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The Electromagnetic Three-Body Problem With Radiation Terms – Existence-Uniqueness of Periodic Orbit (II)

Vasil G. Angelov[a]

[a] *University of Mining and Geology "St. I. Rilski", Department of Mathematics, 1700 Sofia, Bulgaria.*

Abstract

The primary goal of the present paper is to prove an existence-uniqueness of periodic solution of the equations of motion for the 3-body problem of classical electrodynamics. The equations of motion were derived in a recent paper of the author. Particular case of this problem is the He-atom – the simplest multi-electronic atom. We have applied our previous results to 3-body problem introducing radiation terms and in this manner we have obtained a system of 12 equations of motion. We have proved that three equations are a consequence of the first 9 ones, so that we consider 9 equations for 9 unknown functions. We introduce a suitable operator in a specific function space and formulate conditions for the existence-uniqueness of fixed point of this operator that is a periodic solution of the 3-body equations of motion. Finally, we verify the conditions obtained for the He-atom.

Keywords: Classical electrodynamics, Three-body problem, Radiation terms, He-atom.
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1. Introduction

The main purpose of the present paper is to prove an existence-uniqueness of periodic solution of the equations of motion for 3-body problem of classical electrodynamics. The equations of motion were derived in a recent paper [10]. Particular case of this problem is the He-atom – the simplest multi-electronic atom. The general model of N bodies without radiation terms was derived in [2]. In [10] we have applied these results to 3-body problem introducing radiation terms [5], [7], [8]. So we have obtained a system of 12 equations of motion introducing radiation terms. It turns out that three of equations are consequence of the rest ones. In this manner we obtain nine equations for nine unknown trajectories.

Email address: angelov@mgu.bg (Vasil G. Angelov)

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The paper consists of four sections and Appendix. In the Introduction we recall the basic results from [10]. In Section 2 we give an operator formulation of the 3-body periodic problem in suitable function space and prove some preliminary results. Section 3 contains the basic result – the existence-uniqueness of a periodic solution of the 3-body equations of motion. Fixed point theorem is applied to the suitable operator whose fixed point is a periodic solution. We note that we improve and precise some results from [8]. Section 4 is a conclusion where the conditions obtaining in the Main Theorem to the He-atom are applied.

In [10] we have derived the system

$$\begin{aligned} m_1\frac{d\lambda_r^{(1)}}{ds_1} &= \frac{e_1}{c^2}\left(F_{rl}^{(12)}\lambda_l^{(1)}+F_{rl}^{(13)}\lambda_l^{(1)}+F_{rl}^{(1)rad}\lambda_l^{(1)}\right),\\ m_2\frac{d\lambda_r^{(2)}}{ds_2} &= \frac{e_2}{c^2}\left(F_{rl}^{(21)}\lambda_l^{(2)}+F_{rl}^{(23)}\lambda_l^{(2)}+F_{rl}^{(2)rad}\lambda_l^{(2)}\right),\\ m_3\frac{d\lambda_r^{(3)}}{ds_3} &= \frac{e_3}{c^2}\left(F_{rl}^{(31)}\lambda_l^{(3)}+F_{rl}^{(32)}\lambda_l^{(3)}+F_{rl}^{(3)rad}\lambda_l^{(3)}\right), \end{aligned} \tag{1.1}$$

$(r=1,2,3,4)$ where c is the speed of light, $m_k (k=1,2,3)$ – the masses, $e_k (k=1,2,3)$ – the charges of the particles. Recall that there is a summation in repeating l in the right-hand sides of (1.1). The elements of the electromagnetic tensors $F_{rl}^{(kn)}=\frac{\partial\Phi_l^{(n)}}{\partial x_r^{(k)}}-\frac{\partial\Phi_r^{(n)}}{\partial x_l^{(k)}}$ can be calculated by the retarded Lienard-Wiechert potentials $\Phi_r^{(n)}=-\frac{e_n\lambda_r^{(n)}}{\left\langle\lambda^{(n)},\xi^{(kn)}\right\rangle_4}$ (cf. [14], [18], [2]-[7], [9]), while the radiation terms – as a half a difference of retarded and advance potentials in accordance with the Dirac assumption:

$$F_{mn}^{(k)rad}=\frac{1}{2}\left[\left(\frac{\partial A_n^{(k)ret}}{\partial x_m^{(k)ret}}-\frac{\partial A_m^{(n)ret}}{\partial x_n^{(k)ret}}\right)-\left(\frac{\partial A_n^{(k)adv}}{\partial x_m^{(k)adv}}-\frac{\partial A_m^{(n)adv}}{\partial x_n^{(k)adv}}\right)\right],$$

where

$$A_n^{(k)ret}=-\frac{e_k\lambda_n^{(k)ret}}{\left\langle\lambda^{(k)ret},\xi^{(k)ret}\right\rangle_4},\ A_n^{(k)adv}=-\frac{e_k\lambda_n^{(k)adv}}{\left\langle\lambda^{(k)adv},\xi^{(k)adv}\right\rangle_4}.$$

In previous papers [3], [4] and [8] is proved that every fourth equation of (1.1) is a consequence of the first three ones. Recall that by $\left\langle\vec{u}^{(k)},\vec{u}^{(k)}\right\rangle$ the scalar product in the 3-dimensional Euclidean space is denoted. So under assumption **(C)** $\sqrt{\left\langle\vec{u}^{(k)}(t),\vec{u}^{(k)}(t)\right\rangle}\le\bar{c}<c\ (k=1,2,3)$ the equations of motion can be solved with respect to the unknown accelerations (cf. [10]):

$$\begin{aligned} \dot{u}_1^{(1)} &= \frac{c^2-\left(u_1^{(1)}\right)^2}{c^2}\left(G_1^{(12)}+G_1^{(13)}+G_1^{(1)rad}\right)-\frac{u_1^{(1)}u_2^{(1)}}{c^2}\left(G_2^{(12)}+G_2^{(13)}+G_2^{(1)rad}\right)\\ &\quad-\frac{u_1^{(1)}u_3^{(1)}}{c^2}\left(G_3^{(12)}+G_3^{(13)}+G_3^{(1)rad}\right)\equiv U_1^1,\\ \dot{u}_2^{(1)} &= -\frac{u_1^{(1)}u_2^{(1)}}{c^2}\left(G_1^{(12)}+G_1^{(13)}+G_1^{(1)rad}\right)+\frac{c^2-(u_2^{(1)})^2}{c^2}\left(G_2^{(12)}+G_2^{(13)}+G_2^{(1)rad}\right)\\ &\quad-\frac{u_2^{(1)}u_3^{(1)}}{c^2}\left(G_3^{(12)}+G_3^{(13)}+G_3^{(1)rad}\right)\equiv U_2^1,\\ \dot{u}_3^{(1)} &= -\frac{u_1^{(1)}u_3^{(1)}}{c^2}\left(G_1^{(12)}+G_1^{(13)}+G_1^{(1)rad}\right)-\frac{u_2^{(1)}u_3^{(1)}}{c^2}\left(G_2^{(12)}+G_2^{(13)}+G_2^{(1)rad}\right)\\ &\quad+\frac{c^2-(u_3^{(1)})^2}{c^2}\left(G_3^{(12)}+G_3^{(13)}+G_3^{(1)rad}\right)\equiv U_3^1,\\ \dot{u}_1^{(2)} &= \frac{c^2-\left(u_1^{(2)}\right)^2}{c^2}\left(G_1^{(21)}+G_1^{(23)}+G_1^{(2)rad}\right)-\frac{u_1^{(2)}u_2^{(2)}}{c^2}\left(G_2^{(21)}+G_2^{(23)}+G_2^{(2)rad}\right)\\ &\quad-\frac{u_1^{(2)}u_3^{(2)}}{c^2}\left(G_3^{(21)}+G_3^{(23)}+G_3^{(2)rad}\right)\equiv U_1^2, \end{aligned}$$

$$\dot{u}_2^{(2)} = -\frac{u_1^{(2)}u_2^{(2)}}{c^2}\left(G_1^{(21)}+G_1^{(23)}+G_1^{(2)rad}\right)+\frac{c^2-(u_2^{(2)})^2}{c^2}\left(G_2^{(21)}+G_2^{(23)}+G_2^{(2)rad}\right) -\frac{u_2^{(2)}u_3^{(2)}}{c^2}\left(G_3^{(21)}+G_3^{(23)}+G_3^{(2)rad}\right)\equiv U_2^2,$$

$$\dot{u}_3^{(2)} = -\frac{u_1^{(2)}u_3^{(2)}}{c^2}\left(G_1^{(21)}+G_1^{(23)}+G_1^{(2)rad}\right)-\frac{u_2^{(2)}u_3^{(2)}}{c^2}\left(G_2^{(21)}+G_2^{(23)}+G_2^{(2)rad}\right) +\frac{c^2-(u_3^{(2)})^2}{c^2}\left(G_3^{(21)}+G_3^{(23)}+G_3^{(2)rad}\right)\equiv U_3^2\dot{u}_1^{(3)} = \frac{c^2-\left(u_1^{(3)}\right)^2}{c^2}\left(G_1^{(31)}+G_1^{(32)}+G_1^{(3)rad}\right) -\frac{u_1^{(3)}u_2^{(3)}}{c^2}\left(G_2^{(31)}+G_2^{(32)}+G_2^{(3)rad}\right)-\frac{u_1^{(3)}u_3^{(3)}}{c^2}\left(G_3^{(31)}+G_3^{(32)}+G_3^{(3)rad}\right)\equiv U_1^3,$$

$$\dot{u}_2^{(3)} = -\frac{u_1^{(3)}u_2^{(3)}}{c^2}\left(G_1^{(31)}+G_1^{(32)}+G_1^{(3)rad}\right)+\frac{c^2-(u_2^{(3)})^2}{c^2}\left(G_2^{(31)}+G_2^{(32)}+G_2^{(3)rad}\right) -\frac{u_2^{(3)}u_3^{(3)}}{c^2}\left(G_3^{(31)}+G_3^{(32)}+G_3^{(3)rad}\right)\equiv U_2^3,$$

$$\dot{u}_3^{(3)} = -\frac{u_1^{(3)}u_3^{(3)}}{c^2}\left(G_1^{(31)}+G_1^{(32)}+G_1^{(3)rad}\right)-\frac{u_2^{(3)}u_3^{(3)}}{c^2}\left(G_2^{(31)}+G_2^{(32)}+G_2^{(3)rad}\right) +\frac{c^2-(u_3^{(3)})^2}{c^2}\left(G_3^{(31)}+G_3^{(32)}+G_3^{(3)rad}\right)\equiv U_3^3, t\geq 0, \tag{1.2}$$

$$u_\alpha^{(1)}(t) = u_\alpha^{(10)}(t),\ u_\alpha^{(2)}(t)=u_\alpha^{(20)}(t),\ u_\alpha^{(3)}(t)=u_\alpha^{(30)}(t),\ \dot{u}_\alpha^{(1)}(t)=\dot{u}_\alpha^{(10)}(t), \dot{u}_\alpha^{(2)}(t) = \dot{u}_\alpha^{(20)}(t),\ \dot{u}_\alpha^{(3)}(t)=\dot{u}_\alpha^{(30)}(t),\ t\leq 0,$$

where $u_\alpha^{(k0)}(t)$ are prescribed functions. The above 3-body system of equations of motion consists of neutral differential equations with deviating arguments. The delays depend on the unknown trajectories. Such type of equations generate specific difficulties noted in [11], [12] as in the case of two-body problem, overcome by our operator method.

The system (1.2) can be rewritten briefly as

$$\dot{u}_l^{(k)} = \sum_{n=1,n\neq k}^{3}\left(G_l^{(kn)}\right)+G_l^{(l)rad}-\sum_{\gamma=1}^{3}\frac{u_l^{(k)}u_\gamma^{(k)}}{c^2}\left(\sum_{n=1,n\neq k}^{3}G_\gamma^{(kn)}+G_\gamma^{(l)rad}\right)\equiv U_l^k(u),\ (k=1,2,3;\ \ l=1,2,3)$$

where

$$G_\alpha^{(12)}=\frac{e_1e_2\Delta_1}{m_1c^2}\left(A_{12}\xi_\alpha^{(12)}-B_{12}u_\alpha^{(2)}+C_{12}\dot{u}_\alpha^{(2)}\right);\ G_\alpha^{(13)}=\frac{e_1e_3\Delta_1}{m_1c^2}\left(A_{13}\xi_\alpha^{(13)}-B_{13}u_\alpha^{(3)}+C_{13}\dot{u}_\alpha^{(3)}\right);$$

$$G_\alpha^{(21)}=\frac{e_2e_1\Delta_2}{m_2c^2}\left(A_{21}\xi_\alpha^{(21)}-B_{21}u_\alpha^{(1)}+C_{21}\dot{u}_\alpha^{(1)}\right);\ G_\alpha^{(23)}=\frac{e_2e_3\Delta_2}{m_2c^2}\left(A_{23}\xi_\alpha^{(23)}-B_{23}u_\alpha^{(3)}+C_{23}\dot{u}_\alpha^{(3)}\right); \tag{1.3}$$

$$G_\alpha^{(31)}=\frac{e_3e_1\Delta_3}{m_3c^2}\left(A_{31}\xi_\alpha^{(31)}-B_{31}u_\alpha^{(1)}+C_{31}\dot{u}_\alpha^{(1)}\right);\ G_\alpha^{(32)}=\frac{e_3e_2\Delta_3}{m_3c^2}\left(A_{32}\xi_\alpha^{(32)}-B_{32}u_\alpha^{(2)}+C_{32}\dot{u}_\alpha^{(2)}\right)$$

$(\alpha=1,2,3)$ and

$$\xi^{(kn)} = \left(\xi_1^{(kn)},\xi_2^{(kn)},\xi_3^{(kn)},\xi_4^{(kn)}\right) = \left(x_1^{(k)}(t)-x_1^{(n)}(t-\tau_{kn}),x_2^{(k)}(t)-x_2^{(n)}(t-\tau_{kn}),x_3^{(k)}(t)-x_3^{(n)}(t-\tau_{kn}),ic\tau_{kn}(t)\right).$$

Since $\xi^{(kn)}$ are isotropic 4-vectors, i.e. $\left\langle\xi^{(kn)},\xi^{(kn)}\right\rangle_4=0$, the retarded functions $\tau_{kn}(t)$ satisfy the functional equations

$$\tau_{kn}(t) = \frac{1}{c}\sqrt{\left\langle\vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\right\rangle}\equiv\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(k)}(t)-x_\alpha^{(n)}(t-\tau_{kn}(t)\right]^2},(k\,n)=(12),\ (13),\ (21),\ (23),\ (31),\ (32). \tag{1.4}$$

For the 3-vectors we recall the denotations from [10]:

$$\vec{\xi}^{(kn)}=\left(\xi_1^{(kn)},\xi_2^{(kn)},\xi_3^{(kn)}\right);\vec{u}^{(k)}=\left(u_1^{(k)}(t),u_2^{(k)}(t),u_3^{(k)}(t)\right);$$

$$\begin{aligned}\vec{u}^{(n)}(t-\tau_{kn}) &= \left(u_1^{(n)}(t-\tau_{kn}),u_2^{(n)}(t-\tau_{kn}),u_3^{(n)}(t-\tau_{kn})\right)\\ \dot{\vec{u}}^{(n)}(t-\tau_{kn}) &= \left(\dot{u}_1^{(n)}(t-\tau_{kn}),\dot{u}_2^{(n)}(t-\tau_{kn}),\dot{u}_3^{(n)}(t-\tau_{kn})\right);\end{aligned}$$

$$\Delta_k=\sqrt{c^2-\left\langle\vec{u}^{(k)}(t),\vec{u}^{(k)}(t)\right\rangle},\Delta_{kn}=\sqrt{c^2-\left\langle\vec{u}^{(n)}(t-\tau_{kn}),\vec{u}^{(n)}(t-\tau_{kn})\right\rangle};$$

$$D_{kn}=\frac{c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}{c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle};H_{kn}=\Delta_{kn}^2+D_{kn}\left(\left\langle\vec{\xi}^{(kn)},\dot{\vec{u}}^{(n)}\right\rangle+\frac{\left(\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-c^2\tau_{kn}\right)\left\langle\vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle}{\Delta_{kn}^2}\right);$$

$$A_{kn}=\frac{H_{kn}\left(c^2-\left\langle\vec{u}^{(k)},\vec{u}^{(n)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}-D_{kn}\frac{\Delta_{kn}^2\left\langle\vec{u}^{(k)},\dot{\vec{u}}^{(n)}\right\rangle+\left(\left\langle\vec{u}^{(k)},\vec{u}^{(n)}\right\rangle-c^2\right)\left\langle\vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle}{\Delta_{kn}^2\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^2}; \tag{1.5}$$

$$\begin{aligned}B_{kn} &= \frac{H_{kn}\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}-\frac{D_{kn}\left(\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-c^2\tau_{kn}\right)\left\langle\vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle}{\Delta_{kn}^2\left(\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-c^2\tau_{kn}\right)^2};\\ C_{kn} &= \frac{D_{kn}\left(\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-c^2\tau_{kn}\right)}{\left(\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-c^2\tau_{kn}\right)^2}.\end{aligned}$$

Following the Dirac assumption that $\tau^{(p)r}=\tau^{(p)a}=\tau$, we consider τ as a small parameter since $\tau=\tau_0\sqrt{1-\beta^2(t)}$, $\tau_0=r_e/c\approx10^{-24}\,\text{sec}(\ \tau_0=r_e/c\approx10^{-24}\,\text{sec})$. In [5] we have derived a new form of the Dirac radiation terms:

$$G_\alpha^{(k)rad}=-\frac{e_k^2}{m_kc^2}\left(\frac{u_\alpha^{(k)}(t)}{{\Delta_k}^3}\left\langle\vec{u}^{(k)}(t),\frac{\dot{\vec{u}}^{(k)}(t+\tau)-\dot{\vec{u}}^{(k)}(t-\tau)}{2\tau}\right\rangle+\frac{1}{\Delta_k}\frac{\dot{u}_\alpha^{(k)}(t+\tau)-\dot{u}_\alpha^{(k)}(t-\tau)}{2\tau}\right). \tag{1.6}$$

2. Operator Formulation of the Periodic Problem in Suitable Function Spaces and Preliminary Results

The main goal of the present section is to prove the equivalence between the existence-uniqueness of T-periodic solution of the system (1.2) and the existence of fixed point of a suitable operator acting on specific function space. The formulation of the Main Lemma 8 by several Lemmas is preceded.
By $C_T^\infty[0,\infty)$ we denote the set of all infinite differentiable T-periodic functions. Introduce 9 sets of functions:

$$M_\alpha^k=\left\{u_\alpha^{(k)}\in C_T^\infty[0,\infty)\ :\ \left|\left(u_\alpha^{(k)}\right)^{(m)}(t)\right|\le U_0\omega^m,\ t\in[pT,(p+1)T]\ \text{and}\right.$$
$$\left.\int_{pT}^{(p+1)T}u_\alpha^{(k)}(t)dt=0\ ,\left(u_\alpha^{(k)}\right)^{(m)}(0)=0;\ p,m=0,1,2,...\right\},$$

where $U_0,\ \omega,\ T,\mu$ are positive constants and $\left(u_\alpha^{(k)}\right)^{(m)}(0)$ is the m-th derivative of $u_\alpha^{(k)}(t)$ at 0.

