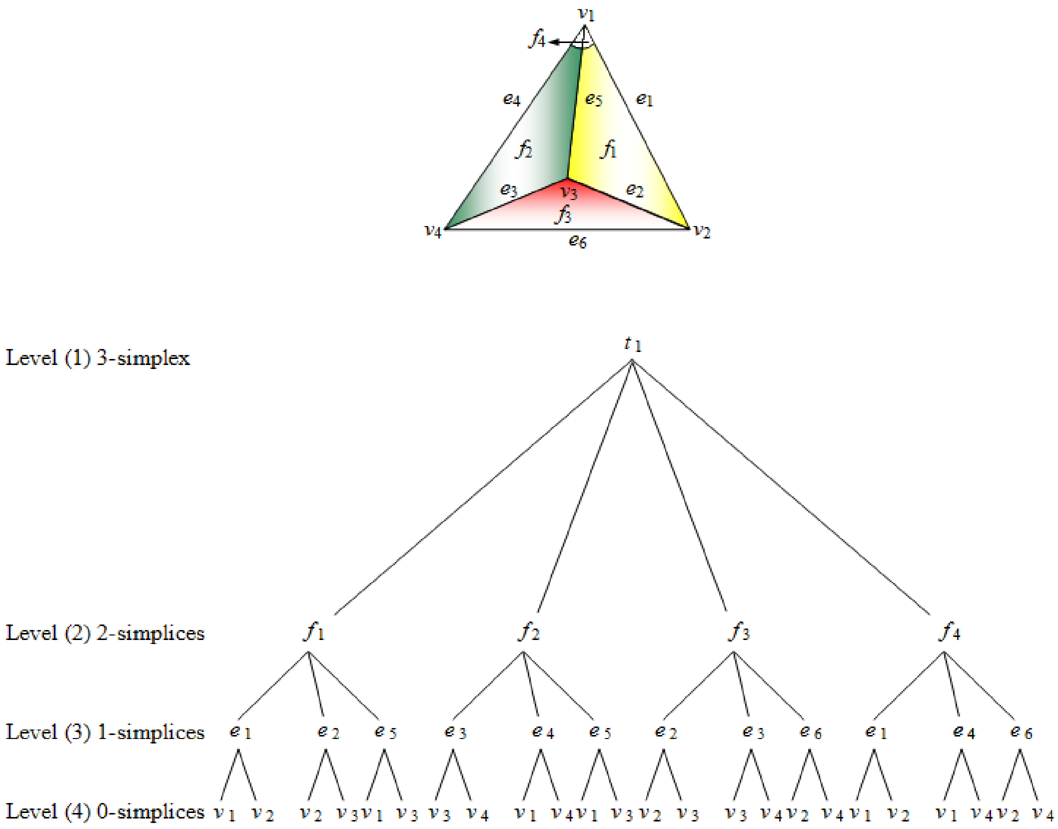

Figure 3. The 3-simplicial complex σ_3 and its approximation space Δ_3.

Right neighborhoods (smallest upper sets) of U_{σ_3} are as follows:

$t_1R_{\Delta_3}(<) = \{t_1, f_1, f_2, f_3, f_4, e_1, e_2, e_3, e_4, e_5, e_6, v_1, v_2, v_3, v_4\}$,
$f_1R_{\Delta_3}(<) = \{f_1, e_1, e_2, e_5, v_1, v_2, v_3\}$,
$f_2R_{\Delta_3}(<) = \{f_2, e_3, e_4, e_5, v_1, v_3, v_4\}$,
$f_3R_{\Delta_3}(<) = \{f_3, e_2, e_3, e_6, v_2, v_3, v_4\}$,
$f_4R_{\Delta_3}(<) = \{f_4, e_1, e_4, e_6, v_1, v_2, v_4\}$,
$e_1R_{\Delta_3}(<) = \{e_1, v_1, v_2\}$,
$e_2R_{\Delta_3}(<) = \{e_2, v_2, v_3\}$,
$e_3R_{\Delta_3}(<) = \{e_3, v_3, v_4\}$,
$e_4R_{\Delta_3}(<) = \{e_4, v_1, v_4\}$,
$e_5R_{\Delta_3}(<) = \{e_5, v_1, v_3\}$,
$e_6R_{\Delta_3}(<) = \{e_6, v_2, v_4\}$,
$v_1R_{\Delta_3}(<) = \{v_1\}$,
$v_2R_{\Delta_3}(<) = \{v_2\}$,
$v_3R_{\Delta_3}(<) = \{v_3\}$,
$v_4R_{\Delta_3}(<) = \{v_4\}$.

The basis is $\mathcal{B}_{R_{\Delta_3}(<)} = \{U_{\sigma_3}, \emptyset, \{f_1, e_1, e_2, e_5, v_1, v_2, v_3\}, \{f_2, e_3, e_4, e_5, v_1, v_3, v_4\}, \{f_3, e_2, e_3, e_6, v_2, v_3, v_4\}, \{f_4, e_1, e_4, e_6, v_1, v_2, v_4\}, \{e_1, v_1, v_2\}, \{e_2, v_2, v_3\}, \{e_3, v_3, v_4\}, \{e_4, v_1, v_4\}, \{e_5, v_1, v_3\}, \{e_6, v_2, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$. A union of members of $\mathcal{B}_{R_{\Delta_3}(<)}$ gives a topology $\tau_{R_{\Delta_3}(<)}$.

3. A betweenness relation on complexes

In this section, Zhang's concept of a betweenness relation [21] is used. This concept can be used on approximation space Δ of a simplicial complex σ. An explanation of Definition 1.4 according to the approximation space (U_σ, Δ) is given as follows.

Remark 3.1. (i) An element (u, v, w) means that v lies between u and w. For instance, in Figure 1, each e_j is between f_1 and v_i, where $i, j \in \{1, 2, 3\}$. In Figure 2, each e_j is between f_i and v_k, where $i \in \{1, 2\}$, $j \in \{1, 2, 3, 4, 5, 6\}$, and

$k \in \{1,2,3,4,5\}$. In Figure 3, f_i and e_j are between t_1 and v_k, each f_i is between t_1 and e_j, and each e_j is between f_i and v_k, where $i, k \in \{1,2,3,4\}$ and $j \in \{1,2,3,4,5,6\}$.

(ii) If $u, v \in U_\sigma$, $(u,v,u) \in B_\Delta$, then $u = v$. In other words, if we have two distinct points $u, v \in U_\sigma$, then (u,v,u) is not in B_Δ.

(iii) If a triple $(v,u,u) \in B_\Delta$, then $(u,u,v) \in B_\Delta$.

(iv) The simplest betweenness relation is denoted by $(B_0)_\Delta$ and is of the form $(B_0)_\Delta = \{(u,v,w) \in U_\sigma^3 : u = v \vee v = w\}$. It is called a minimum betweenness relation.

(ii), (iii), and (iv) can be represented as in Figures 4, 5, and 6, respectively.

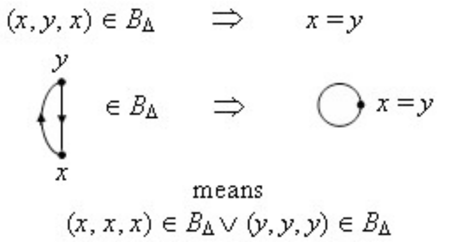

Figure 4. Remark 3.1(ii)

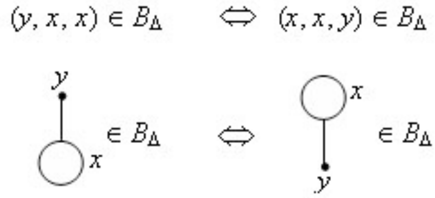

Figure 5. Remark 3.1 (iii)

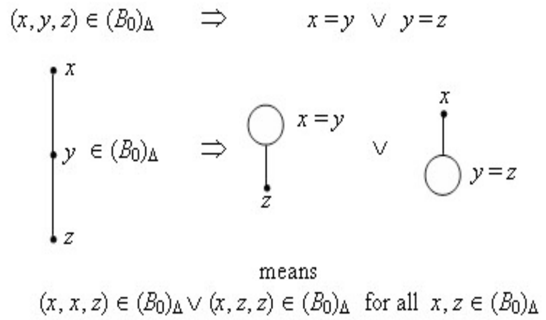

Figure 6. Remark 3.1 (iv)

Now, we investigate some basic characteristics for a betweenness relation on (U_σ, Δ).

Proposition 3.2. *In any approximation space (U_σ, Δ), $(B_0)_\Delta$ is a subclass of B_Δ.*

Proof. Assume that $(B_0)_\Delta \nsubseteq B_\Delta$. Then, there exists $(x,y,z) \in (B_0)_\Delta$ and $(x,y,z) \notin B_\Delta$. So, either $x = y$ or $y = z$ is satisfied. If $x = y$, using Definition 1.4(ii), then $(z,y,x) \in B_\Delta$, and using (i), then $(x,y,z) \in B_\Delta$ which gives a contradiction. Similarly, for $y = z$, $(x,y,z) \in B_\Delta$, which is also a contradiction. □

Proposition 3.3. *Let (U_σ, Δ) be an approximation space. The ternary relation of the form $B_\Delta(<) = \{(x,y,z) \in {U_\sigma}^3 : x = y \;\vee\; y = z \;\vee\; x < y < z \;\vee\; z < y < x$ is held $\}$ is a betweenness by an order relation $R_\Delta(<)$.*

Proof. It is sufficient to prove the conditions of betweenness.

(i) For any $(x,y,z) \in B_\Delta(<)$, $x = y \vee y = z \vee x < y < z \vee z < y < x$ implies that $z = y \vee y = x \vee z < y < x \vee x < y < z$. Therefore, $(z,y,x) \in B_\Delta(<)$.

(ii) If $y = z$, then $(x,y,z) \in B_\Delta(<)$ and $(x,z,y) \in B_\Delta(<)$. Conversely, assume that (x,y,z), $(x,z,y) \in B_\Delta(<)$, where y and z are distinct. For the point (x,y,z), one of the cases $x = y$, $x < y < z$, and $z < y < x$ holds. Similarly, for (x,z,y), one of the cases $x = z$, $x < z < y$, and $y < z < x$ is satisfied. Therefore, there are four cases: $x < y < z$, $x < z < y$; $x < y < z$, $y < z < x$; $z < y < x$, $x < z < y$; $z < y < x$, $y < z < x$. These cases lead to a contradiction and then $y = z$.

(iii) Consider that both (o, x, y) and (o, y, z) are in $B_\Delta(<)$. There are four cases:
i'. If $o < x < y$ and $o < y < z$, then $o < x < z$, and so $(o, x, z) \in B_\Delta(<)$.
ii'. If $o < x < y$ and $z < y < o$, this is impossible since $y \neq o$.
iii'. If $y < x < o$ and $o < y < z$, this is impossible since $y \neq o$.
iv'. If $y < x < o$ and $z < y < o$, then $(o, x, z) \in B_\Delta(<)$.