We form the Cartesian product $M_{10}^1\times M_{20}^1\times M_{30}^1\times M_{10}^2\times M_{20}^2\times M_{30}^2\times M_{10}^3\times M_{20}^3\times M_{30}^3$, where

$$M_{\alpha0}^k=\left\{u_\alpha^{(k)}\in M_\alpha^k\ :\ u_\alpha^{(k)}(t)=u_\alpha^{(k0)}(t)\,,t\in[-\theta_0,0]\right\},(k=1,2,3\ ;\ \alpha=1,2,3).$$

Introduce the family of pseudo-metrics

$$\rho_{(p,m)}\left(u_\alpha^{(k)},\bar{u}_\alpha^{(k)}\right)=ess\sup\left\{e^{-\mu(t-pT)}\omega^{-m}\left|\frac{d^m u_\alpha^{(k)}(t)}{dt^m}-\frac{d^m\bar{u}_\alpha^{(k)}(t)}{dt^m}\right|:t\in[pT,(p+1)T]\right\},\tag{2.1}$$

for $p=0,1,2,...$; $m=0,1,2,...$ Every set$M_{\alpha0}^k$ can be considered as a space completed with respect to $\rho_{(p,m)}\left(u_\alpha^{(k)},\bar{u}_\alpha^{(k)}\right)$. So we obtain spaces with generalized derivatives in the sense of Sobolev-Schwartz (cf. [15], [16]).

It is easy to see that the following inequalities are satisfied for every (p,m) and $t\in[pT,(p+1)T]$:

$$e^{-\mu(t-pT)}\omega^{-m}\left|\frac{d^m u(t)}{dt^m}-\frac{d^m\bar{u}(t)}{dt^m}\right|\le e^{-\mu(t-pT)}\omega^{-m}\,2\,\omega^m e^{\mu(t-pT)}U_0=2U_0<\infty.\tag{2.2}$$

Therefore $\sup\left\{\rho_{(p,m)}(u,\bar{u}):m=0,1,2,...\right\}<\infty$. This property plays an essential role in the proof of the Main Theorem.

Assumption (C): If $\left(u_1^{(k)}(t),u_2^{(k)}(t),u_3^{(k)}(t)\right)\in M_{10}^k\times M_{20}^k\times M_{30}^k\ (k=1,2,3)$ then $\sqrt{\sum_{\gamma=1}^{3}\left(u_\gamma^{(k)}(t)\right)^2}\le\bar{c}<c$ and $U_0e^{\mu T}\le\bar{c}<c$.

Following A. Sommerfeld [17] we denote by $\beta(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left(u_\gamma^{(k)}(t)\right)^2}\le\bar{\beta}<1$, where $\bar{\beta}=\frac{\bar{c}}{c}<1$.

Assumption (T): We assume that $\tau_{kn}(0)=p_{kn}T$ for some positive integer $\nu_{kn}\in N$. But $x_\gamma^{(n0)}(t),t\in[-\theta_0,0]$ is T-periodic function that implies $x_\gamma^{(n0)}(-p_{kn}T)=x_\gamma^{(n0)}(0)\equiv x_{\gamma0}^{(n0)}$ and consequently

$$\sqrt{\sum_{\gamma=1}^{3}\left(x_\gamma^{(k0)}(0)-x_\gamma^{(n0)}(-p_{kn}T)\right)}=\sqrt{\sum_{\gamma=1}^{3}\left(x_\gamma^{(k0)}(0)-x_\gamma^{(n0)}(0)\right)}=c\tau_{kn}(0).$$

Lemma 1. *The condition* $\int_{pT}^{(p+1)T}u(t)dt=0\ (p=0,1,2,3,\,...)$ *implies that* $x(t)=\int_0^t u(s)ds=0$ *is* T-*periodic function.*

Remark 2. *We have, however, already proved in [4] that every equation (1.4) has a unique continuous solution for every Lipschitz continuous trajectories and* $\tau_{kn}\ge r_{kn}(t)/2c$. *If* $\left(x_1^{(k)}(t),x_2^{(k)}(t),x_3^{(k)}(t)\right),(k=1,2,3)$ *are* T-*periodic functions, then* $\tau_{kn}(t)$ *are* T-*periodic functions, too.*

We prove some additional properties of $\tau_{kn}(t)$.

Lemma 3. *The derivative of* τ_{kn} *is* $\dot{\tau}_{kn}(t)=\frac{1}{c}\frac{\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\rangle-\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\rangle}{\sqrt{\langle\vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\rangle}-\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\rangle}$ *and satisfies the inequalities* $\dot{\tau}_{kn}(t)<1$ *and* $\frac{1}{1-\dot{\tau}_{kn}(t)}\le\frac{1+\bar{\beta}}{1-\bar{\beta}}$.

Proof. Differentiating $\tau_{kn}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(k)}(t)-x_\gamma^{(n)}(t-\tau_{kn}(t)\right]^2}$ and solving with respect to $\dot{\tau}_{kn}(t)$ we obtain $\dot{\tau}_{kn}(t)=\frac{1}{c}\frac{\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\rangle-\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\rangle}{\sqrt{\langle\vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\rangle}-\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\rangle}$. Using that (1.4) has a unique solution we have

$$1-\dot{\tau}_{kn}(t)=1-\frac{\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}{c^2\tau_{kn}(t)-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}=\frac{c^2\tau_{kn}(t)-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle+\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}{c^2\tau_{kn}(t)-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}=$$

$=\frac{c^2\tau_{kn}(t)-\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\rangle}{c^2\tau_{kn}(t)-\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\rangle}\ge\frac{c^2\tau_{kn}(t)-c\tau_{kn}(t)\bar{c}}{c^2\tau_{kn}(t)+c\tau_{kn}(t)\bar{c}}=\frac{1-\bar{\beta}}{1+\bar{\beta}}>0$. Obviously $1-\dot{\tau}_{pq}(t)>0$ and therefore $\frac{1}{1-\dot{\tau}_{pq}(t)}\le\frac{1+\bar{\beta}}{1-\bar{\beta}}$.

Lemma (3) is thus proved. □

Introduce the operator B as a 9-tuple

$$B(t)=\left(B_1^{(1)}(u)(t),\ B_2^{(1)}(u)(t),B_3^{(1)}(u)(t),B_1^{(2)}(u)(t),B_2^{(2)}(u)(t),B_3^{(2)}(u)(t),B_1^{(3)}(u)(t),B_2^{(3)}(u)(t),B_3^{(3)}(u)(t)\right),$$

where

$$B_\alpha^{(1)}(u)(t):=\begin{cases}\int_{pT}^{t}U_\alpha^1(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U_\alpha^1(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{v}U_\alpha^1(s)dsdv, & t\in[pT,(p+1)T],\\ & (p=0,\,1,\,2,\,...)\\ u_\alpha^{(10)}(t),\ t\in[-\theta_0,0]\end{cases}$$

$$B_\alpha^{(2)}(u)(t):=\begin{cases}\int_{pT}^{t}U_\alpha^2(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U_\alpha^2(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{v}U_\alpha^2(s)dsdv, & t\in[pT,(p+1)T],\\ & (p=0,\,1,\,2,\,...)\\ u_\alpha^{(20)}(t),\ t\in[-\theta_0,0]\end{cases}\tag{2.3}$$

$$B_\alpha^{(3)}(u)(t):=\begin{cases}\int_{pT}^{t}U_\alpha^3(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U_\alpha^3(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{v}U_\alpha^3(s)dsdv, & t\in[pT,(p+1)T],\\ & (p=0,\,1,\,2,\,...)\\ u_\alpha^{(30)}(t),\ t\in[-\theta_0,0]\end{cases}$$

$(\alpha=1,2,3)$, where $u_\alpha^{(k0)}(t)$ are prescribed infinite differentiable T-periodic functions defined on the initial set $[-\theta_0,0]$, where $\theta_0>0$ is sufficiently large.

For every $u_\alpha^{(k)}(.)\in M_\alpha^{(k)}$ we define the extensions $\tilde{u}_\alpha^{(k)}(t)=\begin{cases}u_\alpha^{(k)}(t),\ t\in[0,\infty)\\ u_\alpha^{(k0)}(t),\ \ t\in[-\theta_0,0]\end{cases}$, where $[-\theta_0,0]$ is the initial set, $\theta_0>0,\ \ \theta_0=qT$ for some positive integer q. In the right-hand-sides $U_\alpha^k(u)(p)$ we substitute the functions with retarded arguments by the initial functions translated to the right on the interval $[0,\infty)$.

Remark 4. *We suppose that the initial functions are such that the translated on* $[0,\infty)$ *function* $\tilde{u}_\alpha^{(k)}(t)$ *belongs to* M_α^k.

Lemma 5. *[6] If the translated function* $\tilde{u}_\alpha^{(k)}(t)$ *on the set* $[0,\infty)$ *satisfies* $\int_{pT}^{(p+1)T}\tilde{u}_\alpha^{(k)}(t)dt=0$ $(p=0,1,2,...)$, *then* $U_\alpha^k(t)$ *is* T*-periodic function.*

Lemma 6. *[6] For every* $u_\alpha^{(k)}\in M_\alpha^k$ *it follows* $\int_{pT}^{(p+1)T}\int_{pT}^{s}U_\alpha^k(\theta)d\theta ds=\int_{(p+1)T}^{(p+2)T}\int_{(p+1)T}^{s}U_\alpha^k(\theta)d\theta ds$ $(p=0,1,2,\dots)$.

Lemma 7. *[6] The functions* $B_\alpha^{(k)}(t)$ *belong to* M_α^k.

Proof. Indeed, $\int_{pT}^{t}U_\alpha^k(u)(s)ds=\int_{(p+1)T}^{t}U_\alpha^k(u)(s)ds+\int_{pT}^{(p+1)T}U_\alpha^k(u)(s)ds=\int_{(p+1)T}^{t}U_\alpha^k(u)(s)ds$ and the assertion of Lemma 6 implies Lemma 7. □

Lemma 8. *(Main Lemma) The periodic problem* (1.2), *has a unique solution*

$$(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)},u_1^{(3)},u_2^{(3)},u_3^{(3)})\in M_0=M_{10}^1\times M_{20}^1\times M_{30}^1\times M_{10}^2\times M_{20}^2\times M_{30}^2\times M_{10}^3\times M_{20}^3\times M_{30}^3$$

if and only if the operator

$$B=\left(B_1^{(1)}(u)(t),\ B_2^{(1)}(u)(t),B_3^{(1)}(u)(t),B_1^{(2)}(u)(t),B_2^{(2)}(u)(t),B_3^{(2)}(u)(t),B_1^{(3)}(u)(t),B_2^{(3)}(u)(t),B_3^{(3)}(u)(t)\right)$$

has a fixed point, belonging to M_0.

Proof. Let $(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, u_1^{(3)}, u_2^{(3)}, u_3^{(3)})$ be a T-periodic solution of

$$\frac{du_\alpha^{(k)}(t)}{dt} = U_\alpha^k(t),\ \ t \in [0,\infty), u_\alpha^{(k)}(t) = u_\alpha^{(k0)}(t),\ \ \frac{du_\alpha^{(k)}(t)}{dt} = \frac{du_\alpha^{(k0)}(t)}{dt}\ ,\ t \in [-\theta,\, 0]. \tag{2.4}$$

Then after integration

$$u_\alpha^{(k)}(t) = \int_{pT}^t U_\alpha^k(t)(t)dt \ \Rightarrow\ 0 = u_\alpha^{(k)}((p+1)T) = \int_{pT}^{(p+1)T} U_\alpha^k(t)dt \ \Rightarrow\ \int_{pT}^{(p+1)T} U_\alpha^k(t)dt = 0$$

and in view of $u_\alpha^{(k)}(pT) = 0$ $(p = 0,1,2,..)$, the components of the operator B become

$$B_\alpha^k(p)(t) = \int_{pT}^t U_\alpha^k(s)ds - \left(\frac{t-pT}{T} - \frac{1}{2}\right)\int_{pT}^{(p+1)T} U_\alpha^k(s)ds\,.$$

Since $U_\alpha^k(.)$, $u_\alpha^k(.)$ are T-periodic functions, changing the order of integration and taking into account $u_\alpha^{(k)}(pT) = 0$ we obtain

$$\begin{aligned}\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)dsd\theta &= \int_{pT}^{(p+1)T}[\,(p+1)T - s]\,U_\alpha^k(s)ds = (p+1)T\int_{pT}^{(p+1)T} U_\alpha^k(s)ds - \int_{pT}^{(p+1)T} sU_\alpha^k(s)ds\\ &= -\int_{pT}^{(p+1)T} sU_\alpha^k(s)ds.\end{aligned}$$

But $\frac{du_\alpha^{(k)}(s)}{ds} = U_\alpha^k(s)$ and therefore

$$\begin{aligned}\int_{pT}^{(p+1)T} sU_\alpha^k(s)ds &= \int_{pT}^{(p+1)T} s\frac{du_\alpha^{(k)}(s)}{ds}ds = \int_{pT}^{(p+1)T} sd\left(u_\alpha^{(k)}(s)\right)\\ &= \left[(p+1)T\,u_\alpha^{(k)}((p+1)T) - pT\,u_\alpha^{(k)}(pT)\right] - \int_{pT}^{(p+1)T} u_\alpha^{(k)}(s)ds = 0.\end{aligned}$$

Consequently $\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)dsd\theta = 0$. Therefore the operator B has the form

$$B_\alpha^k(.)(t) = \int_{pT}^t U_\alpha^{(k)}(s)\,ds - \left(\frac{t-pT}{T} - \frac{1}{2}\right)\int_{pT}^{(p+1)T} U_\alpha^{(k)}(s)\,ds - \frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)\,ds\,d\theta.$$

In other words, the operator B has a fixed point.