Therefore, we conclude that $B_\Delta(<)$ is a betweenness relation on U_σ. □

Remark 3.4. The definition of $B_\Delta(<)$ in Proposition 3.3 is equivalent to
$$B_\Delta(<) = (B_0)_\Delta \cup \{(x, y, z) \in {U_\sigma}^3 : x < y < z \vee z < y < x\}.$$

Example 3.5. Let (U_{σ_1}, Δ_1) be an approximation space in Figure 1. Consider $B'_{\Delta_1}(<) = \{(x, y, z) \in U^3_{\sigma_1} : x < y < z \vee z < y < x\}$; then,

$$\begin{aligned}B'_{\Delta_1}(<) =&\{(v_1, e_1, f_1), (v_2, e_1, f_1), (v_2, e_2, f_1), (v_3, e_2, f_1), (v_1, e_3, f_1), (v_3, e_3, f_1),\\ &(f_1, e_1, v_1), (f_1, e_1, v_2), (f_1, e_2, v_2), (f_1, e_2, v_3), (f_1, e_3, v_1), (f_1, e_3, v_3)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}B_{\Delta_1}(<) =&(B_0)_{\Delta_1} \cup \{(v_1, e_1, f_1), (v_2, e_1, f_1), (v_2, e_2, f_1), (v_3, e_2, f_1), (v_1, e_3, f_1),\\ &(v_3, e_3, f_1), (f_1, e_1, v_1), (f_1, e_1, v_2), (f_1, e_2, v_2), (f_1, e_2, v_3), (f_1, e_3, v_1),\\ &(f_1, e_3, v_3)\}.\end{aligned}$$

Example 3.6. Let (U_{σ_3}, Δ_3) be an approximation space in Figure 3.

$$\begin{aligned}\textit{Consider } B^*_{\Delta_3}(<) =&\{(e_1, f_1, t_1), (v_1, e_1, t_1), (v_1, f_1, t_1), (v_1, e_1, f_1), (v_2, e_1, f_1), (v_2, e_1, t_1),\\ &(v_2, f_1, t_1), (e_2, f_1, t_1), (v_2, e_2, f_1), (v_2, e_2, t_1), (v_2, f_1, t_1), (v_3, e_2, f_1),\\ &(v_3, e_2, t_1), (v_3, f_1, t_1), (v_1, e_5, f_1), (v_1, e_5, t_1), (v_1, f_1, t_1), (e_5, f_1, t_1),\\ &(v_3, e_5, f_1), (v_3, e_5, t_1), (v_3, f_1, t_1), (v_3, e_3, f_2), (e_3, f_2, t_1), (v_3, e_3, t_1),\\ &(v_3, f_2, t_1), (v_4, e_3, f_2), (v_4, e_3, t_1), (v_4, f_2, t_1), (v_1, e_4, f_2), (v_1, e_4, t_1),\\ &(v_1, f_2, t_1), (v_4, e_4, f_2), (v_4, e_4, t_1), (v_4, f_2, t_1), (v_1, e_5, f_2), (e_5, f_2, t_1),\\ &(v_1, e_5, t_1), (v_1, f_2, t_1), (v_3, e_5, f_2), (v_3, e_5, t_1), (v_3, f_2, t_1), (e_2, f_3, t_1),\\ &(v_2, e_2, f_3), (v_2, e_2, t_1), (v_2, f_3, t_1), (v_3, e_2, f_3), (v_3, e_2, t_1), (v_3, f_3, t_1),\\ &(v_4, e_3, f_3), (v_4, e_3, t_1), (v_4, f_3, t_1), (e_6, f_3, t_1), (v_2, e_6, f_3), (v_2, e_6, t_1),\\ &(v_2, f_3, t_1), (v_4, e_6, f_3), (v_4, e_6, t_1), ((v_4, f_3, t_1), (e_1, f_4, t_1), (v_1, e_1, f_4),\\ &(v_1, e_1, t_1), (v_1, f_4, t_1), (v_2, e_1, f_4), (v_2, e_1, t_1), (v_2, f_4, t_1), (v_1, e_4, f_4),\\ &(e_4, f_4, t_1), (v_1, e_4, t_1), (v_1, f_4, t_1), (v_4, e_4, f_4), (v_4, e_4, t_1), (v_4, f_4, t_1),\\ &(e_6, f_4, t_1), (v_2, e_6, f_4), (v_2, e_6, t_1), (v_2, f_4, t_1), (v_4, e_6, f_4), (v_4, e_6, t_1)\}.\end{aligned}$$

Therefore, $B_{\Delta_3}(<)$ is a union of three classes $(B_0)_{\Delta_3}$, $B^*_{\Delta_3}(<)$, and $\{(z, y, x) \in U^3_{\sigma_3} : (x, y, z) \in B^*_{\Delta_3}(<)\}$.

4. Comparison between betweenness and order relations

In this section, a betweenness relation $B_\Delta(<)$ is represented as a class of order relations.

Theorem 4.1. *Let $B_\Delta(<)$ be a betweenness relation in (U_σ, Δ). The binary relation $(O_x)_\Delta$ on U_σ is defined by $(O_x)_\Delta = \{(y, z) \in U^2_\sigma : (x, y, z) \in B_\Delta(<)\}$ and the collection of order relations on U_σ is $\{(O_x)_\Delta : x \in U_\sigma\}$. Then, for any distinct points x, y, and z in U_σ, $(y, z) \in (O_x)_\Delta$ if and only if $(y, x) \in (O_z)_\Delta$.*

Proof. If (y, z) and (z, ℓ) are in $(O_x)_\Delta$, then $(x, y, z), (x, z, \ell) \in B_\Delta(<)$. Using condition (iii) in Definition 1.4, $(x, y, \ell) \in B_\Delta(<)$. Hence, $(y, \ell) \in (O_x)_\Delta$ and so $(O_x)_\Delta$ is transitive. We conclude that the collection $\{(O_x)_\Delta\}_{x\in U_\sigma}$ is considered order relations on U_σ. Now, let $(y, z) \in (O_x)_\Delta$ imply that $(x, y, z) \in B_\Delta(<)$. Using condition (i) in Definition 1.4, $(z, y, x) \in B_\Delta(<)$ and then $(y, x) \in (O_z)_\Delta$. Similarly, if $(y, x) \in (O_z)_\Delta$, then $(y, z) \in (O_x)_\Delta$. □

In Theorem 4.2, we deduce a betweenness relation from an order relation for (U_σ, Δ).

Theorem 4.2. *Let $\{(O_x)_\Delta\}_{x\in U_\sigma}$ be a class of order relations on U_σ and a relation for x be $(B_x)_\Delta = \{(x, y, z) : (y, z) \in (O_x)_\Delta\}$. So, $B_\Delta = \bigcup_{x\in U_\sigma} (B_x)_\Delta$ is a betweenness on U_σ.*

Proof. Let $(x, y, z) \in (B_x)_\Delta$. Using Theorem 4.1, there is $(y, z) \in (O_x)_\Delta$ if and only if $(y, x) \in (O_z)_\Delta$, $(z, y, x) \in (B_x)_\Delta$. So, $(B_x)_\Delta$ satisfies a symmetric condition. To prove a closure property of $(B_x)_\Delta$, let (x, y, z) and $(x, z, y) \in (B_x)_\Delta$ imply that $(y, z), (z, y) \in (O_x)_\Delta$. But $(O_x)_\Delta$ is antisymmetric and so $y = z$. Conversely, if $y = z$, then $(x, y, z), (x, z, y) \in (B_x)_\Delta$. To prove a transitivity, let both (x, y, z) and $(x, z, \ell) \in (B_x)_\Delta$ imply that $(y, z) \in (O_x)_\Delta$ and $(z, \ell) \in (O_x)_\Delta$. Hence, $(y, \ell) \in (O_x)_\Delta$, which leads to $(x, y, \ell) \in (B_x)_\Delta$. □

Remark 4.3. Let $\{(O_x)_\Delta\}_{x\in U_\sigma}$ be a set of order relations. Then, the following hold:

(i) $(x, y) \in (O_x)_\Delta$ for distinct points $x, y \in U_\sigma$, which means that x is a minimum point in $(O_x)_\Delta$, $\bigcup_{x\in U_\sigma} (O_x)_\Delta = {U_\sigma}^2$.

(ii) $\bigcap_{x\in U_\sigma} (O_x)_\Delta = \{(x, x) : x \in U_\sigma\}$.

5. Main results

In this section, we construct a topology on U_σ of (U_σ, Δ) induced by a betweenness relation. For this aim, a right neighborhood of any $y \in U_\sigma$ with respect to $B_\Delta(<)$ is defined.

Definition 5.1. Let (U_σ, Δ) be an approximation space. Then,

(i) a right neighborhood of any $y \in U_\sigma$ with respect to $B_\Delta(<)$ is $((Ry)_x))_{\Delta,<} = \{z \in U_\sigma : (x, y, z) \in B_\Delta(<)\}$;

(ii) a right neighborhood of any $y \in U_\sigma$ with respect to $(O_x)_\Delta$ is $((Ry)_x)_{\Delta,<} = \{z \in U_\sigma : (y, z) \in (O_x)_\Delta\}$.

Proposition 5.2. *Let (U_σ, Δ) be an approximation space. Then, the properties that hold for $((Ry)_x)_{\Delta,<}$, $\forall\, x, y \in U_\sigma$ are as follows:*

(i) $y \in ((Ry)_x)_{\Delta,<}$.

(ii) $((Rx)_x)_{\Delta,<} = U_\sigma$.

(iii) $x \notin ((Ry)_x)_{\Delta,<}$ *if and only if* $x \neq y$.

(iv) $((Ry)_x)_{\Delta,<} \cap ((Rx)_y)_{\Delta,<} = \emptyset$ *if and only if* $x \neq y$.

(v) $\bigcap_{x\in U_\sigma} ((Ry)_x)_{\Delta,<} = \{y\}$.

Proof. Using Remarks 3.1 and 3.4 and Definition 5.1, the proof is obvious. □

Note that the class $((Ry)_x)_{\Delta,<}$, $\forall\, y \in U_\sigma$ is a basis for a topology called $(\tau_x)_{\Delta,<}$. In this topology, the set $((Ry)_x)_{\Delta,<}$ is the smallest neighborhood of y. Each of these topologies $\{(\tau_x)_{\Delta,<} : x \in U_\sigma\}$ is induced by a betweenness relation $B_\Delta(<)$. Also, an order relation $(O_x)_\Delta$ is used to generate other topologies such as the topology of Lashin et al. in [13].

Theorem 5.3. *Let $(\tau_x)_{\Delta,<}$ be a topology equipped with $B_\Delta(<)$ on (U_σ, Δ) with cardinality greater than 1. Then, $(\tau_x)_{\Delta,<}$ is neither discrete nor indiscrete topology, $\forall\, x \in U_\sigma$.*

Proof. It is clear that $\{\emptyset, U_\sigma\} \subset (\tau_x)_{\Delta,<} \subset P(U_\sigma)$, where $P(U_\sigma)$ is the power set (also considered a discrete topology) on U_σ. It is needed to prove that $P(U_\sigma) \neq (\tau_x)_{\Delta,<} \neq \{\emptyset, U_\sigma\}$, for $x \in U_\sigma$. Assume that $\{x\} \in (\tau_x)_{\Delta,<}$; then, $\{x\}$ is the smallest neighborhood of x with respect to $(\tau_x)_{\Delta,<}$. Also, $((R\ x)_x)_{\Delta,<} = U_\sigma$ is the smallest neighborhood of x with respect to $(\tau_x)_{\Delta,<}$. Hence, $\{x\} = U_\sigma$, which contradicts the fact that the cardinality of U_σ is greater than 1. So, $P(U_\sigma) \neq (\tau_x)_{\Delta,<}$. Now, let $y \in U_\sigma/\{x\}$. Since $((R\ y)_x)_{\Delta,<}$ contains y but does not contain x, $((R\ y)_x)_{\Delta,<}$ is a nonempty set and is not U_σ. Moreover, since $((R\ y)_x)_{\Delta,<} \in (\tau_x)_{\Delta,<}$, $(\tau_x)_{\Delta,<}$ is not an indiscrete topology. □