Conversely, let

$$(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, u_1^{(3)}, u_2^{(3)}, u_3^{(3)}) \in M_0$$

be a fixed point of B, that is, $u_\alpha^{(k)} = B_\alpha^{(k)}(u_1^{(1)}, ..., u_3^{(3)})$, $(k = 1,\,2,\,3;\ \alpha = 1,\,2,\,3)$. Therefore

$$u_\alpha^{(k)}(pT) = B_\alpha^{(k)}(u_1^{(1)}, ...,\ u_3^{(3)})(pT)$$

or

$$\begin{aligned}0 &= u_\alpha^{(k)}(pT) = B_\alpha^{(k)}(u_1^{(1)}, ...,\ u_3^{(3)})\,(pT) = \int_{pT}^{pT} U_\alpha^k(s)\,ds - \left(\frac{pT-pT}{T} - \frac{1}{2}\right)\int_{pT}^{(p+1)T} U_\alpha^k(s)\,ds -\\ &\quad -\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)\,ds\,d\theta\\ &= \frac{1}{2}\int_{pT}^{(p+1)T} U_\alpha^k(s)ds - \frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)\,ds\ d\theta\,.\end{aligned}$$

It follows

$$\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)\,ds\,d\theta = \frac{1}{2}\int_{pT}^{(p+1)T} U_\alpha^k(s)\,ds. \tag{2.5}$$

We show that $\int_{pT}^{(p+1)T} U_\alpha^k(s)ds = 0$. Indeed, in view of Inequalities **17)** and **18)** from the Appendix we get

$$\begin{aligned}&\left|\int_{pT}^{(p+1)T} U_\alpha^k ds\right| \le \sum_{\gamma=1}^3 \sum_{n=1,n\neq k}^3 \left|\int_{pT}^{(p+1)T} G_\gamma^{(kn)} ds\right| \\ &\le \sum_{\gamma=1}^3 \sum_{n=1,n\neq k}^3 \frac{|e_k e_n|}{m_k c^2} \left|\int_{pT}^{(p+1)T} \Delta_k \left(A_{kn}\xi_\gamma^{(kn)} - B_{kn}u_\gamma^{(n)} + C_{kn}\dot{u}_\gamma^{(n)}\right) ds\right| \\ &\le \frac{|e_k e_n|}{m_k}\left(\frac{8\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3} + \frac{36\omega\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2} + \frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}} + \frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)\frac{e^{\mu T}-1}{\mu}.\end{aligned}$$

So, if $\delta = \left|\int_{pT}^{(p+1)T} U_\alpha^k(s)ds\right| > 0$ for sufficiently large $\mu > \omega$ and $h \in N$ the inequality might be violated. Therefore in view of (2.5) the operator

$$B_\alpha^{(k)}(u_1^{(1)},...,u_3^{(3)})(t) := \int_{pT}^t U_\alpha^k(s)\,ds - \left(\frac{t-pT}{T} - \frac{1}{2}\right)\int_{pT}^{(p+1)T} U_\alpha^k(s)\,ds - \frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta} U_\alpha^k(s)\,ds\,d\theta$$

becomes $B_\alpha^{(k)}(u_1^{(1)},..., u_3^{(3)})(t) = \int_{pT}^t U_\alpha^k(s)\,ds$. Differentiating the last equalities we obtain that the fixed point of the operator B is a T-periodic solution of (2.4).

Lemma 8 is thus proved. □

3. Existence-Uniqueness of Periodic Solution of the System of Equations of Motion

Here we prove the main result:

Theorem 9. *(Main Result) Let Assumptions (C) and (T) be satisfied, the initial trajectories $x_\alpha^{(k0)}(t)$ and velocities $u_\alpha^{(k0)}(t)$, $t \in [-\theta_0; 0]$ be T-periodic infinitely differentiable functions such that*

$$r_{kn}(t) = \sqrt{\sum_{\gamma=1}^3\left(x_\gamma^{(k0)}(t) - x_\gamma^{(n0)}(t)\right)} \ge r_{kn}^{(0)} > 0\,,\ t \in [-\theta_0,\, 0]$$

and their translations on $[0,\infty)$ belong to M_0, where $r_{kn}^{(0)}$ is the minimal distance between the k-th and n-th particles. If the following inequalities are satisfied:

$$\begin{aligned}&3\frac{e^{\mu T}}{\mu}\left[\sum_{n=1,n\neq k}^3 \frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3} + \frac{36\omega\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2} + \frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}} + \frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right) + \right.\\ &\qquad\left. + \frac{e_k^2}{m_k c^2}\frac{\sqrt{3}+1}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right] \le U_0\end{aligned}$$

$$\begin{aligned}&3\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left[\sum_{n=1,n\neq k}^3 \frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3} + \frac{36\omega\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2} + \frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}} + \frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right) + \right.\\ &\qquad\left. + \frac{e_k^2}{m_k c^2}\frac{\sqrt{3}+1}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right] \le \omega U_0\end{aligned}$$

then there exists a unique T-periodic solution, $(u_1^{(1)}, u_2^{(1)}, ..., u_3^{(3)}) \in M_{10}^1 \times M_{20}^1 \times ... \times M_{30}^3$ of (1.2).

Proof. With accordance of the Main Lemma 8 we have to prove that the operator defined by (2.3) possesses a unique fixed point which means that the 3-body problem has a unique periodic solution.

First we show that the operator B maps M_0 into itself.

We note that the set M_0 can be considered as a uniform space with saturates family of pseudo-metrics formed by the following way

$$\left\{\rho_{(p,m)}\left((u_1, u_2, ..., u_9), (\bar u_1, \bar u_2, ..., \bar u_9)\right) = \sum_{q=1}^9 \rho_{(p,m)}\left(u_q, \bar u_q\right)\ :\ p = 0, 1, ...\ ;\ m = 0, 1, ...\right\},$$

where $\left\{\rho_{(p,m)}\left(u_\alpha^{(k)},\bar{u}_\alpha^{(k)}\right): p=0,1,...\ ;\ m=0,1,...\right\}$ and

$$(u_1,u_2,...,u_9)=(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)},u_1^{(3)},u_2^{(3)},u_3^{(3)}).$$

In view of (2.3)

$$\begin{aligned}B_\alpha^{(k)}(u_1^{(1)},...,u_3^{(3)})(0) &= \int_0^0 U_\alpha^k(s)ds-\left(\tfrac{0}{T}-\tfrac12\right)\int_0^T U_\alpha^k(s)ds-\tfrac1T\int_0^T\int_0^\theta U_\alpha^k(s)dsd\theta\\ &= \tfrac12\int_0^T U_\alpha^k(s)ds-\tfrac1T\int_0^T\int_0^\theta U_\alpha^k(s)dsd\theta=0.\end{aligned}$$

Since, in general, $\frac{dB_\alpha^{(k)}(u_1^{(1)},...,u_3^{(3)})(0)}{dt}=U_\alpha^k(0)-\frac1T\int_0^T U_\alpha^k(s)ds\neq0$, we form for every derivative of B a convolution with the "hat" function $\frac{d^{(m)}B_\alpha^{(k)}(t,\varepsilon)}{dt^m}=\eta_\varepsilon(t)*\frac{d^{(m)}B_\alpha^{(k)}(u_1^{(1)},...,u_3^{(3)})(t)}{dt^m}$, where $\eta_\varepsilon(t)\in C^\infty$. Then $\frac{d^{(m)}B_\alpha^{(k)}(0,\varepsilon)}{dt^m}=0$ and the set of such convolutions is dense in $M_{\alpha0}^k$, that is, $\lim_{\varepsilon\to0}\frac{d^{(m)}B_\alpha^{(k)}(t,\varepsilon)}{dt^m}$ exists in M_α^k (cf. [14], [15]).

In view of $\int_{pT}^{(p+1)T}\left(\frac{t-pT}{T}-\frac12\right)dt=0$ we obtain

$$\begin{aligned}\int_{pT}^{(p+1)T}B_\alpha^{(k)}(p)(u_1^{(1)},...,u_3^{(3)})(t)dt &= \int_{pT}^{(p+1)T}\int_{pT}^\theta U_\alpha^k(s)dsd\theta-\int_{pT}^{(p+1)T}\left(\tfrac{t-pT}{T}-\tfrac12\right)dt\int_{pT}^{(p+1)T}U_\alpha^k(s)ds-\\ &\quad -T\tfrac1T\int_{pT}^{(p+1)T}\int_{pT}^\theta U_\alpha^k(s)dsd\theta=\ 0,\end{aligned}$$

that is, $B_\alpha^{(k)}(p)(u_1^{(1)},...,u_3^{(3)})(.)\in M_{\alpha0}^k$.

The inequalities from Theorem 9 imply (cf.Appendix):

$$\begin{aligned}\left|B_\alpha^{(k)}(u)(t)\right| &\le \left|\int_{pT}^t U_\alpha^k(s)ds\right|+\left|\int_{pT}^{(p+1)T}U_\alpha^k(s)ds\right|\\ &\le e^{\mu(t-pT)}3\frac{e^{\mu T}}{\mu}\left[\sum_{n=1,n\neq k}^3\frac{|e_ke_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3r_{kn}^{(0)}}+\frac{8}{c^2r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)\right.\\ &\quad\left.+\frac{e_k^2}{m_kc^2}\frac{3}{\sqrt{(1-\bar\beta^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]\le U_0e^{\mu(t-pT)}.\end{aligned}$$

For the first derivative we have

$$\begin{aligned}\left|\dot B_\alpha^{(k)}(t)\right| &\le \left|U_\alpha^k(t)\right|+\left|\frac1T\int_0^T U_\alpha^k(s)ds\right|\\ &\le e^{\mu(t-pT)}\times3\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left[\sum_{n=1,n\neq k}^3\frac{|e_ke_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\right.\right.\\ &\quad\left.\left.+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3r_{kn}^{(0)}}+\frac{8}{c^2r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+\frac{e_k^2}{m_kc^2}\frac{3}{\sqrt{(1-\bar\beta^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]\\ &\le \omega U_0e^{\mu(t-pT)}\end{aligned}$$

and so, on. Obviously for sufficiently large μ and h and sufficiently small initial dates we have $\left|\dot B_\alpha^{(k)}(t)\right|\le\omega U_0e^{\mu(t-pT)}$.

Further on, for higher order derivatives we proceed in a similar way.

Consequently, B maps M_0 into itself.

It remains to show that B is a contractive operator in the sense of [1].

Indeed, the set M_0 turns out into a uniform space. We estimate the difference

$$\left|B_\alpha^{(k)}(u)(t)-B_\alpha^{(k)}(\bar u)(t)\right| \le \int_{pT}^{t}\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+\left|\frac{t-pT}{T}-\frac12\right|\int_{pT}^{(p+1)T}\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+$$

$$+\frac1T\int_{pT}^{(p+1)T}\int_{pT}^{\theta}\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|dsd\theta$$

$$\le \int_{pT}^{t}\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+\frac1T\int_{pT}^{(p+1)T}\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds$$

$$\le \sum_{n=1,n\ne k}^{3}\Xi_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{t}\left|\xi_\gamma^{(kn)}(s)-\bar\xi_\gamma^{(kn)}(s)\right|ds$$

$$+\left(\sum_{n=1,n\ne k}^{3}V_{kn}+\frac{5_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar\beta\right)^{5/2}}\right)\sum_{\gamma=1}^{3}\int_{pT}^{t}\left|u_\gamma^{(k)}(s)-\bar u_\gamma^{(k)}(s)\right|ds+$$

$$+\sum_{n=1,n\ne k}^{3}U_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{t}\left|u_\gamma^{(n)}(s-\tau_{kn})-\bar u_\gamma^{(n)}(s-\tau_{kn})\right|ds$$

$$+\sum_{n=1,n\ne k}^{3}\dot U_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{t}\left|\dot u_\gamma^{(n)}(s-\tau_{kn})-\dot{\bar u}_\gamma^{(n)}(s-\tau_{kn})\right|ds$$

$$+\frac{e_k^2}{m_k}\frac{4}{c^3\left(1-\bar\beta\right)^{1/2}}\sum_{\gamma=1}^{3}\int_{pT}^{t}\left|\ddot u_\gamma{}^{(k)}(s)-\ddot{\bar u}_\gamma{}^{(k)}(s)\right|ds$$

$$+\frac1T\left[\sum_{n=1,n\ne k}^{3}\Xi_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|\xi_\gamma^{(kn)}(s)-\bar\xi_\gamma^{(kn)}(s)\right|ds\right.$$

$$+\left(\sum_{n=1,n\ne k}^{3}V_{kn}+\frac{5_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar\beta\right)^{5/2}}\right)\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|u_\gamma^{(k)}(s)-\bar u_\gamma^{(k)}(s)\right|ds$$

$$+\sum_{n=1,n\ne k}^{3}U_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|u_\gamma^{(n)}(s-\tau_{kn})-\bar u_\gamma^{(n)}(s-\tau_{kn})\right|ds$$

$$+\sum_{n=1,n\ne k}^{3}\dot U_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|\dot u_\gamma^{(n)}(s-\tau_{kn})-\dot{\bar u}_\gamma^{(n)}(s-\tau_{kn})\right|ds$$

$$\left.+\frac{e_k^2}{m_k}\frac{4}{c^3\left(1-\bar\beta\right)^{1/2}}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|\ddot u_\gamma{}^{(k)}(s)-\ddot{\bar u}_\gamma{}^{(k)}(s)\right|ds\right]$$

$$\le e^{\mu(t-pT)}\rho_{(p,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(3)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(3)})\right)\times$$

$$\times\left(\frac{2\omega^hU_0}{\mu^{h+2}}\sum_{n=1,n\ne k}^{3}\Xi_{kn}+\frac{2\omega^h}{\mu^{h+1}}\left(\sum_{n=1,n\ne k}^{3}V_{kn}+\frac{e_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar\beta\right)^{5/2}}\right)\right.$$

$$\left.+\frac{3\omega^h}{\mu^{h+1}}\sum_{n=1,n\ne k}^{3}U_{kn}+\frac{3\omega^h}{\mu^h}\sum_{n=1,n\ne k}^{3}\dot U_{kn}+\frac{e_k^2}{m_k}\frac{3\omega^{h+1}}{\mu^h}\frac{4}{c^3\left(1-\bar\beta\right)^{1/2}}\right)+$$

$$+e^{\mu(t-pT)}\rho_{(p,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(3)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(3)})\right)\times$$

$$\times\frac{e^{\mu T}-1}{T}\left(\frac{2\omega^hU_0}{\mu^{h+2}}\sum_{n=1,n\ne k}^{3}\Xi_{kn}+\frac{2\omega^h}{\mu^{h+1}}\left(\sum_{n=1,n\ne k}^{3}V_{kn}+\frac{5_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar\beta\right)^{5/2}}\right)\right.$$

$$\left.+\frac{3\omega^h}{\mu^{h+1}}\sum_{n=1,n\ne k}^{3}U_{kn}+\frac{3\omega^h}{\mu^h}\sum_{n=1,n\ne k}^{3}\dot U_{kn}+\frac{5_k^2}{m_k}\frac{3\omega^{h+1}}{\mu^h}\frac{4}{c^3\left(1-\bar\beta\right)^{1/2}}\right)$$

$$\le e^{\mu(t-pT)}\rho_{(p,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(3)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(3)})\right)\times$$

$$\times\left(1+\frac{e^{\mu T}-1}{T}\right)\left(\frac{2\omega^hU_0}{\mu^{h+2}}\sum_{n=1,n\ne k}^{3}\Xi_{kn}+\frac{2\omega^h}{\mu^{h+1}}\left(\sum_{n=1,n\ne k}^{3}V_{kn}+\frac{e_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar\beta\right)^{5/2}}\right)\right.$$

$$\left.+\frac{3\omega^h}{\mu^{h+1}}\sum_{n=1,n\ne k}^{3}U_{kn}+\frac{3\omega^h}{\mu^h}\sum_{n=1,n\ne k}^{3}\dot U_{kn}+\frac{e_k^2}{m_k}\frac{3\omega^{h+1}}{\mu^h}\frac{4}{c^3\left(1-\bar\beta\right)^{1/2}}\right)$$

Therefore

$$\rho_{(p,0)}\left(B_1^{(1)},B_2^{(1)},B_3^{(1)},B_1^{(2)},B_2^{(2)},B_3^{(2)},B_1^{(3)},B_2^{(3)},B_3^{(3)}\right),\left(\bar{B}_1^{(1)},\bar{B}_2^{(1)},\bar{B}_3^{(1)},\bar{B}_1^{(2)},\bar{B}_2^{(2)},\bar{B}_3^{(2)},\bar{B}_1^{(3)},\bar{B}_2^{(3)},\bar{B}_3^{(3)}\right)\leq$$

$$\leq K\rho_{(p,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(3)}),(\bar{u}_1^{(1)},\bar{u}_2^{(1)},...,\bar{u}_3^{(3)})\right)$$

where

$$K \;=\frac{\omega^h}{\mu^h}\left(1+\frac{e^{\mu T}-1}{T}\right)\left(\frac{2U_0}{\mu^2}\sum_{n=1,n\neq k}^{3}\Xi_{kn}+\frac{2}{\mu}\left(\sum_{n=1,n\neq k}^{3}V_{kn}+\frac{e_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar{\beta}\right)^{5/2}}\right)\right.$$
$$\left.+\frac{3}{\mu}\sum_{n=1,n\neq k}^{3}U_{kn}+3\sum_{n=1,n\neq k}^{3}\dot{U}_{kn}+\frac{e_k^2}{m_k}\frac{`12\omega}{c^3\left(1-\bar{\beta}\right)^{1/2}}\right)<1$$

for sufficiently large $h\in N$and $\mu>\omega$.