Theorem 5.4. *Let (U_σ, Δ) be an approximation space and $(\tau_x)_{\Delta,<}$, $\forall\ x \in U_\sigma$ be a topology obtained by $B_\Delta(<)$. Then, $\bigcap_{x\in U_\sigma} (\tau_x)_{\Delta,<} = \{\emptyset, U_\sigma\}$.*

Proof. Obviously, $\{\emptyset, U_\sigma\} \subseteq \bigcap_{x\in U_\sigma} (\tau_x)_{\Delta,<}$ is verified. Suppose that $F \in \bigcap_{x\in U_\sigma} (\tau_x)_{\Delta,<}$ and $F \neq \emptyset$. If $u \in F$, then $F \in (\tau_u)_{\Delta,<}$. So, F is a neighborhood of u with respect to $(\tau_u)_{\Delta,<}$. Since $((Ru)_u)_{\Delta,<} = U_\sigma$ is the smallest neighborhood of u with respect to $(\tau_u)_{\Delta,<}$, then $F = U_\sigma$ implies that $\bigcap_{x\in U_\sigma} (\tau_x)_{\Delta,<} = \{\emptyset, U_\sigma\}$. □

Example 5.5. In Example 3.5, the set of order relations $(O_x)_{\Delta_1}$, $\forall\ x \in U_{\sigma_1}$, which is induced by a betweennees relation $B_{\Delta_1}(<)$, is as follows:
$(O_{f_1})_{\Delta_1} = \{(f_1, f_1), (e_1, e_1), (e_2, e_2), (e_3, e_3), (v_1, v_1), (v_2, v_2), (v_3, v_3), (f_1, e_1), (f_1, e_2), (f_1, e_3), (f_1, v_1), (f_1, v_2), (f_1, v_3), (e_1, v_1), (e_1, v_2), (e_2, v_2), (e_2, v_3), (e_3, v_1), (e_3, v_3)\}$,
$(O_{e_1})_{\Delta_1} = \{(f_1, f_1), (e_1, e_1), (e_2, e_2), (e_3, e_3), (v_1, v_1), (v_2, v_2), (v_3, v_3), (e_1, f_1), (e_1, e_2), (e_1, e_3), (e_1, v_1), (e_1, v_2), (e_1, v_3)\}$,
$(O_{e_2})_{\Delta_1} = \{(f_1, f_1), (e_1, e_1), (e_2, e_2), (e_3, e_3), (v_1, v_1), (v_2, v_2), (v_3, v_3), (e_2, f_1), (e_2, e_1), (e_2, e_3), (e_2, v_1), (e_2, v_2), (e_2, v_3)\}$,
$(O_{e_3})_{\Delta_1} = \{(f_1, f_1), (e_1, e_1), (e_2, e_2), (e_3, e_3), (v_1, v_1), (v_2, v_2), (v_3, v_3), (e_3, f_1), (e_3, e_1), (e_3, e_2), (e_3, v_1), (e_3, v_2), (e_3, v_3)\}$,
$(O_{v_1})_{\Delta_1} = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (f_1, f_1), (e_1, e_1), (e_2, e_2), (e_3, e_3), (v_1, f_1), (v_1, e_1), (v_1, e_2), (v_1, e_3), (v_1, v_2), (v_1, v_3), (e_1, f_1), (e_3, f_1)\}$,
$(O_{v_2})_{\Delta_1} = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (e_1, e_1), (e_2, e_2), (e_3, e_3), (f_1, f_1), (v_2, f_1), (v_2, e_1), (v_2, e_2), (v_2, e_3), (v_2, v_1), (v_2, v_3), (e_1, f_1), (e_2, f_1)\}$,
$(O_{v_3})_{\Delta_1} = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (e_1, e_1), (e_2, e_2), (e_3, e_3), (f_1, f_1), (v_3, f_1), (v_3, e_1), (v_3, e_2), (v_3, e_3), (v_3, v_1), (v_3, v_2), (e_2, f_1), (e_3, f_1)\}$.

Right neighborhoods (smallest upper sets) for each $x \in U_{\sigma_1}$ are as follows:
$((R\ f_1)_{f_1})_{\Delta_1,<} = \{f_1, e_1, e_2, e_3, v_1, v_2, v_3\}$,
$((R\ e_1)_{f_1})_{\Delta_1,<} = \{e_1, v_1, v_2\}$,
$((R\ e_2)_{f_1})_{\Delta_1,<} = \{e_2, v_2, v_3\}$,
$((R\ e_3)_{f_1})_{\Delta_1,<} = \{e_3, v_1, v_3\}$,
$((R\ v_1)_{f_1})_{\Delta_1,<} = \{v_1\}$,
$((R\ v_2)_{f_1})_{\Delta_1,<} = \{v_2\}$,
$((R\ v_3)_{f_1})_{\Delta_1,<} = \{v_3\}$.
Therefore, the basis is $(\beta_{f_1})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{e_1, v_1, v_2\}, \{e_2, v_2, v_3\}, \{e_3, v_1, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$, which is used to generate $(\tau_{f_1})_{\Delta_1,<}$. Similarly, the bases for other points of U_{σ_1} are as follows:
$(\beta_{e_1})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_2\}, \{e_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{e_2})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{e_3})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_2\}, \{v_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{v_1})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1, f_1\}, \{e_2\}, \{e_3, f_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{v_2})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1, f_1\}, \{e_2, f_1\}, \{e_3\}, \{v_1\}, \{v_3\}\}$,
$(\beta_{v_3})_{\Delta_1,<} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_2, f_1\}, \{e_3, f_1\}, \{v_1\}, \{v_2\}\}$.