Since the operator B is continuous and defined on a dense subset of M_0, then it can be extended on the whole space in view of the know continuation result (cf. [13]).

Define a map of the index set into itself $j(p,m)\to(p,m+h)$.

The inequality (2.2) implies that the space M_0 is j-bounded in the sense given in [1]. Consequently, the operator B is contractive one and has a unique fixed point. It is a T-periodic solution of the 3-body problem in view of Lemma 8.

The Theorem 9 is thus proved. □

4. Conclusion: Numerical Example Concerning *He*-Atom

Let us consider the inequalities implying an existence-uniqueness of periodic solution

$$3\frac{e^{\mu T}}{\mu}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar{\beta}\right)^5}\left(\frac{8\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+\right.$$
$$\left.+\frac{5_k^2}{m_k c^2}\frac{\sqrt{3}+1}{(1-\bar{\beta}^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]\leq U_0.$$

Let $m_1=1836\,m_2=1836\,m_3$, that is, the first particle is the nucleus of the He-atom, while the second and third are moving electrons. We have $m_1=1836m_2,\ m_1=1836m_3;e_1=2e_0,\ e_2=-e_0\,,\ \ e_3=-e_0$, where $e_0=1,6.10^{-19}C$;

$$m_2=m_3=9.10^{-31}kg;\ \frac{|e_1e_2|}{m_1}=\frac{|e_1e_3|}{m_1}=\frac{2e_0{}^2}{2.1836m_2+2m_2}=\frac{(1,6.10^{-19})^2}{1837.9.10^{-31}}\approx 1,55.10^{-11};$$

$$\frac{|e_2e_1|}{m_2}=\frac{2e_0^2}{m_2}=\frac{2.(1,6.10^{-19})^2}{9.10^{-31}}\approx 5,5.10^{-8};\ \frac{|e_2e_3|}{m_2}=2,75.10^{-8};\ \frac{|e_3e_1|}{m_3}=\frac{|e_2e_1|}{m_2}=5,5.10^{-8};$$

$$\frac{|e_3e_2|}{m_3}=2,75.10^{-8};\tau=\tau_0\sqrt{1-\bar{\beta}^2}=9,4.10^{-24}.\sqrt{1-(1/137^2)}\approx 9,39.10^{-24}.$$

Since $\omega<\frac{2}{\tau}=\frac{2}{9{,}39.10^{-24}}\approx 0,21.10^{24}=2,1.10^{23}$ we have to check the angular velocities of the electrons. Indeed, the radius of the He-atom is $r_{12}=r_{13}=3,1.10^{-11}m$, then the velocity is $u=\frac{c}{137}=r_{12}\omega\Rightarrow\ \omega=\frac{3.10^8}{137.3{,}1.10^{-11}}\approx 7.10^{16}\ \Rightarrow\ T=\frac{2\pi}{\omega}=\frac{6{,}28}{7.10^{16}}\approx 8,9.10^{-17}\ \Rightarrow\ f=\frac{1}{T}=1,1.10^{16}Hz\ \Rightarrow\ \lambda=\frac{c}{f}=\frac{3.10^8}{1{,}1.10^{16}}\approx 2,73.10^{-8}m$. Obviously $\omega=7.10^{16}<2,1.10^{23}$.

Let us take $\mu=8.10^{16}>\omega=7.10^{16}\Rightarrow\omega/\mu<1$. It remains to calculate $1-\bar{\beta}\approx 1-(1/137)\approx 0,993$ and then

$\frac{1}{\left(1-\bar{\beta}\right)^5}\approx\frac{1}{0{,}993^5}=1,04;\ \mu T=8.10^{16}.8,9.10^{-17}\approx 7,12\Rightarrow 5^{7,12}\approx 1236,5.$ Since $\frac{e_1^2}{m_1c^2}\frac{3}{(1-\bar{\beta}^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}$ and

$\frac{8}{c^2 5,3.10^{-11}}\frac{\omega^h}{\mu^h}\omega$ can be made sufficiently small, we consider only

$$3\tfrac{1236,5}{8.10^{16}}1,04\sum_{n=2}^{3}1,55.10^{-11}\left(\frac{8\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(n0)}\right\|}{(3,1.10^{-11})^3}+\frac{36.7.10^{16}\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(n0)}\right\|}{3.10^8(3,1.10^{-11})^2}+\right.$$
$$\left.+\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{(3.10^8)^3(3,1.10^{-11})^2}+\frac{16.7.10^{16}\left\|\vec{u}_0^{(n0)}\right\|}{(3.10^8)^3 3,1.10^{-11}}\right)\leq U_0;$$

$$3\tfrac{1236,5}{8.10^{16}}1,04\sum_{n=1,3}5,5.10^{-8}\left(\frac{8\left\|\vec{x}_0^{(20)}-\vec{x}_0^{(n0)}\right\|}{(3,1.10^{-11})^3}+\frac{36.7.10^{16}\left\|\vec{x}_0^{(20)}-\vec{x}_0^{(n0)}\right\|}{3.10^8(3,1.10^{-11})^2}+\right.$$
$$\left.+\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{(3.10^8)^3(3,1.10^{-11})^2}+\frac{16.7.10^{16}\left\|\vec{u}_0^{(n0)}\right\|}{(3.10^8)^3 3,1.10^{-11}}\right)\leq U_0;$$

$$3\tfrac{1236,5}{8.10^{16}}1,04\sum_{n=1,2}5,5.10^{-8}\left(\frac{8\left\|\vec{x}_0^{(30)}-\vec{x}_0^{(n0)}\right\|}{(3,1.10^{-11})^3}+\frac{36.7.10^{16}\left\|\vec{x}_0^{(30)}-\vec{x}_0^{(n0)}\right\|}{3.10^8(3,1.10^{-11})^2}+\right.$$
$$\left.+\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{(3.10^8)^3(3,1.10^{-11})^2}+\frac{16.7.10^{16}\left\|\vec{u}_0^{(n0)}\right\|}{(3.10^8)^3 3,1.10^{-11}}\right)\leq U_0.$$

To estimate the order of U_0 we take into account $U_0\leq\bar{c}e^{-\mu T}=\left(3.10^8/137\right).e^{-7,12}\approx 2,19.10^6.8.10^{-4}\approx$ 1832. Since $\left\|\vec{u}_0^{(n0)}\right\|\leq 3.10^8\ \Rightarrow\ 10^{-21}.3.10^8\approx 0$, then

$$6,3.10^8\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(20)}\right\|+10^{-21}\left\|\vec{u}_0^{(20)}\right\|+6,3.10^8\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(30)}\right\|+10^{-21}\left\|\vec{u}_0^{(30)}\right\|\leq U_0$$

becomes

$$6,3.10^8\left(\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(20)}\right\|+\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(30)}\right\|\right)\leq U_0;$$
$$6,3.10^{11}\left(\left\|\vec{x}_0^{(20)}-\vec{x}_0^{(10)}\right\|+\left\|\vec{x}_0^{(20)}-\vec{x}_0^{(30)}\right\|\right)\leq U_0;$$
$$6,3.10^{11}\left(\left\|\vec{x}_0^{(30)}-\vec{x}_0^{(10)}\right\|+\left\|\vec{x}_0^{(30)}-\vec{x}_0^{(20)}\right\|\right)\leq U_0.$$

Consequently for sufficiently small initial data $\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(20)}\right\|$, $\left\|\vec{x}_0^{(10)}-\vec{x}_0^{(30)}\right\|$, $\left\|\vec{x}_0^{(20)}-\vec{x}_0^{(30)}\right\|$the inequalities are satisfied.

If we substitute a larger radius of excited states, then obviously the inequalities are moreover satisfied.

5. Appendix

5.1. Some Inequalities (A1)

Let $r_{kn}(t)$ be the distance between the k-th and the n-th particle. Since $r_{kn}(t)=r_{nk}(t)$ it is obviously that
$(kn)=(12),(13),(23)$. We use the inequalities $\left|\xi_\alpha^{(kn)}\right|\leq\sqrt{\left\langle\xi^{(kn)},\xi^{(kn)}\right\rangle}=c\tau_{kn}$,$\tau_{kn}\geq\frac{r_{kn}(t)}{2c}\geq\frac{r_{kn}^{(0)}}{2c}>0$ for $(k=1,2,3)$, $n\neq k$.
Introduce the denotation

$$\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|=\sqrt{\left\langle\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)},\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\rangle},$$
$$\left\|\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\|=\sqrt{\left\langle\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)},\vec{u}_0^{(k0)}-\vec{u}_0^{(n0)}\right\rangle}.$$

If $r_{kn}(t) \geq r_{kn}^{(0)} > 0$ and $U_0 e^{\mu T} \leq \bar{c} < c$, then the following statements hold good:

1) $c^2\tau_{kn} - \left\langle \xi^{(kn)}, u^{(n)} \right\rangle \geq c^2\tau_{kn} - \sqrt{\sum_{\alpha=1}^{3}\left(\xi_\alpha^{(kn)}\right)^2}\sqrt{\sum_{\alpha=1}^{3}\left(u_\alpha^{(n)}\right)^2} \geq c^2\tau_{kn} - c\tau_{kn}\bar{c} \geq c^2\frac{r_{kn}(t)}{2c}\left(1-\bar{\beta}\right)$
$> \frac{c r_{kn}^{(0)}\left(1-\bar{\beta}\right)}{2} > 0 \Rightarrow \frac{1}{\tau_{kn}} \leq \frac{2c}{r_{kn}^{(0)}}.$

$$\begin{aligned} 2)\ \left|u_\alpha^{(k)}(t)\right| &\leq \left|u_{\alpha 0}^{(k0)} + \int_{pT}^{t}\frac{du_\alpha^{(k)}(t_1)}{dt_1}dt_1\right| = \left|u_{\alpha 0}^{(k0)} + \int_{pT}^{t}\left(\int_0^{t_1}\frac{d^2u_\alpha^{(k)}(t_2)}{dt_2{}^2}dt_2\right)ds\right| \\ &= \left|u_{\alpha 0}^{(k0)} + \int_{pT}^{t}\left(\int_{pT}^{t_1}\left(\int_{pT}^{t_2}\frac{d^3u_\alpha^{(k)}(t_3)}{dt_3{}^3}dt_3\right)dt_2\right)dt_1\right| \leq \ldots \leq \Gamma_k e^{\mu(t-pT)}, \\ \Gamma_k &= \left\|\vec{u}_0^{(k0)}\right\| + \frac{\omega^h}{\mu^h}U_0; \end{aligned}$$

3) In view of $x_{\alpha 0}^{(n0)} = x_\alpha^{(n0)}(0) = x_\alpha^{(n0)}(pT - p_{kn}T) = x_\alpha^{(n)}(pT)$ we obtain

$$\begin{aligned} \xi^{(kn)} &= x_\alpha^{(k)}(t) - x_\alpha^{(n)}(t-\tau_{kn}) = x_{\alpha 0}^{(k0)} + \int_{pT}^{t}u_\alpha^{(k)}(t_1)dt_1 - x_{\alpha 0}^{(n0)} - \int_{pT}^{t-\tau_{kn}}u_\alpha^{(n)}(t_1)dt_1 \\ &= x_{\alpha 0}^{(k0)} - x_{\alpha 0}^{(n0)} + \int_{pT}^{t}\left(u_{\alpha 0}^{(k0)} + \int_{pT}^{t_1}\dot{u}_\alpha^{(k)}(t_2)dt_2\right)dt_1 - \int_{pT}^{t-\tau_{kn}}\left(u_{\alpha 0}^{(n0)} + \int_{pT}^{t_1}\dot{u}_\alpha^{(n)}(t_2)dt_2\right)dt_1 \\ &= x_{\alpha 0}^{(k0)} - x_{\alpha 0}^{(n0)} + \int_{pT}^{t}u_{\alpha 0}^{(k0)}dt_1 - \int_{pT}^{t-\tau_{kn}}u_{\alpha 0}^{(n0)}dt_1 + \int_{pT}^{t}\int_{pT}^{t_1}\left(\int_{pT}^{t_2}\ddot{u}_\alpha^{(k)}(t_3)dt_3\right)dt_2dt_1 \\ &\quad - \int_{pT}^{t-\tau_{kn}}\int_{pT}^{t_1}\left(\int_{pT}^{t_2}\ddot{u}_\alpha^{(n)}(t_3)dt_3\right)dt_2dt_1 \\ &= x_{\alpha 0}^{(k0)} - x_{\alpha 0}^{(n0)} + \int_{pT}^{t}u_{\alpha 0}^{(k0)}dt_1 - \int_{pT}^{t-\tau_{kn}}u_{\alpha 0}^{(n0)}dt_1 + \int_{pT}^{t}\int_{pT}^{t_1}\dot{u}_{\alpha 0}^{(k0)}dt_2dt_1 - \int_{pT}^{t-\tau_{kn}}\int_{pT}^{t_1}\dot{u}_{\alpha 0}^{(n0)}dt_2dt_1 \\ &\quad + \int_{pT}^{t}\int_{pT}^{t_1}\int_{pT}^{t_2}\ddot{u}_\alpha^{(k)}(t_3)dt_3dt_2dt_1 - \int_{pT}^{t-\tau_{kn}}\int_{pT}^{t_1}\int_{pT}^{t_2}\ddot{u}_\alpha^{(n)}(t_3)dt_3dt_2dt_1. \end{aligned}$$

Therefore

$$\left|\xi^{(kn)}\right| \leq \left\|\vec{x}_0^{(k0)} - \vec{x}_0^{(n0)}\right\| e^{\mu(t-pT)},$$

where

$$\left\|\vec{x}_0^{(k0)} - \vec{x}_0^{(n0)}\right\| + \frac{\left\|\vec{u}_0^{(k0)} - \vec{u}_0^{(n0)}\right\| + \vec{u}_0^{(n0)}}{\mu} + \frac{2\omega^h U_0}{\mu^{h+1}} \approx \left\|\vec{x}_0^{(k0)} - \vec{x}_0^{(n0)}\right\|.$$

4) $\left|u_\gamma^{(n)}(s-\tau_{kn})\right| = \left|u_\gamma^{(n0)}(-\tau_{kn}(0)) + \int_{pT}^{s-\tau_{kn}}\dot{u}_\gamma^{(n)}(s_1)ds_1\right| \leq \left\|\vec{u}_0^{(n0)}\right\| e^{\mu(s-pT)},\ \left\|\vec{u}_0^{(n0)}\right\| + \frac{\omega^h}{\mu^h}U_0 \approx \left\|\vec{u}_0^{(n0)}\right\|.$

5) $\left|\dot{u}_\alpha^{(k)}(t)\right| \leq \frac{\omega^h}{\mu^h}\omega U_0 e^{\mu(t-pT)}$; 7) $\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})\right| \leq \frac{\omega^h}{\mu^h}\omega U_0 e^{\mu(t-pT)}.$

6) $|D_{kn}| = \left|\frac{c^2\tau_{kn} - \left\langle\vec{\xi}^{(kn)}, \vec{u}^{(n)}\right\rangle}{c^2\tau_{kn} - \left\langle\vec{\xi}^{(kn)}, \vec{u}^{(k)}\right\rangle}\right| \leq \frac{c^2\tau_{kn} + c\tau_{kn}\bar{c}}{c^2\tau_{kn} - c\tau_{kn}\bar{c}} = \frac{1+\bar{\beta}}{1-\bar{\beta}}$; 9) $|H_{kn}| \leq c^2 + c\tau_{kn}\frac{3\omega U_0 e^{\mu(t-pT)}}{(1-\bar{\beta})^2} \leq c^2 + \frac{3\omega\, c^2\tau_{kn}}{(1-\bar{\beta})^2}.$

7) $|A_{kn}| \leq \frac{2c^2\left(c^2 + \frac{3\omega c^2\tau_{kn}}{(1-\bar{\beta})^2}\right)}{c^6\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3} + \frac{1+\bar{\beta}}{1-\bar{\beta}}\frac{c^2c\omega + \left(2c^2\right)c\omega}{c^2\left(1-\bar{\beta}^2\right)c^4\left(1-\bar{\beta}\right)^2\tau_{kn}{}^2}U_0e^{\mu(t-pT)} \leq \frac{2}{c^2\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3} + \frac{9\omega}{c^2\left(1-\bar{\beta}\right)^5\tau_{kn}{}^2}.$

8) $|B_{kn}| \leq \frac{2c^2\left(\tau_{kn}c^2 + \tau_{kn}\frac{3\omega c^2\tau_{kn}}{(1-\bar{\beta})^2}\right)}{c^6\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3} + \frac{1+\bar{\beta}}{1-\bar{\beta}}\frac{2c^3\tau_{kn}\omega U_0e^{\mu(t-pT)}}{c^2\left(1-\bar{\beta}^2\right)c^4\left(1-\bar{\beta}\right)^2\tau_{kn}{}^2} \leq \frac{2}{c^2\left(1-\bar{\beta}\right)^5\tau_{kn}{}^2} + \frac{8\omega}{\left(1-\bar{\beta}\right)^5\tau_{kn}} \leq \frac{8}{c^2\left(1-\bar{\beta}\right)^5\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega}{c^2\left(1-\bar{\beta}\right)^5 r_{kn}^{(0)}}.$

9) $|C_{kn}| \leq \frac{4}{c^2\left(1-\bar{\beta}\right)^3\tau_{kn}} \leq \frac{8}{c\left(1-\bar{\beta}\right)^3 r_{kn}^{(0)}}.$

10) $\left|G_\alpha^{(k)rad}\right| \leq \frac{5_k^2}{m_kc^2}\left(\frac{c^2}{c^3\sqrt{(1-\bar{\beta}^2)^3}}\sqrt{\sum_{\gamma=1}^{3}\left(\frac{\dot{u}_\gamma{}^{(k)}(t+\tau) - \dot{u}_\gamma{}^{(k)}(t-\tau)}{2\tau}\right)^2} + \frac{1}{c\sqrt{1-\bar{\beta}^2}}\left|\frac{\dot{u}_\alpha^{(k)}(t+\tau) - \dot{u}_\alpha^{(k)}(t-\tau)}{2\tau}\right|\right)$

$\leq \frac{e_k^2}{m_k}\frac{3}{c^3\sqrt{(1-\bar{\beta}^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}.$

11) $\left|u_\alpha^{(k)}(t)-u_\alpha^{(k)}(t)\right|\leq\frac{\omega^h}{\mu^h}\rho_{(p,h)}(u,\bar{u})e^{\mu(t-pT)};\left|\dot{u}_\alpha^{(k)}(t)-\dot{\bar{u}}_\alpha^{(k)}(t)\right|\leq\ ...\leq\frac{\omega^h}{\mu^h}\omega\rho_{(p,h)}(u,\bar{u})e^{\mu(t-pT)}.$

12) $\left|u_\gamma^{(n)}(\theta-\tau_{kn})-\bar{u}_\gamma^{(n)}(\theta-\tau_{kn})\right|=\left|\int_{pT}^{\theta-\tau_{kn}}\left(\dot{u}_\gamma^{(n)}(\theta_1)-\dot{\bar{u}}_\gamma^{(n)}(\theta_1)\right)d\theta_1\right|$
$\leq\left|\int_{pT}^{\theta-\tau_{kn}}\left(\dot{u}_\gamma^{(n)}(\theta_1)-\dot{\bar{u}}_\gamma^{(n)}(\theta_1)\right)d\theta_1\right|+\left|\int_{\theta-\tau_{kn}}^{\theta}\left(\dot{u}_\gamma^{(n)}(\theta_1)-\dot{\bar{u}}_\gamma^{(n)}(\theta_1)\right)d\theta_1\right|$
$\leq\left|\int_{pT}^{\theta-\tau_{kn}}\int_{pT}^{\theta_1}\left(\ddot{u}_\gamma^{(n)}(\theta_2)-\ddot{\bar{u}}_\gamma^{(n)}(\theta_2)\right)d\theta_2 d\theta_1\right|+\left|\int_{\theta-\tau_{kn}}^{\theta}\int_{pT}^{\theta_1}\left(\ddot{u}_\gamma^{(n)}(\theta_2)-\ddot{\bar{u}}_\gamma^{(n)}(\theta_2)\right)d\theta_2 d\theta_1\right|\leq ...$
$...\leq\frac{\omega^h}{\mu^{h-1}}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})\left|\int_{pT}^{\theta-\tau_{kn}}e^{\mu(\theta_1-pT)}d\theta_1\right|+\frac{\omega^h}{\mu^{h-1}}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})\left|\int_{\theta-\tau_{kn}}^{\theta}e^{\mu(\theta_1-pT)}d\theta_1\right|\leq$
$...\leq\frac{\omega^h}{\mu^{h-1}}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})\left(\left|\frac{e^{\mu(\theta-\tau_{kn}-pT)}-1}{\mu}\right|+\left|\frac{e^{\mu(\theta-pT)}-e^{\mu(\theta-\tau_{kn}-pT)}}{\mu}\right|\right)$
$\leq\frac{\omega^h}{\mu^{h-1}}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{\mu(\theta-pT)}\left(\left|\frac{\left(e^{-\mu\tau_{kn}}-e^{-\mu(\theta-pT)}\right)}{\mu}\right|+\left|\frac{1-e^{-\mu\tau_{kn}}}{\mu}\right|\right)\leq 3\frac{\omega^h}{\mu^h}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{\mu(\theta-pT)};$

13) $\left|\dot{u}_\gamma^{(n)}(\theta-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(\theta-\tau_{kn})\right|\leq 3\frac{\omega^h}{\mu^h}\omega\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{\mu(\theta-pT)};$

14) $\left|\int_{pT}^{(p+1)T}A_{kn}\xi_\alpha^{(kn)}ds\right|\leq\int_{pT}^{(p+1)T}\left(\frac{2\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{c^2\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}+\frac{9\omega\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{c^2\left(1-\bar{\beta}\right)^5\tau_{kn}{}^2}\right)e^{\mu(s-pT)}ds$
$\leq\left(\frac{8c\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{\left(1-\bar{\beta}\right)^3(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{\left(1-\bar{\beta}\right)^5(r_{kn}^{(0)})^2}\right)\frac{e^{\mu T}-1}{\mu},$

15) $\left|\int_{pT}^{(p+1)T}B_{kn}u_\alpha^{(n)}ds\right|\leq\left(\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^2\left(1-\bar{\beta}\right)^5\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^2\left(1-\bar{\beta}\right)^5 r_{kn}^{(0)}}\right)\frac{e^{\mu T}-1}{\mu};$

16) Since $\left|\int_a^b f(s)\dot{g}(s)ds\right|\leq\left|\int_a^b|f(s)|\,dg(s)\right|\leq\max|f(s)|\,|g(b)-g(a)|$ we obtain

$$\left|\int_{pT}^{(p+1)T}C_{kn}\dot{u}_\alpha^{(n)}ds\right|\leq\frac{1+\bar{\beta}}{1-\bar{\beta}}\left|\int_{pT}^{(p+1)T}\frac{\left(c^2\tau_{kn}+c\tau_{kn}\bar{c}\right)}{\left(c^2\tau_{kn}-c\tau_{kn}\bar{c}\right)^2}du_\alpha^{(2)}(s)\right|\leq\frac{8}{c\left(1-\bar{\beta}\right)^3 r_{12}^{(0)}}\left|\int_{pT}^{(p+1)T}du_\alpha^{(n)}(s)\right|$$
$$=\frac{2(1+\bar{\beta})^2}{c(1-\bar{\beta})^3 r_{kn}^{(0)}}\left|u_\alpha^{(n)}\left((p+1)T-\tau_{kn}((p+1)T)\right)-u_\alpha^{(n)}(pT-\tau_{kn}(pT))\right|$$
$$=\frac{2(1+\bar{\beta})^2}{c(1-\bar{\beta})^3 r_{kn}^{(0)}}\left|u_\alpha^{(n)}\left(pT-\tau_{kn}(pT)\right)-u_\alpha^{(n)}(pT-\tau_{kn}(pT))\right|=0;$$

17) $\left|\int_{pT}^{(p+1)T}G_\alpha^{(k)rad}dt\right|\leq\frac{5_k^2}{m_k c^2}\frac{c}{(c^2-\bar{c}^2)^{3/2}}\left(\left|\sum_{\gamma=1}^3\int_{pT}^{(p+1)T}u_\gamma{}^{(k)}(s)\ddot{u}_\gamma{}^{(k)}(s)ds\right|+\left|\int_{pT}^{(p+1)T}u_\alpha{}^{(k)}(s)\ddot{u}_\alpha{}^{(k)}(s)ds\right|\right),$
$\leq\frac{5_k^2}{m_k c^2}\frac{c^2}{(c^2-\bar{c}^2)^{3/2}}\left(\left|\sum_{\gamma=1}^3\int_{pT}^{(p+1)T}\ddot{u}_\gamma{}^{(k)}(s)ds\right|+\left|\int_{pT}^{(p+1)T}\ddot{u}_\alpha{}^{(k)}(s)ds\right|\right)=0.$

18) $\left|\int_{pT}^{(p+1)T}G_\alpha^{(kn)}ds\right|=\frac{|e_k e_n|\Delta_k}{m_k c^2}\left|\int_{pT}^{(p+1)T}\left(A_{kn}\xi_\alpha^{(kn)}-B_{kn}u_\alpha^{(n)}+C_{kn}\dot{u}_\alpha^{(n)}\right)ds\right|$
$\leq\frac{|e_k e_n|}{m_k}\left(\frac{8\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{\left(1-\bar{\beta}\right)^3(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{c\left(1-\bar{\beta}\right)^5(r_{kn}^{(0)})^2}+\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(1-\bar{\beta}\right)^5\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(1-\bar{\beta}\right)^5 r_{kn}^{(0)}}\right)\frac{e^{\mu T}-1}{\mu}$

19) $\left|G_\alpha^{(kn)}\right|\leq\frac{|e_k e_n|\Delta_k}{m_k c^2}\left(|A_{kn}|\left|\xi_\alpha^{(kn)}\right|+|B_{kn}|\left|u_\alpha^{(n)}\right|+|C_{kn}|\left|\dot{u}_\alpha^{(n)}\right|\right)$
$\leq\frac{|e_k e_n|}{m_k}\left(\frac{8\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{\left(1-\bar{\beta}\right)^3(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{c\left(1-\bar{\beta}\right)^5(r_{kn}^{(0)})^2}+\frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(1-\bar{\beta}\right)^5\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(1-\bar{\beta}\right)^5 r_{kn}^{(0)}}+\frac{8}{c^2\left(1-\bar{\beta}\right)^3 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)e^{\mu(t-pT)}.$

20) $|u(t)-\bar{u}(t)|\leq\omega\int_{pT_0}^{t}\frac{|\dot{u}(s)-\dot{\bar{u}}(s)|}{\omega}e^{-\mu(s-pT)}e^{\mu(s-pT)}ds\leq\omega\rho_{(p,1)}(u,\bar{u})\frac{e^{\mu(t-pT)}}{\mu}$

$\Rightarrow\rho_{(p,0)}(u,\bar{u})\leq\frac{\omega}{\mu}\rho_{(p,1)}(u,\bar{u}).$

21) $\left|U_\alpha^k\right| \le \sum_{\gamma=1}^3 \sum_{n=1,n\neq k}^3 \left|G_\gamma^{(kn)}\right| + \sum_{\gamma=1}^3 \left|G_\gamma^{(k)rad}\right|$

$$\le 3\sum_{n=1,n\neq k}^3 \frac{|e_k e_n|}{m_k}\left(\frac{8\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{\left(1-\bar\beta\right)^3 (r_{kn}^{(0)})^3} + \frac{36\omega\left\|\vec{x}_0^{(k0)}-\vec{x}_0^{(n0)}\right\|}{c\left(1-\bar\beta\right)^5 (r_{kn}^{(0)})^2} + \frac{8\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(1-\bar\beta\right)^5\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega\left\|\vec{u}_0^{(n0)}\right\|}{c^3\left(1-\bar\beta\right)^5 r_{kn}^{(0)}} +\right.$$

$$\left.+\frac{8}{c^2\left(1-\bar\beta\right)^3 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right) e^{\mu(t-pT)} + 3\frac{e_k^2}{m_k}\frac{3}{c^3\sqrt{(1-\bar\beta^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}e^{\mu(t-pT)}.$$

22) $\left|\xi_\alpha{}^{(kn)} - \bar\xi_\alpha{}^{(kn)}\right| = \left|x_\alpha^{(k)}(t) - \bar{x}_\alpha^{(k)}(t) - x_\alpha^{(n)}(t-\tau_{kn}) + \bar{x}_\alpha^{(n)}(t-\tau_{kn})\right|$

$$= \left|\int_{pT}^t \left(u_\alpha^{(k)}(t_1) - \bar{u}_\alpha^{(k)}(t_1)\right)dt_1 - \int_{pT}^{t-\tau_{kn}}\left(u_\alpha^{(n)}(t_1) - \bar{u}_\alpha^{(n)}(t_1)\right)dt_1\right|$$

$$\le \left|\int_{pT}^t\int_{pT}^{t_1}\left(\dot{u}_\alpha^{(k)}(t_2) - \dot{\bar{u}}_\alpha^{(k)}(t_2)\right)dt_2dt_1\right| + \left|\int_{pT}^{t-\tau_{kn}}\int_{pT}^{t_1}\left(\dot{u}_\alpha^{(n)}(t_2) - \dot{\bar{u}}_\alpha^{(n)}(t_2)\right)dt_2dt_1\right| \le \ldots$$

$$\le \frac{\omega^h}{\mu^{h+1}}U_0 e^{\mu(t-pT)} + \left|\int_{pT}^{t-\tau_{kn}}\int_{pT}^{t_1}\left(\dot{u}_\alpha^{(n)}(t_2) - \dot{\bar{u}}_\alpha^{(n)}(t_2)\right)dt_2dt_1\right| + \left|\int_{t-\tau_{kn}}^{t}\int_{pT}^{t_1}\left(\dot{u}_\alpha^{(n)}(t_2) - \dot{\bar{u}}_\alpha^{(n)}(t_2)\right)dt_2dt_1\right|$$

$$\le \frac{\omega^h}{\mu^{h+1}}U_0 e^{\mu(t-pT)}\rho_{(p,h)}(u_\alpha^{(k)},\bar{u}_\alpha^{(k)}) + \frac{\omega^h}{\mu^{h+1}}U_0 e^{\mu(t-pT)}\left|e^{-\mu\tau_{kn}} - e^{-\mu(t-pT)}\right|\rho_{(p,h)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)})$$

$$+\frac{\omega^h}{\mu^{h+1}}e^{\mu(t-pT)}U_0\rho_{(p,h)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)})\left|1-e^{-\mu\tau_{kn}}\right|$$

$$\le e^{\mu(t-pT)}U_0\frac{\omega^h}{\mu^{h+1}}\left(\rho_{(p,h)}(u_\alpha^{(k)},\bar{u}_\alpha^{(k)}) + 2\rho_{(p,h)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)}) + \rho_{(p,h)}(u_\alpha^{(n)},\bar{u}_\alpha^{(n)})\right)$$

$$\le e^{\mu(t-pT)}U_0\frac{2\omega^h}{\mu^{h+1}}\rho_{(p,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(3)}),(\bar{u}_1^{(1)},\bar{u}_2^{(1)},...,\bar{u}_3^{(3)})\right).$$

5.2. Upper Bounds for the Operator Functions (A2)

To show that the operator B maps M_0 into itself, we need the following inequalities

$$\left|B_\alpha^{(k)}(u_1^{(1)},...,u_3^{(3)})(t)\right| = \left|\int_{pT}^t U_\alpha^{(k)}(s)ds - \left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U_\alpha^k(s)ds - \frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta}U_\alpha^p(s)dsd\theta\right|$$
$$\le U_0 e^{\mu(t-pT)}\quad(\alpha=1,2,3;\ k=1,2,3).$$

In view of $\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta}U_\alpha^k(s)dsd\theta = \frac{1}{2}\int_{pT}^{(p+1)T}U_\alpha^k(s)ds$ and $\left|\frac{t-pT}{T}-\frac{1}{2}\right| \le \frac{1}{2},\ t\in[pT,(p+1)T]$ we have

$$\left|B_\alpha^{(k)}(u_1^{(1)},...,u_3^{(3)})(t)\right| \le \left|\int_{pT}^t U_\alpha^k(s)ds\right| + \left|\left|\frac{t-pT}{T}-\frac{1}{2}\right|\int_{pT}^{(p+1)T}U_\alpha^k(s)ds + \frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta}U_\alpha^k(s)dsd\theta\right| =$$

$$= \left|\int_{pT}^t U_\alpha^k(s)ds\right| + \left|\frac{1}{2}\int_{pT}^{(p+1)T}U_\alpha^k(s)ds + \frac{1}{2}\int_{pT}^{(p+1)T}U_\alpha^k(s)dst\right| = \left|\int_{pT}^t U_\alpha^k(s)ds\right| + \left|\int_{pT}^{(p+1)T}U_\alpha^k(s)ds\right|.$$

For $\left|\int_{pT}^{(p+1)T}U_\alpha^k(s)ds\right|$ we have an estimate from Lemma 8. Therefore we need to estimate $\left|\int_{pT}^t U_\alpha^k(s)ds\right|$. In view of

$$\int_{pT}^t\left|G_\alpha^{(k)rad}\right|ds \le \frac{5_k^2}{m_k c^2}\int_{pT}^t\frac{3}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}ds \le \frac{5_k^2}{m_k}\frac{3}{c^2(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\frac{e^{\mu(t-pT)}}{\mu},$$