From Example 5.5, we note the following:

(i) $\{x\} \notin (\tau_x)_{\Delta,<}$, $\forall\ x \in U_\sigma$.

(ii) For any $x \in U_\sigma$, $(\tau_x)_{\Delta,<}$ is neither a discrete nor an indiscrete topology.

(iii) $\bigcap\limits_{x\in U_\sigma} (\tau_x)_{\Delta,<} = \{\emptyset, U_\sigma\}$.

Theorem 5.6. *Let $(\tau_x)_\Delta$ be a topology obtained by $(B_0)_\Delta$ on (U_σ, Δ). Then,* $\bigcup\limits_{x\in U_\sigma} (\tau_x)_\Delta = \bigvee\limits_{x\in U_\sigma} (\tau_x)_\Delta = P(U_\sigma)$.

Proof. It is clear that $\bigcup\limits_{x\in U_\sigma} (\tau_x)_\Delta \subseteq \bigvee\limits_{x\in U_\sigma} (\tau_x)_\Delta \subseteq P(U_\sigma)$. It is needed to prove that $P(U_\sigma) \subseteq \bigcup\limits_{x\in U_\sigma} (\tau_x)_\Delta$. Suppose that $F \in P(U_\sigma)$. Then, either $F = \emptyset$ or $F = U_\sigma$. So, $F \in \bigcup\limits_{x\in U_\sigma} (\tau_x)_\Delta$. If $F \neq \emptyset$ and $F \neq U_\sigma$ (take $u \in U_\sigma/F$), then by Remark 3.1 (iv), we get $((R\ y)_u)_\Delta = \{y\}$ for any $y \in F$, while $((R\ y)_u)_\Delta \in (\tau_u)_\Delta$. Then, $F = \bigcup\limits_{y\in F}\{y\} = \bigcup\limits_{y\in F}((R\ y)_u)_\Delta \in (\tau_u)_\Delta$. It is deduced that $F \in \bigcup\limits_{x\in U_\sigma} (\tau_x)_\Delta$. □

Theorem 5.6 is illustrated in Example 5.7.

Example 5.7. Let (U_{σ_1}, Δ_1) be an approximation space in Figure 1; the set of order relations $(O_x)_{\Delta_1}$, for all $x \in U_{\sigma_1}$, which is induced by $(B_0)_{\Delta_1}$ is as follows:
$(O_{f_1})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (f_1,e_1), (f_1,e_2), (f_1,e_3), (f_1,v_1), (f_1,v_2), (f_1,v_3)\}$,
$(O_{e_1})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (e_1,f_1), (e_1,e_2), (e_1,e_3), (e_1,v_1), (e_1,v_2), (e_1,v_3)\}$,
$(O_{e_2})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (e_2,f_1), (e_2,e_1), (e_2,e_3), (e_2,v_1), (e_2,v_2), (e_2,v_3)\}$,
$(O_{e_3})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (e_3,f_1), (e_3,e_1), (e_3,e_2), (e_3,v_1), (e_3,v_2), (e_3,v_3)\}$,
$(O_{v_1})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (v_1,f_1), (v_1,e_1), (v_1,e_2), (v_1,e_3), (v_1,v_2), (v_1,v_3)\}$,
$(O_{v_2})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (v_2,f_1), (v_2,e_1), (v_2,e_2), (v_2,e_3), (v_2,v_1), (v_2,v_3)\}$,
$(O_{v_3})_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3), (v_3,f_1), (v_3,e_1), (v_3,e_2), (v_3,e_3), (v_3,v_1), (v_3,v_2)\}$.
Right neighborhoods are as follows:

$((R\ f_1)_{f_1})_{\Delta_1} = U_{\sigma_1}$,	$((R\ v_1)_{f_1})_{\Delta_1} = \{v_1\}$,
$((R\ e_1)_{f_1})_{\Delta_1} = \{e_1\}$,	$((R\ v_2)_{f_1})_{\Delta_1} = \{v_2\}$,
$((R\ e_2)_{f_1})_{\Delta_1} = \{e_2\}$,	$((R\ v_3)_{f_1})_{\Delta_1} = \{v_3\}$.
$((R\ e_3)_{f_1})_{\Delta_1} = \{e_3\}$,	

Therefore, $(\beta_{f_1})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$, which is used to construct a topology $(\tau_{f_1})_{\Delta_1}$ on U_{σ_1}.
Similarly, the bases for other points of U_{σ_1} are deduced:
$(\beta_{e_1})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_2\}, \{e_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{e_2})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{e_3})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_2\}, \{v_1\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{v_1})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_2\}, \{e_3\}, \{v_2\}, \{v_3\}\}$,
$(\beta_{v_2})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_2\}, \{e_3\}, \{v_1\}, \{v_3\}\}$,
$(\beta_{v_3})_{\Delta_1} = \{U_{\sigma_1}, \emptyset, \{f_1\}, \{e_1\}, \{e_2\}, \{e_3\}, \{v_1\}, \{v_2\}\}$.

In Theorems 5.8, 5.9, and 5.10, necessary and sufficient conditions that $\bigvee\limits_{x\in U_\sigma} (\tau_x)_{\Delta,<} = P(U_\sigma)$ hold for $B_\Delta(<)$ are given.