($\alpha = 1, 2, 3$) we obtain

$$\begin{aligned}\left|\int_{pT}^{t} U_\alpha^k(s)ds\right| &\le \sum_{\alpha=1}^{3}\sum_{n=1,n\neq k}^{3}\left|\int_{pT}^{t} G_\alpha^{(kn)}ds\right| + \sum_{\alpha=1}^{3}\left|\int_{pT}^{t} G_\alpha^{(k)rad}ds\right| \\ &\le 3\frac{e^{\mu(t-pT)}}{\mu}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}\right.\right. \\ &\quad \left.\left.+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+3\frac{5_k^2}{m_k c^2}\frac{3}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]\left|\int_{pT}^{(p+1)T} U_\alpha^k(s)ds\right| \\ &\le 3\frac{e^{\mu T}-1}{\mu}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}\right.\right. \\ &\quad \left.\left.+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+\frac{5_k^2}{m_k c^2}\frac{3}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right];\end{aligned}$$

$$\begin{aligned}\left|B_\alpha^{(k)}(u)(t)\right| &\le \left|\int_{pT}^{t} U_\alpha^k(s)ds\right| + \left|\int_{pT}^{(p+1)T} U_\alpha^k(s)ds\right| \\ &\le 3\frac{e^{\mu(t-pT)}}{\mu}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}\right.\right. \\ &\quad \left.\left.+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+\frac{5_k^2}{m_k c^2}\frac{\sqrt{3}+1}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]+ \\ &\quad +3\frac{e^{\mu T}-1}{\mu}\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left[\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}\right.\right. \\ &\quad \left.\left.+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+\frac{5_k^2}{m_k c^2}\frac{3}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right] \\ &\le e^{\mu(t-pT)}3\frac{e^{\mu T}}{\mu}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}\right.\right. \\ &\quad \left.\left.+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)+\frac{5_k^2}{m_k c^2}\frac{3}{(1-\bar\beta^2)^{3/2}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right].\end{aligned}$$

5.3. Estimates of the Derivatives (A3)

The derivative is $\dot B_\alpha^{(k)}(u_1^{(1)},...,u_3^{(2)})(t) := U_\alpha^k(t) - \frac{1}{T}\int_{pT}^{(p+1)T} U_\alpha^{(p)}(s)ds$ where $k = 1, 2, 3;\ \alpha = 1, 2, 3;\ p = 0, 1, 2, ...$

We need the inequalities 18) and 9) from (A1) to obtain that

$$\begin{aligned}\left|\dot B_\alpha^{(k)}(t)\right| &\le \left|U_\alpha^k(t)\right| + \left|\frac{1}{T}\int_{pT}^{(p+1)T} U_\alpha^k(s)ds\right| \\ &\le \sum_{\alpha=1}^{3}\sum_{n=1,n\neq k}^{3}\left|G_\alpha^{(kn)}\right| + \sum_{\alpha=1}^{3}\left|G_\alpha^{(k)rad}\right| + \left|\frac{1}{T}\int_{pT}^{(p+1)T} U_\alpha^k(s)ds\right| \\ &\le 3e^{\mu(t-pT)}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3}+\frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2}+\frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2}+\frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}}\right.\right.\end{aligned}$$

$$+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\Bigg) + \frac{5_k^2}{m_k}\frac{3}{c^2\sqrt{(1-\bar\beta^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\Bigg] +$$

$$+3\frac{e^{\mu T}-1}{\mu T}\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3} + \frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2} + \frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}} +\right.\right.$$

$$\left.\left.+\frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right) + \frac{e_k^2}{m_k}\frac{3}{c^2\sqrt{(1-\bar\beta^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]$$

$$\le e^{\mu(t-pT)}3\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\times$$

$$\times\left[\sum_{n=1,n\neq k}^{3}\frac{|e_k e_n|}{m_k\left(1-\bar\beta\right)^5}\left(\frac{8\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{(r_{kn}^{(0)})^3} + \frac{36\omega\left\|\vec u_0^{(k0)}-\vec u_0^{(n0)}\right\|}{c(r_{kn}^{(0)})^2} + \frac{8\left\|\vec u_0^{(n0)}\right\|}{c^3\left(r_{kn}^{(0)}\right)^2} + \frac{16\omega\left\|\vec u_0^{(n0)}\right\|}{c^3 r_{kn}^{(0)}} + \frac{8}{c^2 r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)\right.$$

$$\left.+\frac{e_k^2}{m_k}\frac{3}{c^2\sqrt{(1-\bar\beta^2)^3}}\frac{\omega^h}{\mu^h}\frac{4\omega^2}{4-(\tau\omega)^2}\right]$$

$$\le e^{\mu(t-pT)}\omega U_0.$$

As far as is concerned the higher order derivatives we can always prove an inequality of the type $\left|\frac{d^\nu B_\alpha^{(k)}(t)}{dt^\nu}\right| \le \omega^\nu U_0 e^{\mu(t-pT)}$ because of $\left|\frac{d^\nu u(t)}{dt^\nu}\right| \le 2\frac{\omega^{h+v}}{\mu^h}U_0 e^{\mu T} \le 2c\omega^v\frac{\omega^h}{\mu^h}$ and then the multiplier $\frac{\omega^{h+v}}{\mu^h}$ in $\frac{d^\nu B_\alpha^{(k)}(t)}{dt^\nu}$ we can present in the form $\frac{\omega^{h+v}}{\mu^h} = \omega^\nu\frac{\omega^h}{\mu^h}$. Then the v-th derivative becomes smaller than $\omega^v U_0$ for large $h>0$.

5.4. Lipschitz Estimates (A4)

$$\left|\Delta_k - \vec\Delta_k\right| \le \frac{2}{\sqrt{1-\beta^2}}\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}(t) - \bar u_\gamma^{(k)}(t)\right|;$$

$$\left|\frac{\partial D_{kn}}{\partial\xi_\gamma{}^{(kn)}}\right| = \left|\frac{u_\gamma^{(n)}\left(c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(k)}\right\rangle\right)-u_\gamma^{(k)}\left(c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(n)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(k)}\right\rangle\right)^2}\right| \le \frac{4}{c(1-\beta)^2\tau_{kn}}:$$

$$\left|\frac{\partial D_{kn}}{\partial u_\gamma{}^{(k)}}\right| = \left|-\frac{c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(n)}\right\rangle}{\left(c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(k)}\right\rangle\right)^2}\left(-\xi_\gamma{}^{(kn)}\right)\right|$$

$$\le c\tau_{kn}\frac{c^2\tau_{kn}+c\tau_{kn}\bar c}{\left(c^2\tau_{kn}-c\tau_{kn}\bar c\right)^2} \le \frac{2}{c(1-\beta)^2};$$

$$\left|\frac{\partial D_{kn}}{\partial u_\gamma{}^{(n)}}\right| = \left|\frac{-\xi_\gamma{}^{(kn)}}{c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(k)}\right\rangle}\right| \le \frac{c\tau_{kn}}{c^2\tau_{kn}-c\tau_{kn}\bar c} = \frac{1}{c(1-\beta)};$$

$$\left|D_{kn}-\bar D_{kn}\right| = \left|\frac{c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(n)}\right\rangle}{c^2\tau_{kn}-\left\langle\vec\xi^{(kn)},\vec u^{(k)}\right\rangle} - \frac{c^2\tau_{kn}-\left\langle\vec{\bar\xi}^{(kn)},\vec{\bar u}^{(n)}\right\rangle}{c^2\tau_{kn}-\left\langle\vec{\bar\xi}^{(kn)},\vec{\bar u}^{(k)}\right\rangle}\right|$$

$$\le \sum_{\gamma=1}^{3}\left|\frac{\partial D_{kn}}{\partial\xi_\gamma{}^{(kn)}}\right|\left|\xi_\gamma{}^{(kn)}-\bar\xi_\gamma{}^{(kn)}\right| + \sum_{\gamma=1}^{3}\left|\frac{\partial D_{kn}}{\partial u_\gamma{}^{(k)}}\right|\left|u_\gamma{}^{(k)}(t)-\bar u_\gamma{}^{(k)}(t)\right| +$$

$$+\sum_{\gamma=1}^{3}\left|\frac{\partial D_{kn}}{\partial u_\gamma{}^{(n)}}\right|\left|u_\gamma{}^{(n)}(t-\tau_{kn})-\bar u_\gamma{}^{(n)}(t-\tau_{kn})\right|$$

$$\le \frac{4}{c(1-\beta)^2\tau_{kn}}\sum_{\gamma=1}^{3}\left|\xi_\gamma{}^{(kn)}-\bar\xi_\gamma{}^{(kn)}\right| + \frac{2}{c(1-\beta)^2}\sum_{\gamma=1}^{3}\left|u_\gamma{}^{(k)}(t)-\bar u_\gamma{}^{(k)}(t)\right| +$$

$$+\frac{1}{c(1-\beta)}\sum_{\gamma=1}^{3}\left|u_\gamma{}^{(n)}(t-\tau_{kn})-\bar u_\gamma{}^{(n)}(t-\tau_{kn})\right|;$$

$$
\begin{aligned}
\left|H_{kn}-\bar{H}_{kn}\right| \quad & \le \sum_{\gamma=1}^{3}\left|\frac{\partial H_{kn}}{\partial \xi_{\gamma}{}^{(kn)}}\right|\left|\xi_{\gamma}^{(kn)}-\bar{\xi}_{\gamma}^{(kn)}\right|+\sum_{\gamma=1}^{3}\left|\frac{\partial H_{kn}}{\partial u_{\gamma}{}^{(k)}}\right|\left|u_{\gamma}^{(k)}(t)-\bar{u}_{\gamma}^{(k)}(t)\right|+ \\
& +\sum_{\gamma=1}^{3}\left|\frac{\partial H_{kn}}{\partial u_{\gamma}{}^{(n)}}\right|\left|u_{\gamma}^{(n)}(t-\tau_{kn})-\bar{u}_{\gamma}^{(n)}(t-\tau_{kn})\right|+ \\
& +\sum_{\gamma=1}^{3}\left|\frac{\partial H_{kn}}{\partial \dot{u}_{\gamma}{}^{(n)}}\right|\left|\dot{u}_{\gamma}^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_{\gamma}^{(n)}(t-\tau_{kn})\right| \\
& \le \frac{8c^2\omega}{(1-\bar{\beta})^3}\sum_{\gamma=1}^{3}\left|\xi_{\gamma}^{(kn)}-\bar{\xi}_{\gamma}^{(kn)}\right|+\frac{6c\omega\tau_{kn}}{(1-\bar{\beta})^3}\sum_{\gamma=1}^{3}\left|u_{\gamma}^{(k)}(t)-\bar{u}_{\gamma}^{(k)}(t)\right|+ \\
& +\frac{2c^2\omega\tau_{kn}}{(1-\bar{\beta})^2}\sum_{\gamma=1}^{3}\left|u_{\gamma}^{(n)}(t-\tau_{kn})-\bar{u}_{\gamma}^{(n)}(t-\tau_{kn})\right|+ \\
& +\frac{6c\tau_{kn}}{(1-\bar{\beta})^3}\sum_{\gamma=1}^{3}\left|\dot{u}_{\gamma}^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_{\gamma}^{(n)}(t-\tau_{kn})\right|;
\end{aligned}
$$

$$
\begin{aligned}
& \left|A_{kn}-\bar{A}_{kn}\right| \le \left|H_{kn}\right|\left|\frac{\left(c^2-\left\langle\vec{u}^{(k)},\vec{u}^{(n)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}-\frac{\left(c^2-\left\langle\vec{\bar{u}}^{(k)},\vec{\bar{u}}^{(n)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\bar{\xi}}^{(kn)},\vec{\bar{u}}^{(n)}\right\rangle\right)^3}\right|+\left|H_{kn}-\bar{H}_{kn}\right|\left|\frac{\left(c^2-\left\langle\vec{\bar{u}}^{(k)},\vec{u}^{(n)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\bar{\xi}}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}\right| \\
& \le\left(c^2\frac{2}{c^5\left(1-\bar{\beta}\right)^4\tau_{kn}{}^4}+\frac{3\omega\,c^2\tau_{kn}}{(1-\bar{\beta})^2}\frac{2}{c^5\left(1-\bar{\beta}\right)^4\tau_{kn}{}^4}+\frac{2}{c^4\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}\frac{14\omega U_0e^{\mu(t-pT)}}{\left(1-\bar{\beta}\right)^3}\right)\sum_{\gamma=1}^{3}\left|\xi_{\gamma}{}^{(kn)}-\bar{\xi}_{\gamma}{}^{(kn)}\right|+ \\
& \quad+\left(c^2\frac{1}{c^5\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}+\frac{3\omega\,c^2\tau_{kn}}{(1-\bar{\beta})^2}\frac{1}{c^5\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}+\frac{2}{c^4\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}\frac{6\tau_{kn}\omega U_0e^{\mu(t-pT)}}{\left(1-\bar{\beta}\right)^3}\right)\sum_{\gamma=1}^{3}\left|u_{\gamma}{}^{(k)}(t)-\bar{u}_{\gamma}{}^{(k)}(t)\right|+ \\
& \quad+\left(c^2\frac{8}{c^5\left(1-\bar{\beta}\right)^4\tau_{kn}{}^3}+\frac{3\omega\,c^2\tau_{kn}}{(1-\bar{\beta})^2}\frac{8}{c^5\left(1-\bar{\beta}\right)^4\tau_{kn}{}^3}+2c\frac{2}{c^4\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}+\frac{2}{c^4\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}\frac{9\tau_{kn}\omega U_0e^{\mu(t-pT)}}{(1-\bar{\beta})^3}\right)\times \\
& \quad\times\sum_{\gamma=1}^{3}\left|u_{\gamma}{}^{(n)}(t-\tau_{kn})-\bar{u}_{\gamma}{}^{(n)}(t-\tau_{kn})\right|+\frac{16\omega\sum_{\gamma=1}^{3}\left|\xi_{\gamma}^{(kn)}-\bar{\xi}_{\gamma}^{(kn)}\right|}{c^2\tau_{kn}{}^3(1-\bar{\beta})^6}+\frac{12\omega\sum_{\gamma=1}^{3}\left|u_{\gamma}^{(k)}(t)-\bar{u}_{\gamma}^{(k)}(t)\right|}{c^3\tau_{kn}{}^2(1-\bar{\beta})^6}+ \\
& \quad+\frac{4\omega\sum_{\gamma=1}^{3}\left|u_{\gamma}^{(n)}(t-\tau_{kn})-\bar{u}_{\gamma}^{(n)}(t-\tau_{kn})\right|}{c^2\tau_{kn}{}^2(1-\bar{\beta})^5}+\frac{12\sum_{\gamma=1}^{3}\left|\dot{u}_{\gamma}^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_{\gamma}^{(n)}(t-\tau_{kn})\right|}{c^3\tau_{kn}{}^2(1-\bar{\beta})^6} \\
& \le\left(\frac{2}{c^3\left(1-\bar{\beta}\right)^4\tau_{kn}{}^4}+\frac{50\omega}{c^3\left(1-\bar{\beta}\right)^6\tau_{kn}{}^3}\right)\sum_{\gamma=1}^{3}\left|\xi_{\gamma}{}^{(kn)}-\bar{\xi}_{\gamma}{}^{(kn)}\right|+ \\
& \quad+\left(\frac{1}{c^3\left(1-\bar{\beta}\right)^3\tau_{kn}{}^3}+\frac{27\omega}{c^3\left(1-\bar{\beta}\right)^6\tau_{kn}{}^2}\right)\sum_{\gamma=1}^{3}\left|u_{\gamma}{}^{(k)}(t)-\bar{u}_{\gamma}{}^{(k)}(t)\right|+ \\
& \quad+\left(\frac{12}{c^3\left(1-\bar{\beta}\right)^4\tau_{kn}{}^3}+\frac{43\omega}{c^3\left(1-\bar{\beta}\right)^6\tau_{kn}{}^2}\right)\sum_{\gamma=1}^{3}\left|u_{\gamma}{}^{(n)}(t-\tau_{kn})-\bar{u}_{\gamma}{}^{(n)}(t-\tau_{kn})\right|+ \\
& \quad+\frac{12}{c^3(1-\bar{\beta})^6\tau_{kn}{}^2}\sum_{\gamma=1}^{3}\left|\dot{u}_{\gamma}^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_{\gamma}^{(n)}(t-\tau_{kn})\right|;
\end{aligned}
$$

$$
\begin{aligned}
& \left|B_{kn}-\bar{B}_{kn}\right| \le \left|H_{kn}\right|\left|\frac{\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}-\frac{\left(c^2\tau_{kn}-\left\langle\vec{\bar{\xi}}^{(kn)},\vec{\bar{u}}^{(k)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\bar{\xi}}^{(kn)},\vec{\bar{u}}^{(n)}\right\rangle\right)^3}\right|+\left|H_{kn}-\bar{H}_{kn}\right|\left|\frac{\left(c^2\tau_{kn}-\left\langle\vec{\bar{\xi}}^{(kn)},\vec{u}^{(k)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle\vec{\bar{\xi}}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}\right| \\
& \quad\le\left(c^2+\frac{3\omega\,c^2\tau_{kn}}{(1-\beta)^2}\right)\frac{8}{c^5(1-\beta)^4\tau_{kn}{}^3}\sum_{\gamma=1}^{3}\left|\xi_{\gamma}{}^{(kn)}-\bar{\xi}_{\gamma}{}^{(kn)}\right|+ \\
& \quad\quad+\left(c^2+\frac{3\omega\,c^2\tau_{kn}}{(1-\beta)^2}\right)\frac{1}{c^5(1-\beta)^3\tau_{kn}{}^2}\sum_{\gamma=1}^{3}\left|u_{\gamma}{}^{(k)}(t)-\bar{u}_{\gamma}{}^{(k)}(t)\right|+ \\
& \quad\quad+\left(c^2+\frac{3\omega\,c^2\tau_{kn}}{(1-\beta)^2}\right)\frac{2}{c^5(1-\beta)^4\tau_{kn}{}^2}\sum_{\gamma=1}^{3}\left|u_{\gamma}{}^{(n)}(t-\tau_{kn})-\bar{u}_{\gamma}{}^{(n)}(t-\tau_{kn})\right|+ \\
& \quad\quad+\frac{2}{c^4(1-\beta)^3\tau_{kn}{}^2}\frac{14\omega U_0e^{\mu(t-pT_0)}}{(1-\beta)^3}\sum_{\gamma=1}^{3}\left|\xi_{\gamma}{}^{(kn)}-\bar{\xi}_{\gamma}{}^{(kn)}\right|+
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{2}{c^4(1-\beta)^3{\tau_{kn}}^2}\tfrac{6\tau_{kn}\omega U_0 e^{\mu(t-pT_0)}}{(1-\beta)^3}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(k)}(t)-{\bar u_\gamma}^{(k)}(t)\right|+\\
&+\tfrac{2}{c^4(1-\beta)^3{\tau_{kn}}^2}\left(2c+\tfrac{9\tau_{kn}\omega U_0 e^{\mu(t-pT_0)}}{(1-\beta)^3}\right)\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(n)}(t-\tau_{kn})-{\bar u_\gamma}^{(n)}(t-\tau_{kn})\right|+\\
&+\tfrac{2}{c^4(1-\beta)^3{\tau_{kn}}^2}\tfrac{3c\tau_{kn}}{(1-\beta)^2}\textstyle\sum_{\gamma=1}^{3}\left|{\dot u_\gamma}^{(n)}(t-\tau_{kn})-{\dot{\bar u}_\gamma}^{(n)}(t-\tau_{kn})\right|\\
\le&\left(\tfrac{8}{c^3\left(1-\bar\beta\right)^4{\tau_{kn}}^3}+\tfrac{52\omega}{c^3(1-\bar\beta)^6{\tau_{kn}}^2}\right)\textstyle\sum_{\gamma=1}^{3}\left|{\xi_\gamma}^{(kn)}-{\bar\xi_\gamma}^{(kn)}\right|+\\
&+\left(\tfrac{1}{c^3\left(1-\bar\beta\right)^3{\tau_{kn}}^2}+\tfrac{15\omega}{c^3(1-\bar\beta)^6\tau_{kn}}\right)\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(k)}(t)-{\bar u_\gamma}^{(k)}(t)\right|+\\
&+\left(\tfrac{6}{c^3\left(1-\bar\beta\right)^4{\tau_{kn}}^2}+\tfrac{24\omega}{c^3(1-\bar\beta)^6\tau_{kn}}\right)\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(n)}(t-\tau_{kn})-{\bar u_\gamma}^{(n)}(t-\tau_{kn})\right|+\\
&+\tfrac{6}{c^3\left(1-\bar\beta\right)^5\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{\dot u_\gamma}^{(n)}(t-\tau_{kn})-{\dot{\bar u}_\gamma}^{(n)}(t-\tau_{kn})\right|;
\end{aligned}
$$

$$
\begin{aligned}
&\left|C_{kn}-\bar C_{kn}\right|\le|D_{kn}|\left|\tfrac{\left(\left\langle\vec\xi^{(kn)},\vec u^{(k)}\right\rangle-c^2\tau_{kn}\right)}{\left(\left\langle\vec\xi^{(kn)},\vec u^{(n)}\right\rangle-c^2\tau_{kn}\right)^2}-\tfrac{\left(\left\langle\vec{\bar\xi}^{(kn)},\vec{\bar u}^{(k)}\right\rangle-c^2\tau_{kn}\right)}{\left(\left\langle\vec{\bar\xi}^{(kn)},\vec{\bar u}^{(n)}\right\rangle-c^2\tau_{kn}\right)^2}\right|+\left|\tfrac{\left(\left\langle\vec{\bar\xi}^{(kn)},\vec{\bar u}^{(k)}\right\rangle-c^2\tau_{kn}\right)}{\left(\left\langle\vec{\bar\xi}^{(kn)},\vec{\bar u}^{(n)}\right\rangle-c^2\tau_{kn}\right)^2}\right|\left|D_{kn}-\bar D_{kn}\right|\\
&\le\tfrac{12}{c^3(1-\beta)^4{\tau_{kn}}^2}\textstyle\sum_{\gamma=1}^{3}\left|{\xi_\gamma}^{(kn)}-{\bar\xi_\gamma}^{(kn)}\right|+\tfrac{2}{c^3(1-\beta)^3\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(k)}(t)-{\bar u_\gamma}^{(k)}(t)\right|+\\
&+\tfrac{4}{c^3(1-\beta)^4\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(n)}(t-\tau_{kn})-{\bar u_\gamma}^{(n)}(t-\tau_{kn})\right|+\tfrac{8}{c^3(1-\beta)^4{\tau_{kn}}^2}\textstyle\sum_{\gamma=1}^{3}\left|{\xi_\gamma}^{(kn)}-{\bar\xi_\gamma}^{(kn)}\right|+\\
&+\tfrac{4}{c^3(1-\beta)^4\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(k)}(t)-{\bar u_\gamma}^{(k)}(t)\right|+\tfrac{2}{c^3(1-\beta)^3\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(n)}(t-\tau_{kn})-{\bar u_\gamma}^{(n)}(t-\tau_{kn})\right|\\
&\le\tfrac{20}{c^3\left(1-\bar\beta\right)^4{\tau_{kn}}^2}\textstyle\sum_{\gamma=1}^{3}\left|{\xi_\gamma}^{(kn)}-{\bar\xi_\gamma}^{(kn)}\right|+\tfrac{6}{c^3\left(1-\bar\beta\right)^4\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(k)}(t)-{\bar u_\gamma}^{(k)}(t)\right|+\\
&+\tfrac{6}{c^3\left(1-\bar\beta\right)^4\tau_{kn}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(n)}(t-\tau_{kn})-{\bar u_\gamma}^{(n)}(t-\tau_{kn})\right|;
\end{aligned}
$$

$$
\begin{aligned}
\left|G_\alpha^{(k)rad}-\bar G_\alpha^{(k)rad}\right|&\le\tfrac{5_k^2}{m_kc^2}\left(\left|\tfrac{u_\alpha^{(k)}(t)\left\langle\vec u^{(k)}(t),\ddot{\vec u}^{(k)}(t)\right\rangle}{\left(c^2-\left\langle\vec u^{(k)},\vec u^{(k)}\right\rangle\right)^{3/2}}-\tfrac{\bar u_\alpha^{(k)}(t)\left\langle\vec{\bar u}^{(k)}(t),\ddot{\vec{\bar u}}^{(k)}(t)\right\rangle}{\left(c^2-\left\langle\vec{\bar u}^{(k)},\vec{\bar u}^{(k)}\right\rangle\right)^{3/2}}\right|+\right.\\
&\left.+\left|\tfrac{\ddot u_\alpha^{(k)}(t)}{\left(c^2-\left\langle\vec u^{(k)},\vec u^{(k)}\right\rangle\right)^{1/2}}-\tfrac{\ddot{\bar u}_\alpha^{(k)}(t)}{\left(c^2-\left\langle\vec{\bar u}^{(k)},\vec{\bar u}^{(k)}\right\rangle\right)^{1/2}}\right|\right)\\
&\le\tfrac{e_k^2}{m_k}\left(\tfrac{5\omega^2}{c^3\left(1-\bar\beta\right)^{5/2}}\textstyle\sum_{\gamma=1}^{3}\left|{u_\gamma}^{(k)}(t)-{\bar u_\gamma}^{(k)}(t)\right|+\tfrac{1}{c^3\left(1-\bar\beta\right)^{1/2}}\textstyle\sum_{\gamma=1}^{3}\left|{\ddot u_\gamma}^{(k)}(t)-{\ddot{\bar u}_\gamma}^{(k)}(t)\right|\right);\\
\left|G_\alpha^{(kn)}-\bar G_\alpha^{(kn)}\right|&\le\tfrac{|e_ke_n|}{m_kc^2}\Delta_k\left(|A_{12}|\left|\xi_\alpha^{(kn)}-\bar\xi_\alpha^{(kn)}\right|+\left|A_{kn}-\bar A_{kn}\right|\left|\bar\xi_\alpha^{(kn)}\right|+\left|B_{kn}-\bar B_{kn}\right|\left|u_\alpha^{(n)}\right|+\right.\\
&\left.+\left|u_\alpha^{(n)}-\bar u_\alpha^{(n)}\right|\left|\bar B_{kn}\right|+|C_{kn}|\left|\dot u_\alpha^{(n)}-\dot{\bar u}_\alpha^{(n)}\right|+\left|C_{kn}-\bar C_{kn}\right|\left|\dot{\bar u}_\alpha^{(n)}\right|\right)+\\
&+\tfrac{|e_ke_n|}{m_kc^2}\left|\Delta_k-\bar\Delta_k\right|\left|\bar A_{kn}\bar\xi_\alpha^{(kn)}-\bar B_{kn}\bar u_\alpha^{(n)}+\bar C_{kn}\dot{\bar u}_\alpha^{(n)}\right|;
\end{aligned}
$$

$$
\begin{aligned}
&|A_{12}|\left|\xi_\alpha^{(kn)}-\bar\xi_\alpha^{(kn)}\right|+\left|A_{kn}-\bar A_{kn}\right|\left|\bar\xi_\alpha^{(kn)}\right|+\left|B_{kn}-\bar B_{kn}\right|\left|u_\alpha^{(n)}\right|+\left|u_\alpha^{(n)}-\bar u_\alpha^{(n)}\right|\left|\bar B_{kn}\right|+|C_{kn}|\left|\dot u_\alpha^{(n)}-\dot{\bar u}_\alpha^{(n)}\right|+\\
&+\left|C_{kn}-\bar C_{kn}\right|\left|\dot{\bar u}_\alpha^{(n)}\right|
\end{aligned}
$$

$$\leq \left(\frac{80c}{(1-\bar{\beta})^4 \left(r_{kn}^{(0)}\right)^3} + \frac{324\omega}{(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{32c\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4 \left(r_{kn}^{(0)}\right)^4} + \frac{400\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^3} \right) \left|\xi_\alpha^{(kn)} - \bar{\xi}_\alpha^{(kn)}\right| +$$
$$+ \left(\frac{8\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^3 \left(r_{kn}^{(0)}\right)^3} + \frac{1108\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{4}{(1-\bar{\beta})^3 \left(r_{kn}^{(0)}\right)^2} + \frac{42\omega}{c(1-\bar{\beta})^6 r_{kn}^{(0)}} \right) \sum_{\gamma=1}^3 \left|u_\gamma{}^{(k)}(t) - \bar{u}_\gamma{}^{(k)}(t)\right| +$$
$$+ \left(\frac{96\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4 \left(r_{kn}^{(0)}\right)^3} + \frac{172\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{24}{(1-\bar{\beta})^5 \left(r_{kn}^{(0)}\right)^2} + \frac{60\omega}{c(1-\bar{\beta})^6 r_{kn}^{(0)}} \right) \times$$
$$\times \sum_{\gamma=1}^3 \left|u_\gamma{}^{(n)}(t-\tau_{kn}) - \bar{u}_\gamma{}^{(n)}(t-\tau_{kn})\right| +$$
$$+ \left(\frac{48\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 (r_{kn}^{(0)})^2} + \frac{20}{c(1-\bar{\beta})^6 r_{kn}^{(0)}} \right) \sum_{\gamma=1}^3 \left|\dot{u}_\gamma^{(n)}(t-\tau_{kn}) - \dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right| .$$

Therefore

$$\left|G_\alpha^{(kn)} - \bar{G}_\alpha^{(kn)}\right| \leq \frac{|e_k e_n|}{m_k c} \left[\left(\frac{80c}{(1-\bar{\beta})^4 \left(r_{kn}^{(0)}\right)^3} + \frac{324\omega}{(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{32c\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4 \left(r_{kn}^{(0)}\right)^4} + \frac{400\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^3} \right) \left|\xi_\alpha^{(kn)} - \bar{\xi}_\alpha^{(kn)}\right| + \right.$$
$$+ \left(\frac{8\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^3 \left(r_{kn}^{(0)}\right)^3} + \frac{108\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{4}{(1-\bar{\beta})^3 \left(r_{kn}^{(0)}\right)^2} + \frac{42\omega}{c(1-\bar{\beta})^6 r_{kn}^{(0)}} \right) \sum_{\gamma=1}^3 \left|u_\gamma{}^{(k)}(t) - \bar{u}_\gamma{}^{(k)}(t)\right| +$$
$$+ \left(\frac{96\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4 \left(r_{kn}^{(0)}\right)^3} + \frac{172\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{24}{(1-\bar{\beta})^5 \left(r_{kn}^{(0)}\right)^2} + \frac{60\omega}{c(1-\bar{\beta})^6 r_{kn}^{(0)}} \right) \sum_{\gamma=1}^3 \left|u_\gamma{}^{(n)}(t-\tau_{kn}) - \bar{u}_\gamma{}^{(n)}(t-\tau_{kn})\right| +$$
$$\left. + \left(\frac{48\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 (r_{kn}^{(0)})^2} + \frac{20}{c(1-\bar{\beta})^6 r_{kn}^{(0)}} \right) \sum_{\gamma=1}^3 \left|\dot{u}_\gamma^{(n)}(t-\tau_{kn}) - \dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right| \right] +$$
$$+ \frac{|e_k e_n|}{m_k c} \left(\frac{16\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4 (r_{kn}^{(0)})^3} + \frac{72\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6 (r_{kn}^{(0)})^2} + \frac{16\left\|\vec{x}^{(n0)}\right\|}{c^3(1-\bar{\beta})^6 \left(r_{kn}^{(0)}\right)^2} + \frac{32\omega\left\|\vec{x}^{(n0)}\right\|}{c^3(1-\bar{\beta})^6 r_{kn}^{(0)}} + \frac{16}{c^2(1-\bar{\beta})^4 r_{kn}^{(0)}} \frac{\omega^h}{\mu^h} \omega U_0 \right) \times$$
$$\times \sum_{\gamma=1}^3 \left|u_\gamma^{(k)}(t) - \bar{u}_\gamma^{(k)}(t)\right| ;$$