Theorem 5.8. *Let (U_σ, Δ) be an approximation space. For any $y \in U_\sigma$, there is $\{x_i\}_{i\in I}$, where I is a finite index set, from U_σ such that $(x_i, y, z) \in B_\Delta(<)$ and $i \in I$. Then, $z = y$ if and only if $\bigcap_{i\in I}((R\ y)_{x_i})_{\Delta,<} = \{y\}$.*

Proof. Let $(x_i, y, z) \in B_\Delta(<)$, $\forall\ i \in I$ and $z = y$. Assume that $z \in \bigcap_{i\in I}((Ry)_{x_i})_{\Delta,<}$ implies that (x_i, y, z) is in $B_\Delta(<)$ $\forall\ i \in I$. Since $z = y \in \{y\}$, then $\bigcap_{i\in I}((Ry)_{x_i})_{\Delta,<} \subseteq \{y\}$. Using Proposition 5.2, we get $\{y\} \subseteq \bigcap_{i\in I}((Ry)_{x_i})_{\Delta,<}$. Therefore, $\bigcap_{i\in I}((Ry)_{x_i})_{\Delta,<} = \{y\}$. Conversely, if for any $y \in U_\sigma$, $\exists\ \{x_i\}_{i\in I}$ of U_σ and (x_i, y, z) is in $B_\Delta(<)$, using Definition 5.1, we get $z \in ((Ry)_{x_i})_{\Delta,<}$, while $\bigcap_{i\in I}((Ry)_{x_i})_{\Delta,<} = \{y\}$. Therefore, $z = y$. □

Theorem 5.9. *Let (U_σ, Δ) be an approximation space and $(\tau_x)_{\Delta,<}$ be a topology obtained by $B_\Delta(<)$, for $x \in U_\sigma$. Then, $\forall\ x \in U_\sigma$, $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<} = P(U_\sigma)$ if and only if $B_\Delta(<)$ satisfies Theorem 5.8.*

Proof. Suppose that $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<} = P(U_\sigma)$. Then, $\{y\} \in \bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ for any $y \in U_\sigma$. Since $\bigcup_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ is a subbase for $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$, then there is $\{F_j\}_{j\in J}$ of U_σ such that $\{y\} \in \bigcup_{j\in J} F_j$, where F_j is a finite intersection of elements of $\bigcup_{x\in U_\sigma}(\tau_x)_{\Delta,<}$. Hence, we find $j_0 \in J$ such that $\{y\} = F_{j_0}$ and a finite set $\{w_i : i \in I\}$ of $\bigcup_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ such that $F_{j_0} = \bigcap_{i\in I} w_i$. Hence, $\{y\} = \bigcap_{i\in I} w_i$. Since $w_i \in \bigcup_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ for each $i \in I$, $\exists\ x_i \in U_\sigma$ such that $w_i \in (\tau_{x_i})_{\Delta,<}$, then w_i is a neighborhood of y with respect to $(\tau_{x_i})_{\Delta,<}$. Obviously, $((R\ y)_{x_i})_{\Delta,<}$ is the smallest neighborhood of y with respect to $(\tau_{x_i})_{\Delta,<}$. Then, $y \in ((R\ y)_{x_i})_{\Delta,<} \subseteq w_i$, but $\{y\} = \bigcap_{i\in I} w_i$, and so $\bigcap_{i\in I}((R\ y)_{x_i})_{\Delta,<} = \{y\}$. Therefore, $B_\Delta(<)$ satisfies Theorem 5.8. Conversely, let $B_\Delta(<)$ satisfy Theorem 5.8. Then, for any $y \in U_\sigma$, there is a finite subset $\{x_i : i \in I\}$ of U_σ such that $\bigcap_{i\in I}((Ry)_{x_i})_{\Delta,<} = \{y\}$, but $((R\ y)_{x_i})_{\Delta,<} \in (\tau_{x_i})_{\Delta,<} \subseteq \bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ for $i \in I$ implies that $\{y\} \in \bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$. Therefore, $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<} = P(U_\sigma)$. □

Alexandroff spaces [1] are topological spaces, where each element is contained in the smallest open set. In Alexandroff spaces, an arbitrary intersection of open sets is open.

Theorem 5.10. *Let (U_σ, Δ) be an approximation space and $(\tau_x)_{\Delta,<}$, $\forall\ x \in U_\sigma$ be topologies obtained by $B_\Delta(<)$. Then, $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<} = P(U_\sigma)$ if and only if $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ is an Alexandroff topology.*

Proof. If $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<} = P(U_\sigma)$, then it is clear that $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ is Alexandroff. Conversely, let $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ be Alexandroff, then, for distinct points x and y in U_σ, $((R\ y)_x)_{\Delta,<} \in (\tau_x)_{\Delta,<} \subseteq \bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$. From Proposition 5.2, we know that $\bigcap_{x\in U_\sigma}((R\ y)_x)\Delta,< = \{y\}$. Since $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ is Alexandroff, it implies that $\{y\} \in \bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<}$ for any $y \in U_\sigma$. Hence, $\bigvee_{x\in U_\sigma}(\tau_x)_{\Delta,<} = P(U_\sigma)$. □

In Theorem 5.11, a betweenness relation must satisfy Theorems 5.8 and 5.9. Moreover, a relationship between the topology $\tau_{R_\Delta}(<)$ and topologies $\{(\tau_x)_{\Delta,<}\}_{x\in U_\sigma}$ is studied.

Theorem 5.11. *Let (U_σ, Δ) be an approximation space and induce both topologies $\tau_{R_\Delta(<)}$ and $(\tau_u)_{\Delta,<}$, $\forall\ u \in U_\sigma$. Then,*

(i) $\tau_{R_\Delta(<)} \subseteq \bigvee_{u\in U_\sigma} (\tau_u)_{\Delta,<}$;

(ii) $\tau_{R_\Delta} = \bigvee_{u\in U_\sigma} (\tau_u)_{\Delta,<}$ *if and only if* $R_\Delta = \{(u,u) : u \in U_\sigma\}$.

Proof. (i) Since $\bigvee_{u\in U_\sigma} (\tau_u)_{\Delta,<} = P(U_\sigma)$, by Theorem 5.9, then $\tau_{R_\Delta}(<) \subseteq \bigvee_{u\in U_\sigma} (\tau_u)_{\Delta,<}$.

(ii) Let $\bigvee_{u\in U_\sigma} (\tau_u)_{\Delta,<} = P(U_\sigma)$. It is needed to prove that $\tau_{R_\Delta} = P(U_\sigma)$ if and only if $R_\Delta = \{(u,u) : u \in U_\sigma\}$. If $R_\Delta = \{(u,u) : u \in U_\sigma\}$ implies that $\{u\} \in \tau_{R_\Delta}$ $\forall$ $u \in U_\sigma$, then $\tau_{R_\Delta} = P(U_\sigma)$. On the other hand, if $\tau_{R_\Delta} = P(U_\sigma)$, then from a one-one correspondence between order relations and topological spaces for Alexandroff on U_σ, we get $R_\Delta = \{(u,u) : u \in U_\sigma\}$.

□

Example 5.12. From Examples 2.7 and 5.5, it is clear that $\tau_{R_{\Delta_1}(<)} \subseteq \bigvee_{x\in U_{\sigma_1}} (\tau_x)_{\Delta_1,<}$.

Example 5.13. If $R_{\Delta_1} = \{(u,u) : u \in U_{\sigma_1}\}$, where U_{σ_1} is shown in Figure 1, then $R_{\Delta_1} = \{(f_1,f_1), (e_1,e_1), (e_2,e_2), (e_3,e_3), (v_1,v_1), (v_2,v_2), (v_3,v_3)\}$. The set of right neighborhoods is as follows:

$f_1\ R_{\Delta_1} = \{f_1\}$, $v_1\ R_{\Delta_1} = \{v_1\}$,
$e_1\ R_{\Delta_1} = \{e_1\}$, $v_2\ R_{\Delta_1} = \{v_2\}$,
$e_2\ R_{\Delta_1} = \{e_2\}$, $v_3\ R_{\Delta_1} = \{v_3\}$.
$e_3\ R_{\Delta_1} = \{e_3\}$,

Obviously, if $R_{\Delta_1} = \{(u,u) : u \in U_{\sigma_1}\}$, then the corresponding betweenness relation is $(B_0)_{\Delta_1}$. Also, $\tau_{R_{\Delta_1}} = \bigvee_{x\in U_{\sigma_1}} (\tau_x)_{\Delta_1}$.

Theorem 5.14. *Let* (U_σ, Δ) *be an approximation space and induce both topologies* $\tau_{R_\Delta(<)}$ *and* $(\tau_x)_{\Delta,<}$. *Then, for any* $y \in U_\sigma$, $(\tau_y)_{\Delta,<} = \tau_{R_\Delta(<)}$ *if and only if* y *is a minimum element with respect to* $R_\Delta(<)$.

Proof. Let $(\tau_y)_{\Delta,<} = \tau_{R_\Delta(<)}$. So, it is concluded that $(O_y)_\Delta = R_\Delta(<)$. This means that y is a minimum element with respect to $R_\Delta(<)$. Conversely, let y be the minimum element with respect to $R_\Delta(<)$. It is equivalent to show that $(O_y)_\Delta = R_\Delta(<)$. Let $(\ell,z) \in (O_y)_\Delta$ imply that $(y,\ell,z) \in B_\Delta(<)$. So, there are four cases:

(i) $\ell = y$ implies that $(y,z) \in R_\Delta(<)$ since y is a minimum element in $R_\Delta(<)$ and so $(\ell,z) \in R_\Delta(<)$.

(ii) $\ell = z$ implies that $(\ell,z) \in R_\Delta(<)$, by the reflexitivity of $R_\Delta(<)$.

(iii) $yR_\Delta(<)\ell R_\Delta(<)z$ implies that $(\ell,z) \in R_\Delta(<)$.

(iv) $zR_\Delta(<)\ell R_\Delta(<)y$. Since y is a minimum element in $R_\Delta(<)$ and $R_\Delta(<)$ antisymmetric, then $z = \ell = y$. By the reflexitivity of $R_\Delta(<)$, we have $(\ell,z) \in R_\Delta(<)$. Hence, $(O_y)_\Delta \subseteq R_\Delta(<)$. Conversely, if $(\ell,z) \in R_\Delta(<)$, then $y\ R_\Delta(<)\ell R_\Delta(<)z$ since y is the minimum element in $R_\Delta(<)$. Then, $(y,\ell,z) \in B_\Delta(<)$, and so $(\ell,z) \in (O_y)_\Delta$. Thus, $R_\Delta(<) \subseteq (O_y)_\Delta$. Therefore, $(O_y)_\Delta = R_\Delta(<)$.

□

Example 5.15. From Examples 2.7 and 5.5, since f_1 is the minimum element with respect to $R_{\Delta_1}(<)$, then $R_{\Delta_1}(<) = (O_{f_1})_{\Delta_1}$. It is clear that $(\tau_{f_1})_{\Delta_1,<} = \tau_{R_{\Delta_1}}(<)$ since f_1 is the minimum element with respect to $R_{\Delta_1}(<)$.

6. Conclusions

In this paper, we begin with a simplicial complex σ. An approximation space (U_σ, Δ) is established. The universal set U_σ of a simplicial complex σ is represented by a set of points from the vertices, edges, triangles, tetrahedrons, and so on. A betweenness relation is used to establish a new class of order relations. From the set of order relations, the researchers have a set of topologies. Moreover, a relationship between the topology induced by $R_{\Delta}(<)$ and the topologies generated by $(O_x)_{\Delta}$ is studied.

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