Recall

$$U_l^k(u) = \sum_{n=1,n\neq k}^3 \left(G_l^{(kn)}\right) + G_l^{(l)rad} - \sum_{\gamma=1}^3 \frac{u_l^{(k)} u_\gamma^{(k)}}{c^2} \left(\sum_{n=1,n\neq k}^3 G_\gamma^{(kn)} + G_\gamma^{(l)rad} \right), \quad (k=1,2,3; l=1,2,3),$$

then

$$\left|U_l^k(u) - \bar{U}_l^k(\bar{u})\right| \leq \sum_{n=1,n\neq k}^3 \left|G_l^{(kn)} - \bar{G}_l^{(kn)}\right| + \left|G_l^{(l)rad} - \bar{G}_l^{(l)rad}\right| +$$
$$+ \frac{1}{c^2} \left| \sum_{\gamma=1}^3 u_l^{(k)} u_\gamma^{(k)} \sum_{n=1,n\neq k}^3 G_\gamma^{(kn)} - \sum_{\gamma=1}^3 \bar{u}_l^{(k)} \bar{u}_\gamma^{(k)} \sum_{n=1,n\neq k}^3 \bar{G}_\gamma^{(kn)} + \right.$$
$$\left. + \sum_{\gamma=1}^3 u_l^{(k)} u_\gamma^{(k)} G_\gamma^{(l)rad} - \sum_{\gamma=1}^3 \bar{u}_l^{(k)} \bar{u}_\gamma^{(k)} \bar{G}_\gamma^{(l)rad} \right| \qquad ll$$
$$\leq \sum_{n=1,n\neq k}^3 \left|G_l^{(kn)} - \bar{G}_l^{(kn)}\right| + \left|G_l^{(l)rad} - \bar{G}_l^{(l)rad}\right| + \frac{1}{c^2} \left| \sum_{\gamma=1}^3 u_l^{(k)} u_\gamma^{(k)} \sum_{n=1,n\neq k}^3 G_\gamma^{(kn)} \right.$$
$$\left. - \sum_{\gamma=1}^3 \bar{u}_l^{(k)} \bar{u}_\gamma^{(k)} \sum_{n=1,n\neq k}^3 G_\gamma^{(kn)} \right| + \frac{1}{c^2} \left| \sum_{\gamma=1}^3 \bar{u}_l^{(k)} \bar{u}_\gamma^{(k)} \sum_{n=1,n\neq k}^3 G_\gamma^{(kn)} \right.$$

$$-\sum_{\gamma=1}^{3}\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}\sum_{n=1,n\neq k}^{3}\bar{G}_\gamma^{(kn)}\Big|+\frac{1}{c^2}\left|\sum_{\gamma=1}^{3}u_l^{(k)}u_\gamma^{(k)}G_\gamma^{(l)rad}-\sum_{\gamma=1}^{3}\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}G_\gamma^{(l)rad}\right|$$

$$+\frac{1}{c^2}\left|\sum_{\gamma=1}^{3}\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}G_\gamma^{(l)rad}-\sum_{\gamma=1}^{3}\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}\bar{G}_\gamma^{(l)rad}\right|$$

$$\leq\sum_{n=1,n\neq k}^{3}\left|G_l^{(kn)}-\bar{G}_l^{(kn)}\right|+\left|G_l^{(l)rad}-\bar{G}_l^{(l)rad}\right|+\frac{1}{c^2}\left(\sum_{\gamma=1}^{3}\left|u_l^{(k)}u_\gamma^{(k)}-\bar{u}_l^{(k)}u_\gamma^{(k)}\right|\left|\sum_{n=1,n\neq k}^{3}G_\gamma^{(kn)}\right|+\right.$$

$$\left.+\sum_{\gamma=1}^{3}\left|\bar{u}_l^{(k)}u_\gamma^{(k)}-\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}\right|\left|\sum_{n=1,n\neq k}^{3}G_\gamma^{(kn)}\right|\right)+\frac{1}{c^2}\sum_{\gamma=1}^{3}\left|\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}\right|\sum_{n=1,n\neq k}^{3}\left|G_\gamma^{(kn)}-\bar{G}_\gamma^{(kn)}\right|+$$

$$+\frac{1}{c^2}\left(\sum_{\gamma=1}^{3}\left|u_l^{(k)}u_\gamma^{(k)}-\bar{u}_l^{(k)}u_\gamma^{(k)}\right|\left|G_\gamma^{(l)rad}\right|+\sum_{\gamma=1}^{3}\left|\bar{u}_l^{(k)}u_\gamma^{(k)}-\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}\right|\left|G_\gamma^{(l)rad}\right|\right)+$$

$$+\frac{1}{c^2}\sum_{\gamma=1}^{3}\left|\bar{u}_l^{(k)}\bar{u}_\gamma^{(k)}\right|\left|G_\gamma^{(l)rad}-\bar{G}_\gamma^{(l)rad}\right|$$

$$\leq\left(\sum_{n=1,n\neq k}^{3}\left|G_l^{(kn)}-\bar{G}_l^{(kn)}\right|+\left|G_l^{(l)rad}-\bar{G}_l^{(l)rad}\right|+\sum_{\gamma=1}^{3}\sum_{n=1,n\neq k}^{3}\left|G_\gamma^{(kn)}-\bar{G}_\gamma^{(kn)}\right|+\right.$$

$$\left.+\sum_{\gamma=1}^{3}\left|G_\gamma^{(l)rad}-\bar{G}_\gamma^{(l)rad}\right|\right)+\frac{1}{c}\left(3\left|u_l^{(k)}-\bar{u}_l^{(k)}\right|+\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\bar{u}_\gamma^{(k)}\right|\right)\left(\left|\sum_{n=1,n\neq k}^{3}G_\gamma^{(kn)}\right|+\left|G_\gamma^{(l)rad}\right|\right)$$

$$\leq\sum_{n=1,n\neq k}^{3}\frac{|e_ke_n|}{m_kc}\left[\left(\frac{80c}{(1-\bar{\beta})^4\left(r_{kn}^{(0)}\right)^3}+\frac{324\omega}{(1-\bar{\beta})^6\left(r_{kn}^{(0)}\right)^2}+\frac{32c\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4\left(r_{kn}^{(0)}\right)^4}+\frac{400\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^6\left(r_{kn}^{(0)}\right)^3}\right)\left|\xi_\alpha^{(kn)}-\bar{\xi}_\alpha^{(kn)}\right|+\right.$$

$$+\left(\frac{8\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^3\left(r_{kn}^{(0)}\right)^3}+\frac{108\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6\left(r_{kn}^{(0)}\right)^2}+\frac{4}{(1-\bar{\beta})^3\left(r_{kn}^{(0)}\right)^2}+\frac{42\omega}{c(1-\bar{\beta})^6r_{kn}^{(0)}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma{}^{(k)}(t)-\bar{u}_\gamma{}^{(k)}(t)\right|+$$

$$+\left(\frac{96\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4\left(r_{kn}^{(0)}\right)^3}+\frac{172\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6\left(r_{kn}^{(0)}\right)^2}+\frac{24}{(1-\bar{\beta})^5\left(r_{kn}^{(0)}\right)^2}+\frac{60\omega}{c(1-\bar{\beta})^6r_{kn}^{(0)}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma{}^{(n)}(t-\tau_{kn})-\bar{u}_\gamma{}^{(n)}(t-\tau_{kn})\right|+$$

$$\left.+\left(\frac{48\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6(r_{kn}^{(0)})^2}+\frac{20}{c(1-\bar{\beta})^6r_{kn}^{(0)}}\right)\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|\right]+$$

$$+\sum_{n=1,n\neq k}^{3}\frac{|e_ke_n|}{m_kc}\left(\frac{16\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^4(r_{kn}^{(0)})^3}+\frac{72\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^6(r_{kn}^{(0)})^2}+\frac{16\left\|\vec{x}^{(n0)}\right\|}{c^3(1-\bar{\beta})^6\left(r_{kn}^{(0)}\right)^2}+\frac{32\omega\left\|\vec{x}^{(n0)}\right\|}{c^3(1-\bar{\beta})^6r_{kn}^{(0)}}\right.$$

$$\left.+\frac{16}{c^2(1-\bar{\beta})^4r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}(t)-\bar{u}_\gamma^{(k)}(t)\right|+\frac{e_k^2}{m_k}\frac{20\omega^2}{c^3(1-\bar{\beta})^{5/2}}\sum_{\gamma=1}^{3}\left|u_\gamma{}^{(k)}(t)-\bar{u}_\gamma{}^{(k)}(t)\right|+$$

$$+\frac{e_k^2}{m_k}\frac{4}{c^3(1-\bar{\beta})^{1/2}}\sum_{\gamma=1}^{3}\left|\ddot{u}_\gamma{}^{(k)}(t)-\ddot{\bar{u}}_\gamma{}^{(k)}(t)\right|+$$

$$+\sum_{n=1,n\neq k}^{3}\frac{|e_ke_n|}{m_kc}\left(\frac{32\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{(1-\bar{\beta})^3(r_{kn}^{(0)})^3}+\frac{144\omega\left\|\vec{x}^{(k0)}-\vec{x}^{(n0)}\right\|}{c(1-\bar{\beta})^5(r_{kn}^{(0)})^2}+\frac{32\left\|\vec{x}^{(n0)}\right\|}{c^3(1-\bar{\beta})^5\left(r_{kn}^{(0)}\right)^2}+\frac{64\omega\left\|\vec{x}^{(n0)}\right\|}{c^3(1-\bar{\beta})^5r_{kn}^{(0)}}+\right.$$

$$\left.+\frac{32}{c^2(1-\bar{\beta})^3r_{kn}^{(0)}}\frac{\omega^h}{\mu^h}\omega U_0\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\bar{u}_\gamma^{(k)}\right|.$$

Therefore

$$\left|U_\alpha^k(u)-U_\alpha^k(\bar{u})\right|\leq\sum_{n=1,n\neq k}^{3}\Xi_{kn}\sum_{\gamma=1}^{3}\left|\xi_\gamma^{(kn)}(t)-\bar{\xi}_\gamma^{(kn)}(t)\right|+$$

$$+\left(\sum_{n=1,n\neq k}^{3}V_{kn}+\frac{e_k^2}{m_k}\frac{20\omega^2}{c^3(1-\bar{\beta})^{5/2}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}(t)-\bar{u}_\gamma^{(k)}(t)\right|+$$

$$+\sum_{n=1,n\neq k}^{3}U_{kn}\sum_{\gamma=1}^{3}\left|u_\gamma^{(n)}(t-\tau_{kn})-\bar{u}_\gamma^{(n)}(t-\tau_{kn})\right|+$$

$$+\sum_{n=1,n\neq k}^{3}\dot{U}_{kn}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|+$$

$$+\frac{e_k^2}{m_k}\frac{4}{c^3(1-\bar{\beta})^{1/2}}\sum_{\gamma=1}^{3}\left|\ddot{u}_\gamma{}^{(k)}(t)-\ddot{\bar{u}}_\gamma{}^{(k)}(t)\right|,$$

where

$$\Xi_{kn}=\frac{|e_ke_n|}{m_k}\left(\frac{80c}{(1-\bar\beta)^4\left(r_{kn}^{(0)}\right)^3}+\frac{324\omega}{(1-\bar\beta)^6\left(r_{kn}^{(0)}\right)^2}+\frac{32c\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{(1-\bar\beta)^4\left(r_{kn}^{(0)}\right)^4}+\frac{400\omega\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{(1-\bar\beta)^6\left(r_{kn}^{(0)}\right)^3}\right);$$

$$\begin{aligned}V_{kn}\ &=\tfrac{|e_ke_n|}{m_k}\left(\tfrac{8\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{(1-\bar\beta)^3\left(r_{kn}^{(0)}\right)^3}+\tfrac{108\omega\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{c(1-\bar\beta)^6\left(r_{kn}^{(0)}\right)^2}+\tfrac{4}{(1-\bar\beta)^3\left(r_{kn}^{(0)}\right)^2}+\tfrac{42\omega}{c(1-\bar\beta)^6r_{kn}^{(0)}}+\right.\\&+\tfrac{16\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{(1-\bar\beta)^4(r_{kn}^{(0)})^3}+\tfrac{72\omega\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{c(1-\bar\beta)^6(r_{kn}^{(0)})^2}+\tfrac{16\left\|\vec x^{(n0)}\right\|}{c^3(1-\bar\beta)^6\left(r_{kn}^{(0)}\right)^2}+\tfrac{32\omega\left\|\vec x^{(n0)}\right\|}{c^3(1-\bar\beta)^6r_{kn}^{(0)}}+\\&+\tfrac{16}{c^2(1-\bar\beta)^4r_{kn}^{(0)}}\tfrac{\omega^h}{\mu^h}\omega U_0+\tfrac{32\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{(1-\bar\beta)^3(r_{kn}^{(0)})^3}+\tfrac{144\omega\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{c(1-\bar\beta)^5(r_{kn}^{(0)})^2}+\\&\left.+\tfrac{32\left\|\vec x^{(n0)}\right\|}{c^3(1-\bar\beta)^5\left(r_{kn}^{(0)}\right)^2}+\tfrac{64\omega\left\|\vec x^{(n0)}\right\|}{c^3(1-\bar\beta)^5r_{kn}^{(0)}}+\tfrac{32}{c^2(1-\bar\beta)^3r_{kn}^{(0)}}\tfrac{\omega^h}{\mu^h}\omega U_0\right);\end{aligned}$$

$$U_{kn}=\tfrac{|e_ke_n|}{m_k}\left(\tfrac{96\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{(1-\bar\beta)^4\left(r_{kn}^{(0)}\right)^3}+\tfrac{172\omega\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{c(1-\bar\beta)^6\left(r_{kn}^{(0)}\right)^2}+\tfrac{24}{(1-\bar\beta)^5\left(r_{kn}^{(0)}\right)^2}+\tfrac{60\omega}{c(1-\bar\beta)^6r_{kn}^{(0)}}\right);$$

$$\dot U_{kn}=\tfrac{|e_ke_n|}{m_k}\left(\tfrac{48\left\|\vec x^{(k0)}-\vec x^{(n0)}\right\|}{c(1-\bar\beta)^6(r_{kn}^{(0)})^2}+\tfrac{20}{c(1-\bar\beta)^6r_{kn}^{(0)}}\right).$$

Then

$$\begin{aligned}&\left|B_\alpha^{(k)}(u)(t)-B_\alpha^{(k)}(\bar u)(t)\right|\le\int_{pT}^t\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+\left|\left(\tfrac{t-pT}{T}-\tfrac12\right)\int_{pT}^{(p+1)T}\left(U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right)ds+\right.\\&\quad\left.+\tfrac1T\int_{pT}^{(p+1)T}\int_{pT}^{\theta}\left(U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right)dsd\theta\ \right|\\&=\int_{pT}^t\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+\left|\tfrac{t-pT}{T}\int_{pT}^{(p+1)T}\left(U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right)ds-\right.\\&\quad\left.-\tfrac12\int_{pT}^{(p+1)T}\left(U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right)ds+\tfrac1T\int_{pT}^{(p+1)T}\int_{pT}^{\theta}\left(U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right)dsd\theta\right|\\&=\int_{pT}^t\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+\left|\tfrac{t-pT}{T}\int_{pT}^{(p+1)T}\left(U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right)ds\ \right|\\&\le\int_{pT}^t\left|U_\alpha^k(u)-U_\alpha^k(\bar u)\right|ds+\int_{pT}^{(p+1)T}\left|U_\alpha^k(u)(s)-U_\alpha^k(\bar u)(s)\right|ds\\&\le\sum_{n=1,n\neq k}^3\Xi_{kn}\sum_{\gamma=1}^3\int_{pT}^t\left|\xi_\gamma^{(kn)}(s)-\bar\xi_\gamma^{(kn)}(s)\right|ds+\\&\quad+\left(\sum_{n=1,n\neq k}^3V_{kn}+\tfrac{e_k^2}{m_k}\tfrac{20\omega^2}{c^3(1-\bar\beta)^{5/2}}\right)\sum_{\gamma=1}^3\int_{pT}^t\left|u_\gamma^{(k)}(s)-\bar u_\gamma^{(k)}(s)\right|ds+\\&\quad+\sum_{n=1,n\neq k}^3U_{kn}\sum_{\gamma=1}^3\int_{pT}^t\left|u_\gamma^{(n)}(s-\tau_{kn})-\bar u_\gamma^{(n)}(s-\tau_{kn})\right|ds+\\&\quad+\sum_{n=1,n\neq k}^3\dot U_{kn}\sum_{\gamma=1}^3\int_{pT}^t\left|\dot u_\gamma^{(n)}(s-\tau_{kn})-\dot{\bar u}_\gamma^{(n)}(s-\tau_{kn})\right|ds+\\&\quad+\tfrac{e_k^2}{m_k}\tfrac{4}{c^3(1-\bar\beta)^{1/2}}\sum_{\gamma=1}^3\int_{pT}^t\left|\ddot u_\gamma{}^{(k)}(s)-\ddot{\bar u}_\gamma{}^{(k)}(s)\right|ds+\\&\quad+\left[\sum_{n=1,n\neq k}^3\Xi_{kn}\sum_{\gamma=1}^3\int_{pT}^{(p+1)T}\left|\xi_\gamma^{(kn)}(s)-\bar\xi_\gamma^{(kn)}(s)\right|ds+\right.\end{aligned}$$

$$+\left(\sum_{n=1,n\neq k}^{3} V_{kn}+\frac{e_k^2}{m_k}\frac{20\omega^2}{c^3\left(1-\bar{\beta}\right)^{5/2}}\right)\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|u_\gamma^{(k)}(s)-\bar{u}_\gamma^{(k)}(s)\right|ds+$$

$$+\sum_{n=1,n\neq k}^{3} U_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|u_\gamma^{(n)}(s-\tau_{kn})-\bar{u}_\gamma^{(n)}(s-\tau_{kn})\right|ds+$$

$$+\sum_{n=1,n\neq k}^{3} \dot{U}_{kn}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|\dot{u}_\gamma^{(n)}(s-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(s-\tau_{kn})\right|ds+$$

$$\left.+\frac{e_k^2}{m_k}\frac{4}{c^3\left(1-\bar{\beta}\right)^{1/2}}\sum_{\gamma=1}^{3}\int_{pT}^{(p+1)T}\left|\ddot{u}_\gamma{}^{(k)}(s)-\ddot{\bar{u}}_\gamma{}^{(k)}(s)\right|ds\right].$$

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